RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 6

Term 2 Week 10 Math Focus
Topic: Vectors 1

1 9740/2016/01/Q5

The vectors **u** and **v** are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$, where a and b are constants.

(i) Find $(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v})$ in terms of a and b. [2]

(ii) Given that the **i**- and **k**-components of the answer to part **(i)** are equal, express $(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v})$ in terms of a only. Hence find, in an exact form, the possible values of a for which $(\mathbf{u}+\mathbf{v}) \times (\mathbf{u}-\mathbf{v})$ is a unit vector. [4]

(iii) Given instead that $(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v}) = 0$, find the numerical value of $|\mathbf{v}|$. [2]

Qn 1	Solution
(i)	$(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ $= \mathbf{u}\times\mathbf{u} + \mathbf{v}\times\mathbf{u} - \mathbf{u}\times\mathbf{v} - \mathbf{v}\times\mathbf{v}$ $= 2\mathbf{v}\times\mathbf{u}$ (Since $\mathbf{u}\times\mathbf{u} = \mathbf{0}$, $\mathbf{v}\times\mathbf{v} = \mathbf{0}$ and $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$) $= 2\begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ $= 2\begin{pmatrix} b \\ 2b-2a \\ -a \end{pmatrix} = 2b\mathbf{i} + (4b-4a)\mathbf{j} - 2a\mathbf{k}$
(ii)	Given that $b = -a$ $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = 2\begin{pmatrix} -a \\ -4a \\ -a \end{pmatrix} = 2a\begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$ $\lvert(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})\rvert = 1 \Rightarrow \lvert 2a\rvert\sqrt{18} = 1 \Rightarrow a = \pm\frac{1}{2\sqrt{18}} = \pm\frac{1}{6\sqrt{2}}$
(iii)	$(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v}) = 0 \quad \Rightarrow \mathbf{u.u} - \mathbf{v.v} = 0$ $\Rightarrow \lvert\mathbf{u}\rvert^2 - \lvert\mathbf{v}\rvert^2 = 0$ $\Rightarrow \lvert\mathbf{v}\rvert = \lvert\mathbf{u}\rvert = \sqrt{2^2 + (-1)^2 + 2^2} = 3$

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Vectors **a**, **b** and **c** are such that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \times 3\mathbf{b} = 2\mathbf{a} \times \mathbf{c}$.

(i) Show that $3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, where λ is a constant. [2]

(ii) It is now given that **a** and **c** are unit vectors, that the modulus of **b** is 4 and that the angle between **b** and **c** is 60°. Using a suitable scalar product, find exactly the two possible values of λ. [5]

Qn 2	Solution
(i)	$\mathbf{a} \times (3\mathbf{b} - 2\mathbf{c}) = \mathbf{a} \times 3\mathbf{b} - \mathbf{a} \times 2\mathbf{c}$ $= 2\mathbf{a} \times \mathbf{c} - \mathbf{a} \times 2\mathbf{c}$ $= \mathbf{0}$ $\therefore \mathbf{a} // (3\mathbf{b} - 2\mathbf{c})$ or $3\mathbf{b} - 2\mathbf{c} = \mathbf{0}$ Thus, $3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, where λ is a constant.
(ii)	$\mathbf{b}.\mathbf{c} = \lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert\cos 60^\circ = 2$ $(3\mathbf{b} - 2\mathbf{c}).(3\mathbf{b} - 2\mathbf{c}) = \lambda\mathbf{a}.\lambda\mathbf{a}$ $9\lvert\mathbf{b}\rvert^2 - 12\mathbf{b}.\mathbf{c} + 4\lvert\mathbf{c}\rvert^2 = \lambda^2 \lvert\mathbf{a}\rvert^2$ $144 - 24 + 4 = \lambda^2 \Rightarrow \lambda = \pm\sqrt{124} = \pm 2\sqrt{31}$

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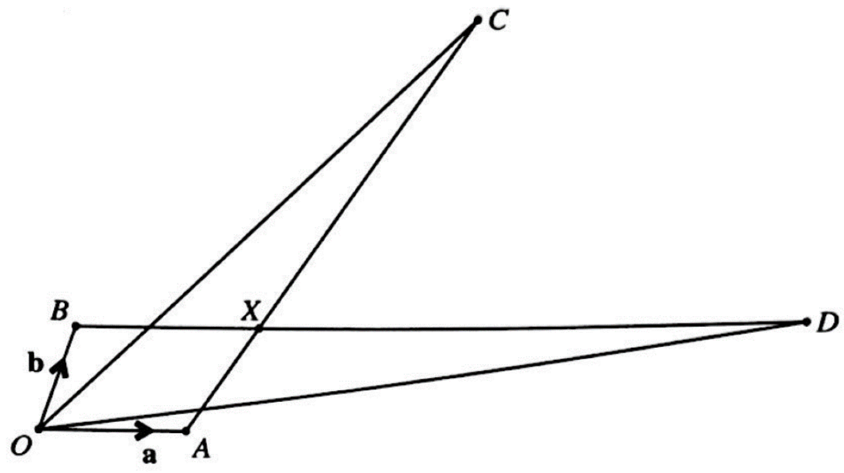

With reference to the origin O. the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = 2\mathbf{a} + 4\mathbf{b}$ and $\overrightarrow{OD} = \mathbf{b} + 5\mathbf{a}$. The lines BD and AC cross at X (see diagram).

(i) Express $\overrightarrow{OX}$ in terms of **a** and **b**. [4]

The point Y lies on CD and is such that the points O. X and Y are collinear.

(ii) Express $\overrightarrow{OY}$ in terms of **a** and **b** and find the ratio $OX : OY$. [6]

Qn 3	Solution
(i)	$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}$ $= \mathbf{b} + \lambda\overrightarrow{BD}$ $= \mathbf{b} + \lambda\left(\overrightarrow{OD} - \overrightarrow{OB}\right)$ $= \mathbf{b} + \lambda\left(\mathbf{b} + 5\mathbf{a} - \mathbf{b}\right)$ $= 5\lambda\mathbf{a} + \mathbf{b}$ $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ $= \mathbf{a} + \mu\overrightarrow{AC}$ $= \mathbf{a} + \mu\left(\overrightarrow{OC} - \overrightarrow{OA}\right)$ $= \mathbf{a} + \mu\left(2\mathbf{a} + 4\mathbf{b} - \mathbf{a}\right)$ $= \left(1 + \mu\right)\mathbf{a} + 4\mu\mathbf{b}$ $\therefore \mathbf{b} + 5\lambda\mathbf{a} = \left(1 + \mu\right)\mathbf{a} + 4\mu\mathbf{b}$ Since $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ and **a** is not parallel to **b**, $4\mu = 1 \Rightarrow \mu = \frac{1}{4}$ $5\lambda = 1 + \mu = \frac{5}{4} \Rightarrow \lambda = \frac{1}{4}$ $\overrightarrow{OX} = \frac{5}{4}\mathbf{a} + \mathbf{b}$

(ii)	Since Y lies on CD, let $CY:YD = 1-\alpha:\alpha$. $\overrightarrow{OY} = \alpha\overrightarrow{OD} + (1-\alpha)\overrightarrow{OC}$ $= \alpha(5\mathbf{a}+\mathbf{b}) + (1-\alpha)(2\mathbf{a}+4\mathbf{b})$ Similarly, O, X and Y are collinear, $\overrightarrow{OY} = \beta\overrightarrow{OX}$ $= \beta\left(\frac{5}{4}\mathbf{a}+\mathbf{b}\right)$ $\therefore \alpha(5\mathbf{a}+\mathbf{b}) + (1-\alpha)(2\mathbf{a}+4\mathbf{b}) = \beta\left(\frac{5}{4}\mathbf{a}+\mathbf{b}\right)$ $\Rightarrow \left[5\alpha + 2(1-\alpha)\right]\mathbf{a} + \left[\alpha + 4(1-\alpha)\right]\mathbf{b} = \frac{5}{4}\beta\mathbf{a} + \beta\mathbf{b}$ Since $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}$ is not parallel to $\mathbf{b}$, $\beta = \alpha + 4(1-\alpha) \Rightarrow \beta + 3\alpha = 4$ ...(1) $\frac{5}{4}\beta = 5\alpha + 2(1-\alpha) \Rightarrow \frac{5}{4}\beta - 3\alpha = 2$ ...(2) $(1)+(2)$: $\frac{9}{4}\beta = 6$ $\therefore \beta = \frac{24}{9} = \frac{8}{3}$ $\overrightarrow{OY} = \beta\overrightarrow{OX} = \frac{8}{3}\left(\frac{5}{4}\mathbf{a}+\mathbf{b}\right) = \frac{10}{3}\mathbf{a} + \frac{8}{3}\mathbf{b}$ $OX:OY = 3:8$