Paper 1 Solutions

1 A cubic curve passes through the points $(2,3)$ and $(-3,-22)$. Find the equation of the curve if it has a stationary point at $(1,-6)$. [4]

[Solution]

Let $y = ax^3 + bx^2 + cx + d$

$a(2)^3 + b(2)^2 + c(2) + d = 3 \quad \Rightarrow \quad 8a + 4b + 2c + d = 3 \quad \ldots\text{Eq(1)}$

$a(-3)^3 + b(-3)^2 + c(-3) + d = -22 \quad \Rightarrow \quad -27a + 9b - 3c + d = -22 \quad \ldots\text{Eq(2)}$

$a(1)^3 + b(1)^2 + c(1) + d = -6 \quad \Rightarrow \quad a + b + c + d = -6 \quad \ldots\text{Eq(3)}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + 2bx + c$

$3a(1)^2 + 2b(1) + c = 0 \quad \Rightarrow \quad 3a + 2b + c = 0 \quad \ldots\text{Eq(4)}$

From GC, $a = 2, b = 1, c = -8, d = -1$

$\therefore$ Equation of curve is $y = 2x^3 + x^2 - 8x - 1$

2 (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3]

(b) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{6}} \sin 2\theta \cos 3\theta \, \mathrm{d}\theta$ exactly. [4]

Solutions

(i)

$$\begin{aligned}
\cos 3\theta &= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \quad \text{(shown)}
\end{aligned}$$

(ii)

$$\int_0^{\frac{\pi}{6}} \sin 2\theta \cos 3\theta \, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{6}} (2\sin\theta\cos\theta)(4\cos^3\theta - 3\cos\theta)\, \mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{6}} (8\sin\theta\cos^4\theta - 6\sin\theta\cos^2\theta)\, \mathrm{d}\theta$$

$$= \left[-\frac{8\cos^5\theta}{5} + \frac{6\cos^3\theta}{3}\right]_0^{\frac{\pi}{6}}$$

$$= \left(-\frac{8}{5}\cos^5\frac{\pi}{6} + 2\cos^3\frac{\pi}{6}\right) - \left(-\frac{8}{5}\cos^5 0 + 2\cos^3 0\right)$$

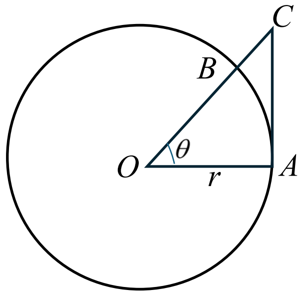
$$= \left(-\frac{8}{5}\left(\frac{\sqrt{3}}{2}\right)^5 + 2\left(\frac{\sqrt{3}}{2}\right)^3\right) - \frac{2}{5}$$

$$= -\frac{8}{5}\left(\frac{9}{16}\right)\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{3}{4}\right)\left(\frac{\sqrt{3}}{2}\right) - \frac{2}{5}$$

$$= \frac{3\sqrt{3}}{10} - \frac{2}{5}$$

3 (a) Given that θ is small, show that $\sec\theta \approx 1+\frac{1}{2}\theta^2$ [2]

(b) The diagram below shows a circle, centre O and radius r, with points A and B on the circumference such that $\angle AOB = \theta$ radians, where θ is small. AC is a tangent to the circle at A and OBC is a straight line.



(i) Show that the length of chord AB can be approximated by $r\theta$. [2]

(ii) Hence show that the perimeter of triangle ABC can be approximated by $r\theta(a+b\theta)$, where a and b are constants to be determined. [4]

[Solution]

(a)

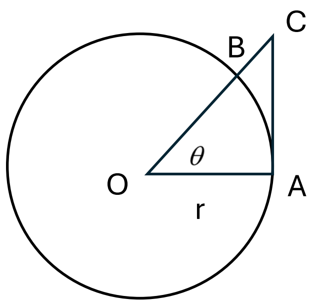
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\approx \left(1-\frac{\theta^2}{2}\right)^{-1} \text{ since } \theta \text{ is small}$$

$$= 1+(-1)\left(-\frac{\theta^2}{2}\right)+\ldots$$

$$\approx 1+\frac{1}{2}\theta^2 \text{ (shown)}$$

(b)(i)



$$AB^2 = r^2+r^2-2(r)(r)\cos\theta$$

$$\Rightarrow AB = \sqrt{2r^2-2r^2\cos\theta}$$

Since θ is a small angle,

$$AB = \left(2r^2 - 2r^2\cos\theta\right)^{\frac{1}{2}}$$

$$\approx \sqrt{2}r\left(1-\left(1-\frac{\theta^2}{2}\right)\right)^{\frac{1}{2}}$$

$$= r\theta$$

b(ii)

$$OC^2 = r^2 + \left(r\tan\theta\right)^2$$
$$= r^2\sec^2\theta$$

$$BC = OC - OB = r\sec\theta - r$$

$$\approx r\left(1+\frac{1}{2}\theta^2 - 1\right)$$
$$= \frac{1}{2}r\theta^2$$

Perimeter of triangle ABC $= AB + BC + AC$

$$= \frac{1}{2}r\theta^2 + r\theta + r\tan\theta$$
$$\approx \frac{1}{2}r\theta^2 + r\theta + r\theta$$

$$= r\theta\left(2+\frac{1}{2}\theta\right)$$

$$\therefore \quad a = 2, \quad b = \frac{1}{2}$$

4 The Folium of Descartes is a curve, defined by the equation $x^3 + y^3 = 3axy$ where a is a real constant. It is given that $a \neq 0$ for this question.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{ay - x^2}{y^2 - ax}$. [2]

(b) The point $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$ lies on this curve. Show that the equation of the normal at this point on the curve is independent of the value of a. [2]

(c) Given that the curve has a stationary point at (pa, qa), where p and q are positive constants, find the exact values of p and q. You need not determine the nature of this stationary point. [4]

[Solution]

(i) Differentiating wrt x: $3x^2 + 3\dfrac{\mathrm{d}y}{\mathrm{d}x}y^2 = 3ay + 3ax\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(y^2 - ax\right) = ay - x^2 \qquad (*)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ay - x^2}{y^2 - ax} \qquad \text{(Shown)}$$

(ii) Gradient of normal at $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right) = -\dfrac{1}{\dfrac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}} = -\dfrac{\frac{3a^2}{4}}{-\frac{3a^2}{4}} = 1$

$\therefore$ Equation of normal is $\quad y - \dfrac{3a}{2} = 1\left(x - \dfrac{3a}{2}\right)$

$y = x$ which is independent of the value of a. (Shown)

(iii) Let $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Rightarrow \quad \dfrac{ay - x^2}{y^2 - ax} = 0$

$$y = \frac{x^2}{a}$$

$$\Rightarrow \quad x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax\left(\frac{x^2}{a}\right)$$

$$x^3\left(1 + \frac{x^3}{a^3} - 3\right) = 0$$

$$\frac{x^3}{a^3}\left(x^3 - 2a^3\right) = 0$$

$x = 0$ (rejected, $a \neq 0$ and $p > 0$) or $x = 2^{\frac{1}{3}}a$

When $x = 2^{\frac{1}{3}}a$, $y = \dfrac{\left(2^{\frac{1}{3}}a\right)^2}{a} = 2^{\frac{2}{3}}a \ (a \neq 0)$

$\Rightarrow \left(2^{\frac{1}{3}}a, 2^{\frac{2}{3}}a\right)$ is a stationary point of the curve, where $p = 2^{\frac{1}{3}}$ and $q = 2^{\frac{2}{3}}$

Method 2

Let $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \Rightarrow \dfrac{ay - x^2}{y^2 - ax} = 0$

At $(pa,\ qa)$, $qa^2 = p^2a^2$

$$\Rightarrow q = p^2$$

At $(pa,\ qa)$, $(pa)^3 + (qa)^3 = 3a(pa)(qa)$

$$p^3 + q^3 = 3pq \ \ (a \neq 0)$$

$p^3 + p^6 = 3p^3$

$p^3\left(p^3 - 2\right) = 0$

$p^3 = 2 \ \ (p > 0)$

$p = 2^{1/3}$ and $q = \left(2^{1/3}\right)^2 = 2^{2/3}$

5 The curve C has equation given by $y = \dfrac{1+3x-18x^2}{9x+3}$.

(a) Without the use of a calculator, find the range of values that y can take. [4]

(b) Sketch the graph of C indicating clearly its asymptotes and all stationary points. [3]

(c) State a sequence of transformations that will transform the curve C to the curve with equation $y = \dfrac{1+3(2x-1)-18(2x-1)^2}{9(2x-1)+3}$. [2]

[Solution]

(i) $9xy + 3y = -18x^2 + 3x + 1$

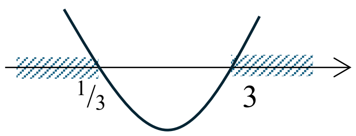
$18x^2 + (9y-3)x + 3y - 1 = 0$

For the equation to have real roots, $(9y-3)^2 - 4(18)(3y-1) \geq 0$

$9(3y-1)^2 - 8(9)(3y-1) \geq 0$

$(3y-1)(3y-1-8) \geq 0$

$(3y-1)(y-3) \geq 0$



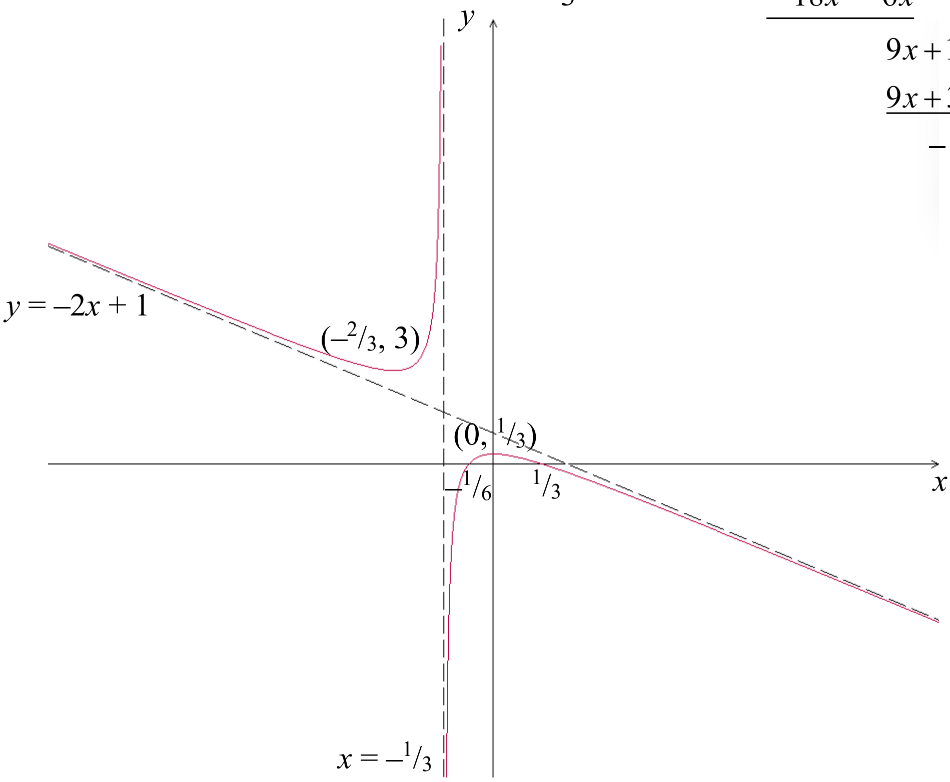
$\Rightarrow \quad y \leq \dfrac{1}{3}$ or $y \geq 3$

Hence the set required is $\{y \in \mathbb{R} : y \leq \dfrac{1}{3}$ or $y \geq 3\}$

(ii) $\dfrac{1+3x-18x^2}{9x+3} = -2x+1-\dfrac{2}{9x+3}$

$$\begin{array}{r|l} & -2x+1 \\ \hline 9x+3 & -18x^2+3x+1 \\ & -18x^2-6x \\ \hline & \quad\quad 9x+1 \\ & \quad\quad 9x+3 \\ \hline & \quad\quad\quad -2 \end{array}$$

$\therefore$ the asymptotes are $y = -2x+1$ and $x = -\dfrac{1}{3}$



(iii) Let $y = \mathrm{f}(x) = \dfrac{1+3x-18x^2}{9x+3}$

$$y = \mathrm{f}(x) \quad \to \quad y = \mathrm{f}\left(x-1\right) = \frac{1+3\left(x-1\right)-18\left(x-1\right)^2}{9\left(x-1\right)+3}$$

$$\to \quad y = \mathrm{f}\left(2x-1\right) = \frac{1+3\left(2x-1\right)-18\left(2x-1\right)^2}{9\left(2x-1\right)+3}$$

The transformations are

(1) Translation of the curve in the positive x direction by 1 unit.

(2) Scaling of scale factor $\dfrac{1}{2}$ along the direction of the x-axis.

OR: (1) Scaling of scale factor $\dfrac{1}{2}$ along the direction of the x-axis.

(2) Translation of the curve in the positive x direction by $\dfrac{1}{2}$ unit.

6 It is given that $\sum_{r=1}^{n} \frac{7r+4}{r(r+1)(r+2)} = \frac{9}{2} - \frac{2}{n+1} - \frac{5}{n+2}$

(a) Find $\sum_{r=7}^{2n} \frac{7r-3}{r^3 - r}$ giving your answers in terms of n. [4]

(b) Show algebraically that $(r+1)^3 > r(r+1)(r+2)$ for all positive integers r. [2]

(c) Hence show that $\sum_{r=1}^{n} \frac{7r+4}{(r+1)^3} < \frac{9}{2}$. [3]

Solution

(a)
$$\begin{aligned}
\sum_{r=7}^{2n} \frac{7r-3}{r^3-r} &= \sum_{r=7}^{2n} \frac{7r-3}{(r-1)(r)(r+1)} \\
&= \sum_{r=6}^{2n-1} \frac{7(r+1)-3}{(r)(r+1)(r+2)} \\
&= \sum_{r=1}^{2n-1} \frac{7r+4}{(r)(r+1)(r+2)} - \sum_{r=1}^{5} \frac{7r+4}{(r)(r+1)(r+2)} \\
&= \frac{9}{2} - \frac{2}{(2n-1)+1} - \frac{5}{(2n-1)+2} - \left(\frac{9}{2} - \frac{2}{5+1} - \frac{5}{5+2}\right) \\
&= \frac{2}{6} + \frac{5}{7} - \frac{2}{2n} - \frac{5}{2n+1} \\
&= \frac{22}{21} - \frac{1}{n} - \frac{5}{2n+1}
\end{aligned}$$

(b) To prove $(r+1)^3 > (r)(r+1)(r+2)$

Method 1

$$\begin{aligned}
\text{LHS} &= (r+1)^3 \\
&= r^3 + 3r^2 + 3r + 1 \\
\text{RHS} &= (r)(r+1)(r+2) \\
&= r^3 + 3r^2 + 2r
\end{aligned}$$

Clearly, $r^3 + 3r^2 + 3r + 1 > r^3 + 3r^2 + 2r$ since $r > 0$.

Therefore, $(r+1)^3 > (r)(r+1)(r+2)$

Method 2

$(r+1)^2 = r^2 + 2r + 1$

$r(r+2) = r^2 + 2r$

Clearly, $(r+1)^2 > r(r+2)$

$(r+1)^3 > r(r+1)(r+2)$ since $r > 0$ (shown)

(c) $\Rightarrow (r+1)^3 > (r)(r+1)(r+2)$

$$\Rightarrow \frac{1}{(r+1)^3} < \frac{1}{(r)(r+1)(r+2)}$$

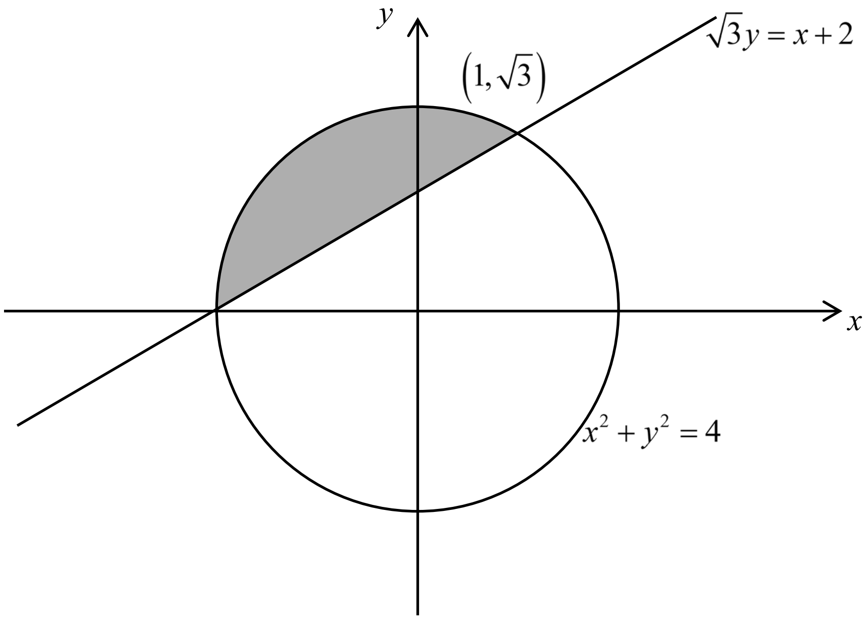
$$\Rightarrow \sum_{r=1}^{n} \frac{1}{(r+1)^3} < \sum_{r=1}^{n} \frac{1}{(r)(r+1)(r+2)}$$

$$\Rightarrow \sum_{r=1}^{n} \frac{7r+4}{(r+1)^3} < \sum_{r=1}^{n} \frac{7r+4}{(r)(r+1)(r+2)}$$

$$= \frac{9}{2} - \frac{2}{n+1} - \frac{5}{n+2}$$

$$< \frac{9}{2}$$ since $\frac{2}{n+1} > 0$ and $\frac{5}{n+2} > 0$

7 Curve C is a circle with radius 2 and center at the origin with equation $x^2 + y^2 = 4$. Line L has the equation $\sqrt{3}y = x + 2$. The diagram below shows the shaded region A which is enclosed between C and L.

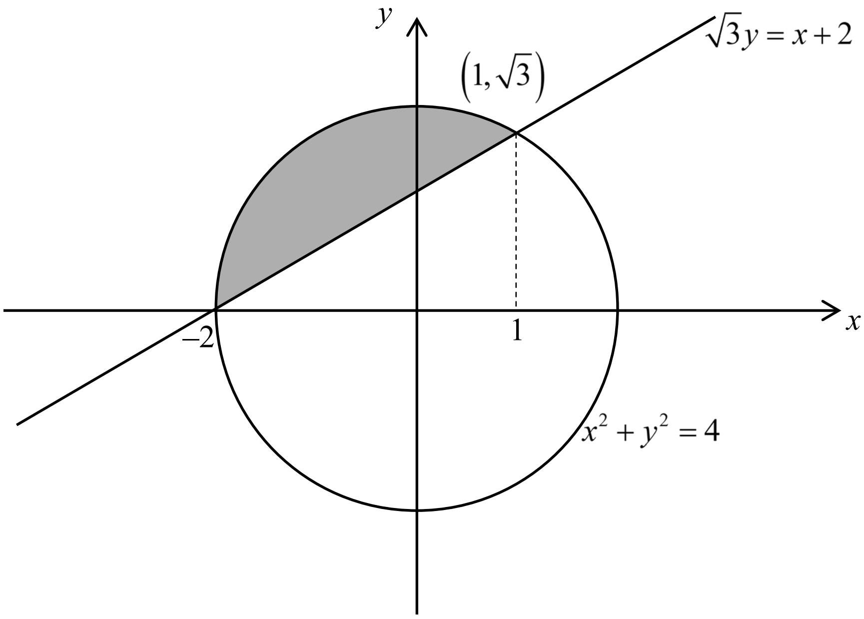


(a) Use the substitution $x = 2\sin\theta$ to show that the area of region A can be written in the expression $\int_b^a 4\cos^2\theta \, \mathrm{d}\theta - c$, where a, b and c are exact constants to be determined. Hence evaluate this area exactly. [6]

(b) The region B is bounded by C, L and the line $x = 2$. Find the volume generated when region B is rotated through 2π radians about the y-axis. Give your answer to two decimal places. [3]

[Solution]

(a)



The circle $x^2 + y^2 = 4$ and line $\sqrt{3}y = x + 2$ intersects at $x = -2$ and $x = 1$

Area of region $A = \int_{-2}^{1} \sqrt{4 - (x)^2}\, \mathrm{d}x -$ area of triangle

$x = 2\sin\theta \quad \Rightarrow \quad \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$

When $x = 1 \quad \Rightarrow \quad \theta = \sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$

When $x = -2 \quad \Rightarrow \quad \theta = \sin^{-1}(-1) = -\dfrac{\pi}{2}$

$$= \int_{-\pi/2}^{\pi/6} \sqrt{4 - (2\sin\theta)^2}\, (2\cos\theta)\, \mathrm{d}\theta - \frac{1}{2}(3)\sqrt{3}$$

$$= \int_{-\pi/2}^{\pi/6} (2\cos\theta)(2\cos\theta)\, \mathrm{d}\theta - \frac{3\sqrt{3}}{2}$$

$$= \int_{-\pi/2}^{\pi/6} 4\cos^2\theta\, \mathrm{d}\theta - \frac{3\sqrt{3}}{2} \text{ (shown)}$$

Where $a = \dfrac{\pi}{6}, b = -\dfrac{\pi}{2}, c = \dfrac{3\sqrt{3}}{2}$

$$4\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \cos^2\theta \,\mathrm{d}\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{\cos 2\theta + 1}{2}\right) \mathrm{d}\theta$$

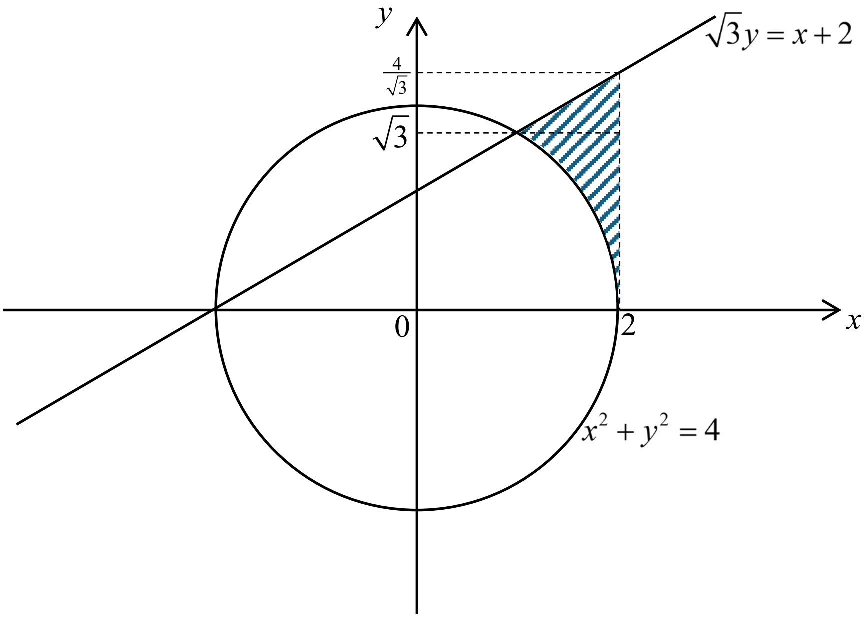
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\cos 2\theta + 1\right) \mathrm{d}\theta$$

$$= 2\left[\frac{\sin 2\theta}{2} + \theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$

Required area $= 2\left[\left(\frac{1}{2}\sin 2\left(\frac{\pi}{6}\right) + \frac{\pi}{6}\right) - \left(\frac{1}{2}\sin 2\left(-\frac{\pi}{2}\right) - \frac{\pi}{2}\right)\right] - \frac{3\sqrt{3}}{2}$

$$= \left(2\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\pi}{2}\right)\right) - \frac{3\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2} + \frac{4\pi}{3} - \frac{3\sqrt{3}}{2} = \frac{4\pi}{3} - \sqrt{3}$$



(b) Volume obtained by rotating region B through 2π radians about y-axis

$$= \text{volume of cylinder} - \pi\int_{\sqrt{3}}^{\frac{4}{\sqrt{3}}}\left(\sqrt{3}y-2\right)^2 \mathrm{d}y - \pi\int_0^{\sqrt{3}}\left(4-y^2\right)\mathrm{d}y$$

$$= \pi(2)^2\left(\frac{4}{\sqrt{3}}\right) - \pi\int_{\sqrt{3}}^{\frac{4}{\sqrt{3}}}\left(\sqrt{3}y-2\right)^2 \mathrm{d}y - \pi\int_0^{\sqrt{3}}\left(4-y^2\right)\mathrm{d}y$$

≈ 8.4644

≈ 8.46 (2 d.p.)

8 (a) (i) The ninth, fifth and second terms of an arithmetic progression are successive terms of a geometric progression with first term a and common ratio r, where $r \neq 1$ and $a > 0$. Find the value of r and deduce that the geometric series is convergent. [3]

(ii) Using the value of r found in (i), find the least value of n such that the sum of all the terms after the nth term of the geometric progression is less than 1% of its sum of the first n terms. [3]

(b) The sum, S_n, of the first n terms of another sequence is given by $S_n = n\ln 2^{q(n+1))}$, where q is a constant. Prove that the sequence follows an arithmetic progression. [4]

[Solution]

(a) Let the first term and common difference of the arithmetic progression be b and d respectively.

Since $b + 8d$, $b + 4d$ and $b + d$ are successive terms of a GP, then $\dfrac{b+4d}{b+8d} = \dfrac{b+d}{b+4d}$ (common ratio).

$$(b+4d)^2 = (b+8d)(b+d)$$

$$8bd + 16d^2 = 9bd + 8d^2$$

$$d(8d - b) = 0$$

$$b = 8d \quad (d \neq 0)$$

$$\Rightarrow r = \frac{8d+4d}{8d+8d} = \frac{3}{4}$$

Since $|r| = \left|\frac{3}{4}\right| < 1,$ the series is convergent.

(ii) $S_\infty - S_n < 0.01S_n \quad \Rightarrow \quad S_\infty < 1.01S_n$

$$\frac{a}{1-\frac{3}{4}} < 1.01\left[\frac{a\left(1-\left(\frac{3}{4}\right)^n\right)}{1-\frac{3}{4}}\right]$$

$$1 < 1.01\left(1-\left(\frac{3}{4}\right)^n\right)$$

$$\left(\frac{3}{4}\right)^n < 0.009901$$

$$n\ln\left(\frac{3}{4}\right) < \ln(0.009901)$$

$$n > \frac{\ln(0.009901)}{\ln\left(\frac{3}{4}\right)} = 16.042$$

Hence least n is 17.

(b) $T_n = S_n - S_{n-1} = n\ln 2^{q(n+1)} - (n-1)\ln 2^{q(n)}$

$$= n(n+1)q\ln 2 - (n-1)(nq)\ln 2$$
$$= nq\ln 2 + nq\ln 2$$
$$= 2nq\ln 2$$

$$T_n - T_{n-1} = 2nq\ln 2 - 2(n-1)q\ln 2$$

$= 2q\ln 2$ (constant $\because$ q is a constant)

$\therefore$ the sequence is an arithmetic progression (Proved).

9 The line ℓ has the equation $\dfrac{x-a}{2}=y-2=\dfrac{1-2z}{b}$, where a and b are real constants. The plane π has the equation $3x-y+4z=10$.

(a) Given that the line ℓ and the plane π do not intersect, show that $a\neq\dfrac{10}{3}$ and $b=\dfrac{5}{2}$

[5]

(b) It is given that $a=1$ and $b=3$. Find the point of intersection between the line ℓ and the plane π. [3]

(c) It is given now that $a=1$, $b=\dfrac{5}{2}$.

(i) Find the distance between the line ℓ and the plane π. [2]

(ii) Determine whether the line ℓ and the origin O lie on the same side of the plane π. State an equation of the other plane that is equidistant from line ℓ and parallel to plane π. [3]

[Solution]

(a)

$$\frac{x-a}{2}=\frac{y-2}{1}=\frac{1-2z}{b} \quad\Rightarrow\quad \frac{x-a}{2}=\frac{y-2}{1}=\frac{z-\frac{1}{2}}{-\frac{b}{2}}$$

Vector equation of line ℓ: $\mathbf{r}=\begin{pmatrix}a\\2\\\frac{1}{2}\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-\frac{b}{2}\end{pmatrix},\ \lambda\in\mathbb{R}$

Vector equation of plane π: $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}=10$

For the line to be parallel to the plane, $\mathbf{d}_\ell\cdot\mathbf{n}_\pi=0$

$$\begin{pmatrix}2\\1\\-\frac{b}{2}\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}=0$$

$$6-1-2b=0$$

$$b=\frac{5}{2}$$

For the line to not intersect the plane, the point must not lie on the plane,

i.e. $\begin{pmatrix} a \\ 2 \\ \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \neq 10$

$3a - 2 + 2 \neq 10$

$a \neq \dfrac{10}{3}$

The line and plane do not intersect when $a \neq \dfrac{10}{3}$ and $b = \dfrac{5}{2}$.

(b)

Equation of line: $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ \frac{1}{2} \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -\frac{3}{2} \end{pmatrix}$, $\lambda \in \mathbb{R}$

Let the position vector of the point of intersection, P be

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ \frac{1}{2} \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ 2+\lambda \\ \frac{1}{2}-\frac{3}{2}\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

Substituting into equation of plane π:

$\begin{pmatrix} 1+2\lambda \\ 2+\lambda \\ \frac{1}{2}-\frac{3}{2}\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = 10$

$3(1+2\lambda) - (2+\lambda) + 4\left(\dfrac{1}{2} - \dfrac{3}{2}\lambda\right) = 10$

$3 + 6\lambda - 2 - \lambda + 2 - 6\lambda = 10$

$\therefore \quad \lambda = -7$

$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ \frac{1}{2} \end{pmatrix} - 7 \begin{pmatrix} 2 \\ 1 \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} -13 \\ -5 \\ 11 \end{pmatrix}$

The point of intersection P is (–13, –5, 11).

(c) (i)

Method 1: Using distance formula

Taking the point $\left(1,\ 2,\ \dfrac{1}{2}\right)$ from the line ℓ:

$$\text{Distance} = \frac{\left|\begin{pmatrix}1\\2\\\frac{1}{2}\end{pmatrix}\cdot\mathbf{n}-d\right|}{|\mathbf{n}|}$$

$$= \frac{\left|\begin{pmatrix}1\\2\\\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}-10\right|}{\sqrt{3^2+(-1)^2+4^2}}$$

$$= \frac{7}{\sqrt{26}} \text{ units}$$

Method 2: Finding length of projection on normal vector

Taking the point $A\left(1,\ 2,\ \dfrac{1}{2}\right)$ from the line ℓ, and the point B (2, 0, 1) from the plane π:

$$\overrightarrow{AB} = \begin{pmatrix}2\\0\\1\end{pmatrix}-\begin{pmatrix}1\\2\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}1\\-2\\\frac{1}{2}\end{pmatrix}$$

$$\text{Required distance } = \frac{\left|\begin{pmatrix}1\\-2\\\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\-1\\4\end{pmatrix}\right|}$$

$$= \frac{|3+2+2|}{\sqrt{3^2+(-1)^2+4^2}}$$

$$= \frac{7}{\sqrt{26}} \text{ units}$$

Method 3: Using distance between 2 parallel planes

Consider a parallel plane containing the line ℓ: $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\2\\\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}=3$

$$\text{Distance}=\left|\frac{10-3}{\sqrt{3^2+(-1)^2+4^2}}\right|$$

$$=\frac{7}{\sqrt{26}}\ \text{units}$$

(c) (ii)

Equation of plane π: $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}=10\Rightarrow\mathbf{r}\cdot\frac{\begin{pmatrix}3\\-1\\4\end{pmatrix}}{\sqrt{26}}=\frac{10}{\sqrt{26}}$

Equation of plane parallel to π containing the line ℓ: $\mathbf{r}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\2\\\frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\4\end{pmatrix}=3$

$$\Rightarrow\mathbf{r}\cdot\frac{\begin{pmatrix}3\\-1\\4\end{pmatrix}}{\sqrt{26}}=\frac{3}{\sqrt{26}}$$

Since they are of the same sign, plane π and line ℓ are on the same side of the origin, therefore both the origin and the line ℓ lie on the same side of the plane.

Another plane π': $\quad 3x-y+4z=-4$

10 The function f is defined by

$$\mathrm{f}: x \to 2x - \frac{1}{2x}, \quad 0 < x < 2$$

It is given that f^{-1} exists.

(a) Define f^{-1} in a similar form. [3]

(b) Sketch, on a single diagram, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$. [3]

(c) The region R is bounded by the curve $y = \mathrm{f}^{-1}(x)$, $y = x$ and the y-axis. Find the exact area of R. [3]

(d) Another function g is defined by

$$\mathrm{g}: x \to \begin{cases} \dfrac{3}{2} + \dfrac{3}{3x-11} & \text{for} \quad x \le 3, \\ \left| x - \dfrac{1}{3}x^2 \right| & \text{for} \quad 3 < x \le 5. \end{cases}$$

Show that the composite function gf exists and find the range of gf. [2]

Solution:

(a) Let $y = 2x - \dfrac{1}{2x}$

$$4x^2 - 2xy - 1 = 0$$

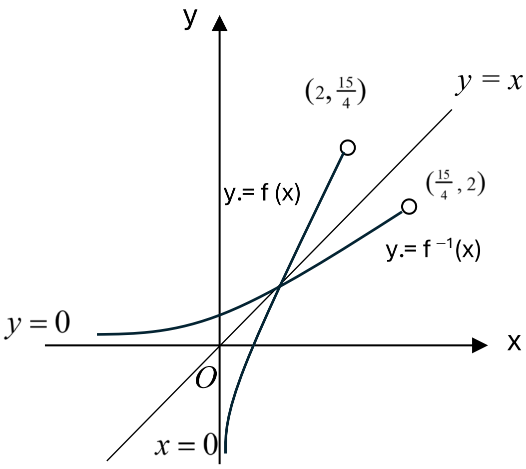
$$\left(2x - \frac{y}{2}\right)^2 = 1 + \frac{y^2}{4}$$

$$x = \frac{y}{4} \pm \frac{1}{4}\sqrt{y^2 + 4}$$

Since $0 < x < 2$, $\therefore x = \dfrac{y}{4} + \dfrac{1}{4}\sqrt{y^2 + 4}$

Hence, $\mathrm{f}^{-1}: x \to \dfrac{x}{4} + \dfrac{1}{4}\sqrt{x^2 + 4}$, $x < \dfrac{15}{4}$

(b)



(c) $x = 2x - \dfrac{1}{2x}$

$$x = \frac{1}{2x}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

Area bounded by the curve $y = \mathrm{f}^{-1}(x)$, $y = \mathrm{f}^{-1}\,\mathrm{f}(x)$ and the y-axis is the same as area bounded by the curve $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}\,\mathrm{f}(x)$ and the x-axis.

Area of the region R

$$= \int_0^{\frac{1}{\sqrt{2}}} x\,\mathrm{d}x - \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} 2x - \frac{1}{2x}\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 - \left[x^2 - \frac{1}{2}\ln|x|\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$

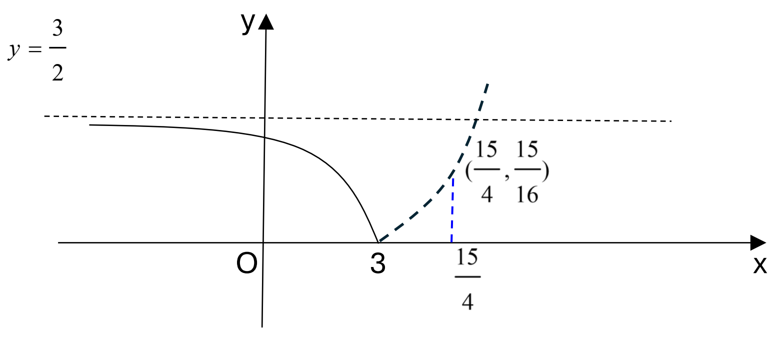
$$= \frac{1}{4} - \left[\left(\frac{1}{2} - \frac{1}{2}\ln\left(\frac{1}{\sqrt{2}}\right)\right) - \left(\frac{1}{4} - \frac{1}{2}\ln\frac{1}{2}\right)\right]$$

$$= -\frac{1}{4}\ln 2 + \frac{1}{2}\ln 2$$

$$= \frac{1}{4}\ln 2 \text{ unit}^2$$

(d) $\mathrm{R_f} = \left(-\infty, \dfrac{15}{4}\right)$, $\mathrm{D_g} = (-\infty, 5]$

Since $\mathrm{R_f} \subseteq \mathrm{D_g}$, gf exists.



$$\mathrm{D_f} = (0,2) \xrightarrow{\mathrm{f}} \left(-\infty, \frac{15}{4}\right) \xrightarrow{\mathrm{g}} \left[0, \frac{3}{2}\right) = \mathrm{R_{gf}}$$

11 In a chemical reaction, compounds X and Y react to form a product. Let x and y denote the concentrations (mol per kilolitres) of X and Y respectively, at time t minutes after the start of the experiment. The initial concentrations of X and Y are given by x_0 and y_0 mol/kL.

(a) In a particular experiment where Y is present in excess, the reaction rate can be modeled as a pseudo-first-order reaction, leading to the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = -ax$, where a is a positive constant.

(i) Solve this differential equation, expressing x in terms of t, x_0 and a. [3]

(ii) The half-life of the reaction, denoted by $t_{0.5}$, is defined as the time taken for the concentration of X to decrease to half its initial value. Show that $t_{0.5} = \frac{\ln 2}{a}$. [2]

(b) In another experiment conducted by a chemist, the rate of the reaction, is directly proportional to the product of the concentration of X and the square of the concentration of Y. It is also known that at any instance during the reaction, every 1 mol of X reacts with every 2 mol of Y giving the equation $y_0 - y = 2(x_0 - x)$.

The initial concentrations of X and Y are 1 mol/kL and 4 mol/kL respectively.

(i) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = -bx(x+1)^2$, where b is a positive constant. [2]

(ii) Given that the concentration of X is 0.5 mol/kL at the instance 1 min after the start of the experiment. Find the concentration of X at the instance 2 min after the start of the experiment, giving your answer to 3 significant figures. [6]

Solution

(a)(i) $\frac{\mathrm{d}x}{\mathrm{d}t} = -ax$

$$\int \frac{1}{x}\,\mathrm{d}x = \int -a\,\mathrm{d}t$$

$\ln x = -at + c$ since $x > 0$

$x = A\mathrm{e}^{-at}$, where $A = \mathrm{e}^{c}$

When $t = 0$, $x = x_0$,

$x_0 = A\mathrm{e}^{-a(0)} \Rightarrow A = x_0$

$\therefore x = x_0\mathrm{e}^{-at}$

(a)(ii)

When $t = t_{0.5}$, $x = \frac{x_0}{2}$,

$$\frac{x_0}{2} = x_0 e^{-a(t_{0.5})}$$
$$e^{a(t_{0.5})} = 2$$
$$a(t_{0.5}) = \ln 2$$
$$t_{0.5} = \frac{\ln 2}{a}$$

(b)(i)

Since the rate of the reaction is directly proportional to the product of the concentration of X and the square of the concentration of Y, $\frac{dx}{dt} = -kxy^2$, $k > 0$

Substituting $y = y_0 - 2(x_0 - x)$ into $\frac{dx}{dt} = -bxy^2$, and letting $x_0 = 1$ and $y_0 = 4$

$$\frac{dx}{dt} = -kx\left(4 - 2(1 - x)\right)^2$$
$$\frac{dx}{dt} = -kx(2x + 2)^2$$
$$\frac{dx}{dt} = -bx(x+1)^2, \text{ where } b = 4k$$

(b)(ii)

$$\frac{dx}{dt} = -bx(x+1)^2 \Rightarrow \int \frac{1}{x(x+1)^2}\,dx = \int -b\,dt$$

Now, consider $\frac{1}{x(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow 1 \equiv A(x+1)^2 + Bx(x+1) + Cx$.

Let $x = -1$: $\quad 1 = -C \Rightarrow C = -1$

Let $x = 0$: $\quad A = 1$

Let $x = 1$: $\quad 1 = 4(1) + 2B - 1 \Rightarrow B = -1$

So, we have $\int \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx = \int -b\,dt$

$$\ln(x) - \ln(x+1) + \frac{1}{x+1} = -bt + c_1, \qquad \because x > 0 \text{ and } x+1 > 0$$
$$\ln\left(\frac{x}{x+1}\right) = -\frac{1}{x+1} - bt + c_1$$
$$\frac{x}{x+1} = c_2 e^{-\frac{1}{x+1} - bt}, \text{ where } c_2 = e^{c_1}$$

When $t = 0$, $\quad x = 1$:

$$\frac{1}{1+1} = c_2 e^{-\frac{1}{1+1}}$$
$$c_2 = 0.5e^{0.5}$$

Substituting back,

$$\frac{x}{x+1}=0.5\mathrm{e}^{0.5}\mathrm{e}^{-\frac{1}{x+1}-bt}$$

When $t=1$, $x=0.5$:

$$\frac{0.5}{0.5+1}=0.5\mathrm{e}^{0.5-\frac{1}{0.5+1}-b}$$

$$\frac{2}{3}=\mathrm{e}^{-\frac{1}{6}-b}$$

$$\ln\left(\frac{2}{3}\right)=-\frac{1}{6}-b$$

$$b=-\frac{1}{6}-\ln\left(\frac{2}{3}\right)$$

Therefore, we have $\frac{x}{x+1}=0.5\mathrm{e}^{0.5}\mathrm{e}^{-\frac{1}{x+1}+\left(\frac{1}{6}+\ln\frac{2}{3}\right)t}$.

Finally, when $t=2$,

$$\frac{x}{x+1}=0.5\mathrm{e}^{0.5-\frac{1}{x+1}+2\left(\ln\frac{2}{3}+\frac{1}{6}\right)}$$

Using GC, we get $x=0.313815$

Two min after the start of the experiment, the concentration of X is 0.314 mol/kL

<u>Method 2</u>

So, we have $$\int\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}\,\mathrm{d}x=\int -b\,\mathrm{d}t$$

$$\ln(x)-\ln(x+1)+\frac{1}{x+1}=-bt+c_1, \qquad \because x>0 \text{ and } x+1>0$$

$$\ln\left(\frac{x}{x+1}\right)=-\frac{1}{x+1}-bt+c_1$$

When $t=0$, $x=1$:

$$\ln\left(\frac{1}{1+1}\right)=-\frac{1}{1+1}-b(0)+c_1$$

$$c_1=\frac{1}{2}+\ln\left(\frac{1}{2}\right)=\frac{1}{2}-\ln 2$$

Substituting back,

$$\ln\left(\frac{x}{x+1}\right)=-\frac{1}{x+1}-bt+\frac{1}{2}-\ln 2$$

When $t=1$, $x=0.5$:

$$\ln\left(\frac{0.5}{0.5+1}\right)=-\frac{1}{0.5+1}-b+\frac{1}{2}-\ln 2$$

$$b=-\ln\left(\frac{2}{3}\right)-\frac{1}{6}$$

Therefore, we have $\ln\left(\frac{x}{x+1}\right)=-\frac{1}{x+1}-\left(\ln\left(\frac{3}{2}\right)-\frac{1}{6}\right)t+\frac{1}{2}-\ln 2$.

Finally, when $t=2$,

$$\ln\left(\frac{x}{x+1}\right)=-\frac{1}{x+1}-2\left(\ln\left(\frac{3}{2}\right)-\frac{1}{6}\right)+\frac{1}{2}-\ln 2$$

Using GC, we get $x=0.313815$

Two min after the start of the experiment, the concentration of X is 0.314 mol/kL