ANDERSON SERANGOON JUNIOR COLLEGE
JC2 Preliminary Examination 2025
Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1 **27 August 2025 (Wednesday), 2 – 5 pm**

3 hours

Additional Materials: Printed Answer Booklet

List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do **not** use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

1 A surface has equation $z = \mathrm{f}(x, y)$ where $2x = u + v$ and $2y = u - v$.

(a) Show that $\dfrac{\partial \mathrm{f}}{\partial u} = \dfrac{\partial \mathrm{f}}{\partial v} = \dfrac{1}{2}$ if the tangent plane at the point $P(a, b, c)$ is parallel to the plane with equation $x - z = 1$ where $a > 0$. [3]

(b) Given that $z = x + y - \sqrt{\dfrac{x+y}{x-y}}$, find the exact coordinates of the point P. [4]

2 **(a)** V denotes the set of vectors of the form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a, b and c are real numbers.

(i) Show that the subset $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V : a^2 = b^2 \right\}$ does not form a linear space. [1]

(ii) Determine whether $\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V : 2a - b + c = 0 \right\}$ forms a linear space, justifying your answer. [3]

(b) It is given that the square matrix $\mathbf{M}$ satisfies $\mathbf{M}^2 + 2\mathbf{M} + 3\mathbf{I} = \mathbf{0}$.

Deduce that $\mathbf{M}^3 = \mathbf{M} + 6\mathbf{I}$. Hence show that $\mathbf{M}^8 + 2\mathbf{M}^7 + 4\mathbf{M}^6 + \mathbf{M}$ can be expressed as $k(\mathbf{M} + 3\mathbf{I})$, where k is a constant to be determined. [4]

3 By using the substitution $w = \dfrac{\mathrm{d}y}{\mathrm{d}x} + 2x$, show that the general solution of the differential equation

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 4\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 8(x+1) = \mathrm{e}^{-2x}$$

is $-\dfrac{1}{8}\mathrm{e}^{-2x}\left(2x^2 + C_1 x + C_2\right) - x^2 + D$, where C_1, C_2 and D are arbitrary constants. [9]

4 Consider the second-order linear recurrence relation

$$a_{n+2} - 7a_{n+1} + 10a_n = 0, \qquad \text{where } n \geq 0, \quad a_0 = 2,\ a_1 = 9.$$

(a) Express this relation in matrix form $\mathbf{v}_{n+1} = \mathbf{A}\mathbf{v}_n$, where $\mathbf{v}_n = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$, $n \geq 0$. [1]

(b) Without using a calculator, find the eigenvalues of **A** and the corresponding eigenvectors. [4]

(c) By writing **A** in the form $\mathbf{PDP}^{-1}$, find a_n in terms of n. [3]

(d) Explain whether the approach used in **(d)** would work for all second-order linear recurrence relations of the form $a_{n+2} + \lambda a_{n+1} + \mu a_n = 0$, $\lambda,\ \mu \in \mathbb{R}$. [1]

5 **(a)** Show that $\displaystyle\int_0^1 \frac{1}{\mathrm{e}^x + \mathrm{e}^{-x}}\,\mathrm{d}x = \tan^{-1}\left(\frac{\mathrm{e}-1}{\mathrm{e}+1}\right)$. [3]

(b) Two students want to obtain numerical approximations of $\displaystyle\int_0^1 \frac{1}{\mathrm{e}^x + \mathrm{e}^{-x}}\,\mathrm{d}x$.

Student A uses Simpson's rule with a step-size of $\dfrac{1}{6}$, while Student B uses standard series (in MF27) to obtain the series expansion of $\dfrac{1}{\mathrm{e}^x + \mathrm{e}^{-x}}$, up to the x^2 term, followed by integration.

Find the percentage errors incurred for each of the two methods. [6]

(c) The two students now want to obtain the numerical approximation of $\displaystyle\int_0^3 \frac{1}{\mathrm{e}^x + \mathrm{e}^{-x}}\,\mathrm{d}x$ using their previously chosen methods. Without performing further computations, suggest with reason which student would obtain a better approximation. [1]

6 **(a)** By considering the series $z + z^2 + z^3 + \cdots + z^n$, where $z = \cos\theta + \mathrm{i}\sin\theta$, show that

$$\cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta = \sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}\operatorname{cosec}\frac{\theta}{2},$$

and obtain a similar expression for $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta$. [5]

(b) Obtain a simplified expression for $\left(\displaystyle\sum_{r=1}^{n} \sin r\theta\right)^2 + \left(\displaystyle\sum_{r=1}^{n} \cos r\theta\right)^2$ in terms of n and θ. [1]

(c) Show that, for all positive integers n, the expression

$$\left(\sin\frac{2\pi}{n+1} + \sin\frac{4\pi}{n+1} + \cdots + \sin\frac{2n\pi}{n+1}\right)^2 + \left(\cos\frac{2\pi}{n+1} + \cos\frac{4\pi}{n+1} + \cdots + \cos\frac{2n\pi}{n+1}\right)^2$$

is equal to a constant independent of n. [2]

7 **(a)** Show that for positive integers $n \geq 2$,

$$\int \sin^n x \, \mathrm{d}x = \left(1 - \frac{1}{n}\right) \int \sin^{n-2} x \, \mathrm{d}x - \frac{\cos x \sin^{n-1} x}{n}.$$ [3]

(b) A polar curve C_1 has equation $r = \sin^4 \theta, \quad 0 \leq \theta \leq \pi$.

(i) Find the exact area enclosed by C_1. [3]

Another polar curve C_2 has equation $r = \cot^4 \theta, \quad 0 < \theta < \pi$.

(ii) Show that the y-coordinate of the points of intersection of the two curves is $\phi^{-\frac{5}{2}}$, where $\phi = \dfrac{1+\sqrt{5}}{2}$. [3]

(iii) Find the length of C_2 that is enclosed within C_1, giving your answer correct to 3 decimal places. [2]

8 Consider a curve segment satisfying $y = \frac{1}{4}\left(x^2 - 2\ln x\right)$, with endpoints $(2, 1-\ln\sqrt{2})$ and $(4, 4-\ln 2)$.

(a) Show that the length of the curve segment is $3 + \ln\sqrt{2}$. [3]

(b) Find the exact area of the surface generated when the curve segment is rotated fully about the x-axis, simplifying your answer. [4]

(c) Find the volume of the solid formed when the region bounded by the curve segment, the y-axis and the horizontal lines $y = 1$ and $y = 3$ is rotated fully about the y-axis. [4]

9 The function $y = y(x)$ satisfies $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = \sin x \cos x$.

The value of $y(h)$ is to be found where h is a small positive number such that terms in h^3 and above are negligible, and $y(0) = y_0$.

(a) Use two steps of Euler's method to determine an approximation to $y(h)$ in terms of h and y_0. [3]

(b) Use one step of the improved Euler's method to determine an alternative approximation to $y(h)$ in terms of h and y_0. [2]

(c) Solve the differential equation analytically and find $y(h)$ in terms of h and y_0. [7]

(d) Comment on the results obtained in **(a)**, **(b)** and **(c)**. [1]

10 Coffee, a major Brazilian export, is subject to year-to-year price swings driven by global demand and speculative activity. Because planting decisions today won't fully bear fruit for several seasons, farmers often use last year's prices as their signal for how much to plant in the coming season.

The price of coffee per kg in the n^{th} year is denoted by P_n. Since farmers decide how much to plant based on the previous year's price, the quantity of coffee (in kg) produced in the n^{th} year, denoted S_n, follows

$$S_n = a + bP_{n-1},$$

where a and b are positive real constants.

On the other side of the market, consumers in the n^{th} year have a demand of

$$D_n = c - dP_n,$$

where c and d are positive real constants.

(a) When the quantity of coffee produced equals to the demand for coffee, write down a recurrence relation for P_n and P_{n-1} in terms of a, b, c and d.

Hence find P_n in terms of α, β, n and P_0, where $\alpha = \dfrac{c-a}{d}$ and $\beta = \dfrac{b}{d}$. [4]

(b) Explain what will happen to the price of coffee in the long run if $0 < \beta < 1$. [2]

For the rest of the question, use $\boldsymbol{P_1 = \frac{6}{5}P_0}$, $\boldsymbol{\alpha = 2P_0}$ and $\boldsymbol{\beta = 1}$.

It is now hypothesised that farmers base their planting decisions on the average price over the previous two years instead.

(c) By rewriting the equation for S_n, show that the new recurrence relation can be written as $2P_n + P_{n-1} + P_{n-2} = 4P_0$. [2]

(d) Hence, if $\dfrac{\alpha}{1+\beta}$ is a solution for the recurrence relation in **(c)**, show that

$$P_n = H^n\left(KP_0 \sin n\theta\right) + P_0,$$

where H, K are exact real constants to be determined and $\theta = \pi - \tan^{-1}\sqrt{7}$. [5]

(e) By comparing the behaviours of the two models, comment on which model is more realistic. [1]

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