RAFFLES INSTITUTION
H2 Mathematics 9758
2025 Year 6 Term 2 Revision (Summary)

Topic: Vectors 1 (Vector Algebra, Ratio Theorem, Scalar and Vector Product)

Summary for Vectors 1

Vector Algebra

With reference to an origin $O(0, 0, 0)$, given points $A\left(a_1, a_2, a_3\right)$ and $B\left(b_1, b_2, b_3\right)$, we have the corresponding (position vectors) expressed in column form $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$.

$$\mathrm{a} \pm \mathrm{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \pm \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_1 \pm b_1\\a_2 \pm b_2\\a_3 \pm b_3\end{pmatrix} \text{ and}$$

$$k\mathrm{a}=k\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}ka_1\\ka_2\\ka_3\end{pmatrix} \text{ with } k \text{ a real number.}$$

The **magnitude** (or modulus) of a vector, $|\mathbf{a}|$, is the non-negative number

$|\mathbf{a}|=\left|\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\right|=\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}$. This value is equal to the distance from O to A.

We say that $\mathbf{a}$ is parallel to $\mathbf{b}$, denoted by $\mathbf{a} \| \mathbf{b}$, if and only if $\mathbf{b}=\lambda\mathbf{a}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, that is, $\mathbf{b}$ is a (non-zero) scalar multiple of $\mathbf{a}$.

- If $\lambda>0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in the **same** direction.
- If $\lambda<0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in **opposite** directions.

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O, such that $\mathbf{b}=\lambda\mathbf{a}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. We then say that the points O, A and B are **collinear**.

The **unit vector** in the **direction** of $\mathbf{a}$ denoted by $\hat{\mathbf{a}}$ is obtained by scaling $\mathbf{a}$ by $\frac{1}{|\mathbf{a}|}$, thus $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.

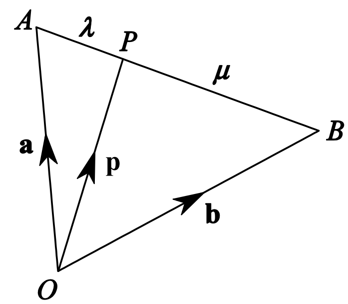
The vectors $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}-2\\-4\\4\end{pmatrix}$ are parallel since $\begin{pmatrix}-2\\-4\\4\end{pmatrix}$ is a scalar multiple of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ $(k=-2)$ but are in opposite directions since $k<0$. The points $(-2,-4,4)$, $(0,0,0)$ and $(1,2,-2)$ are also said to be collinear. The magnitude of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ is $\sqrt{1^2+2^2+(-2)^2}=3$ so the unit vector in the direction of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ is $\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$

Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero and **non-parallel** vectors:

- If $\lambda\mathbf{a} = \mu\mathbf{b}$ for some $\lambda,\ \mu \in \mathbb{R}$, then $\lambda = \mu = 0$.
- If $\alpha\mathbf{a} + \beta\mathbf{b} = s\mathbf{a} + t\mathbf{b}$ for some $\alpha, \beta, s, t \in \mathbb{R}$, then $\alpha = s$, $\beta = t$.

Note the importance of **non-parallel** vectors when comparing coefficients.
Suppose $\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\0\\0\end{pmatrix}$ then $6\mathbf{a}+2\mathbf{b} = 4\mathbf{a}+3\mathbf{b}$ however we cannot "compare coefficients" of vectors **a** and **b** (note that **a** is parallel to **b**) as $6 \neq 4$ and $2 \neq 3$.

Ratio Theorem



Consider a triangle OAB with $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$.
So $\mathbf{a}$ and $\mathbf{b}$ are non-zero and **non-parallel** vectors.
Let P be a point which divides AB in the ratio $\lambda:\mu$,
i.e. $\dfrac{AP}{PB}=\dfrac{\lambda}{\mu}$.

If $\overrightarrow{OP}=\mathbf{p}$, then $\mathbf{p}=\dfrac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$ (MF27)

Note that the Ratio Theorem is an immediate consequence of the addition of vectors.
From diagram, $\mathbf{p}-\mathbf{a}=\dfrac{\lambda}{\lambda+\mu}(\mathbf{b}-\mathbf{a})$, rearranging we have $\mathbf{p}=\dfrac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$. Sometimes, it is easier to use $\mathbf{p}-\mathbf{a}=\dfrac{\lambda}{\lambda+\mu}(\mathbf{b}-\mathbf{a})$ like the following example.

Points A, B and P have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ respectively, relative to the origin O. Given that $\mathbf{a}=\begin{pmatrix}2\\2\\5\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\6\\-3\end{pmatrix}$, find $\mathbf{p}$ if P lies on AB produced such that $\dfrac{AB}{AP}=\dfrac{2}{5}$.

Solution: Easier to find directly, $\mathbf{p}-\mathbf{a}=\dfrac{5}{2}(\mathbf{b}-\mathbf{a})$

[DO NOT WRITE $\dfrac{\mathbf{b}-\mathbf{a}}{\mathbf{p}-\mathbf{a}}=\dfrac{2}{5}$. We cannot divide vectors]

$$\mathbf{p}=\begin{pmatrix}2\\2\\5\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2-2\\6-2\\-3-5\end{pmatrix}=\begin{pmatrix}2\\12\\-15\end{pmatrix}$$

Scalar (Dot) Product

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O.

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where $\theta = \angle AOB$.

Re-arranging, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

- If vectors $\mathbf{a}$ and $\mathbf{b}$ are in the **same direction** then $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$, largest possible value.
- If vectors $\mathbf{a}$ and $\mathbf{b}$ are in the **opposite direction** then $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$, smallest possible value.

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, denoted by $\mathbf{a}\perp\mathbf{b}$, if and only if $\mathbf{a}\cdot\mathbf{b}=0$.
In particular, $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$ as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular.

If $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, then $\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3$.

Find the cosine of the angle between the vectors $a = 2\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $b=\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and determine if the angle is acute or obtuse.
Solution $\cos\theta = \dfrac{(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})}{\sqrt{2^2+3^2+2^2}\sqrt{1^2+2^2+(-1)^2}}$
$\cos\theta = \dfrac{2-6-2}{\sqrt{17}\sqrt{6}} = -\sqrt{\dfrac{6}{17}} < 0$
θ is an obtuse angle

Properties of Scalar Product

(i) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

(ii) $\mathbf{a}\cdot(\mathbf{b}\pm\mathbf{c})=(\mathbf{a}\cdot\mathbf{b})\pm(\mathbf{a}\cdot\mathbf{c})$

(iii) $\lambda(\mathbf{a}\cdot\mathbf{b})=(\lambda\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(\lambda\mathbf{b})$

(iv) $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2 \Leftrightarrow |\mathbf{a}|=\sqrt{\mathbf{a}\cdot\mathbf{a}}$ note the relationship between the scalar product and the modulus

Given that $\lvert\mathbf{a}\rvert=2,\ \lvert\mathbf{b}\rvert=3,\ \mathbf{a}\cdot\mathbf{b}=-1$,	
(i) $(\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})$ $=2\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}$ $=2\lvert\mathbf{a}\rvert^2-\mathbf{a}\cdot\mathbf{b}-\lvert\mathbf{b}\rvert^2$ $=2(2^2)-(-1)-3^2$ $=0$	**(ii)** $(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})$ $=4\mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}$ $=4\lvert\mathbf{a}\rvert^2-\lvert\mathbf{b}\rvert^2$ $=4(2^2)-3^2$ $=7$
(iii) $\lvert\mathbf{a}-2\mathbf{b}\rvert=\sqrt{(\mathbf{a}-2\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})}$ $=\sqrt{\mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{a}+4\mathbf{b}\cdot\mathbf{b}}$ $=\sqrt{\lvert\mathbf{a}\rvert^2-4\mathbf{a}\cdot\mathbf{b}+4\lvert\mathbf{b}\rvert^2}=\ldots=2\sqrt{11}$	

Applications of Scalar Product

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O.
Given that P is the point on the line OB such that $\overrightarrow{AP} \perp \overrightarrow{OB}$, it can be shown that

$$\overrightarrow{OP} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}$$

P is the foot of perpendicular from A to the line OB and P is also the point on the line OB nearest to A.
Thus, the **length of projection of vector a onto vector $\mathbf{b}$** is given by $\left|\overrightarrow{OP}\right| = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$.

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O.

Given that $a = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $b = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, find the length of projection of **a onto vector $\mathbf{b}$.**
Find the position vector of the foot of the perpendicular from A to OB.

Solution

$$OP = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right| = \frac{\left|(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})\right|}{\sqrt{1^2+2^2+(-1)^2}} = \frac{|2-6-2|}{\sqrt{6}} = \sqrt{6}$$

$$\overrightarrow{OP} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}} = -\left(\frac{6}{\sqrt{6}}\right)\frac{\mathbf{i}+2\mathbf{j}-\mathbf{k}}{\sqrt{6}} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

Vector (Cross) Product

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors that are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

$$\mathbf{a}\times\mathbf{b} = \left(|\mathbf{a}||\mathbf{b}|\sin\theta\right)\hat{\mathbf{n}}$$

where $\theta = \angle AOB$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

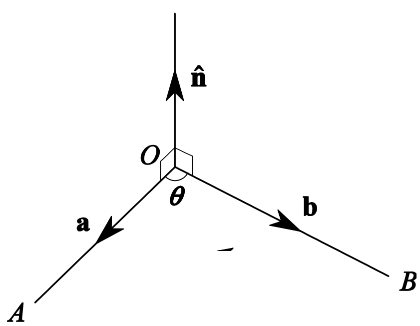


Figure 1

It follows that two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** if and only if $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|$.
In particular, $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$ and $\mathbf{k}\times\mathbf{i} = \mathbf{j}$.

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **parallel** if and only if $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.
In particular, $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$.

Properties of Vector Product

1. $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$.
2. $|\mathbf{a}\times\mathbf{b}| = |\mathbf{b}\times\mathbf{a}| = |\mathbf{a}||\mathbf{b}||\sin\theta| \le |\mathbf{a}||\mathbf{b}|$.
3. $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\pm(\mathbf{a}\times\mathbf{c})$.
4. $\lambda(\mathbf{a}\times\mathbf{b}) = (\lambda\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(\lambda\mathbf{b})$.
5. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$ So $\mathbf{a}\times\mathbf{b}\perp\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}\perp\mathbf{b}$.

Examples of how the properties are used

$$\begin{aligned}|(\mathbf{a}+2\mathbf{b})\times(3\mathbf{a}-\mathbf{b})| &= |3\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+6\mathbf{b}\times\mathbf{a}-2\mathbf{b}\times\mathbf{b}| \\ &= |\mathbf{0}+\mathbf{b}\times\mathbf{a}+6\mathbf{b}\times\mathbf{a}+\mathbf{0}| \\ &= 7|\mathbf{b}\times\mathbf{a}|\end{aligned}$$

$$\begin{aligned}|(\alpha\mathbf{a}+\beta\mathbf{b})\times(\alpha\mathbf{a}-\beta\mathbf{b})| &= |\alpha^2\mathbf{a}\times\mathbf{a}-\alpha\beta\mathbf{a}\times\mathbf{b}+\alpha\beta\mathbf{b}\times\mathbf{a}-\beta^2\mathbf{b}\times\mathbf{b}| \\ &= |\alpha\beta||\mathbf{0}+\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{a}+\mathbf{0}| \qquad \left[\text{Note that:} |\alpha\mathbf{a}| = |\alpha||\mathbf{a}|\right] \\ &= 2|\alpha\beta||\mathbf{b}\times\mathbf{a}|\end{aligned}$$

Given that $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=\mathbf{0}$, what can be deduced about the vectors $\mathbf{u}$ and $\mathbf{v}$?

Solution:

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) &= \mathbf{0} \\ \mathbf{u}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}+\mathbf{v}\times\mathbf{u}-\mathbf{v}\times\mathbf{v} &= \mathbf{0} \\ \mathbf{0}+2\mathbf{v}\times\mathbf{u}-\mathbf{0} &= \mathbf{0} \\ \mathbf{v}\times\mathbf{u} &= \mathbf{0}\end{aligned}$$

Then, $\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$ or $\mathbf{u}\parallel\mathbf{v}$

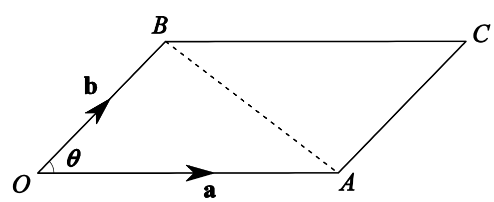
Let $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$ be two points in three-dimensional space and let the position vectors of A and B with respect to the origin O be $\mathbf{a}$ and $\mathbf{b}$ respectively.

Then vector (cross) product $\mathbf{a}\times\mathbf{b}$, is the vector given by

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix} = \begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix} \text{ (MF27)}$$

Applications of Vector Product

Let the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to the origin O, and let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Let C be the point such that $OACB$ is a parallelogram.



Then we have

(1) Area of triangle OAB is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

(2) Area of parallelogram $OACB$ is $|\mathbf{a}\times\mathbf{b}|$.

(3) Distance from B to OA is $|\mathbf{b}\times\hat{\mathbf{a}}|$.

(Recall the **length of projection** of $\overrightarrow{OB}$ onto $\overrightarrow{OA}$ is $|\mathbf{b}\cdot\hat{\mathbf{a}}|$)