ANDERSON SERANGOON JUNIOR COLLEGE
JC2 Preliminary Examination 2025
Higher 2

FURTHER MATHEMATICS 9649/02

Paper 2 **29 August 2025 (Friday), 8 – 11 am**

3 hours

Additional Materials: Printed Answer Booklet

List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do **not** use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [50 marks]

1 The Fibonacci numbers is defined by the recurrence relation $U_{n-2}+U_{n-1}=U_n$ where $n\geq 2$ and $U_0=U_1=1$.

$\{R_n\}$ is a sequence of real numbers where $R_n=\dfrac{U_n}{U_{n-1}}$, $n\in\mathbb{Z}^+$. Given that this sequence converges, prove algebraically that its limit can be expressed in the form $\dfrac{a+\sqrt{b}}{2}$, where a and b are integers to be determined. [5]

2 **(a)** As an alternative to the Newton-Raphson method for numerical root-finding, *Chebyshev*'s method computes successive approximations using the following recurrence relation:

$$x_{n+1}=x_n-\frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}-\frac{\mathrm{f}(x_n)^2\mathrm{f}''(x_n)}{2\left[\mathrm{f}'(x_n)\right]^3},\quad n\in\mathbb{Z}^+.$$

Determine the general form that f(x) must take such that Newton-Raphson method and the Chebyshev's method will lead to the same subsequent approximation given **any** initial value in the domain of f. [2]

The equation $x\ln x=0$ has a root $\alpha=1$.

(b) Using a common initial approximation of $x_1=5$, perform successive iterations of both Newton-Raphson and Chebyshev's methods, and comment on your observations. [4]

(c) Explain why Chebyshev's method fails with an initial approximation of 0.5. [2]

3 The *Himmelblau* function is defined by $\mathrm{f}(x,y)=\left(x^2+y-11\right)^2+\left(x+y^2-7\right)^2$.

(a) Find the unit vector in the direction of the steepest descent at $(1,\,2,\,68)$ and state its directional derivative. [4]

(b) Given that there is a saddle point near $(3,\,0,\,\mathrm{f}(3,\,0))$, use quadratic approximation to find an estimate for the coordinates of the saddle point, showing clearly that it is indeed a saddle point. [6]

4 In this question, you are to show all numerical computation workings.

Linear algebra forms the mathematical foundation of Large Language Models (LLMs) such as ChatGPT. At their core, LLMs use matrices and vectors to predict the next word in a sentence through sophisticated mathematical processes. Words are represented as high-dimensional vectors (typically of hundreds or thousands of dimensions) that capture complex features such as grammar, semantic meaning, and contextual relationships. These are processed through multiple layers of matrix operations involving attention mechanisms and neural networks.

For this problem, we will consider a simplified model where an LLM processes just 3 words from a sentence, with each word represented as a vector in 3-dimensional space.

For the following matrix

$$\mathbf{E} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 3 & 7 \end{pmatrix},$$

the word vectors are its columns, namely $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$.

(a) Determine whether the word vectors are linearly independent. [2]

The column space of matrix **E** represents the *embedding subspace* that the model can understand and work with. Any linguistic concept or word meaning that lies outside this subspace cannot be properly processed by the model, as it lacks the representational capacity to encode such information.

(b) Write down a basis for the column space of **E**. [1]

During language processing, the model encounters a new word or concept represented by the vector $\mathbf{v} = \begin{pmatrix} 6 & 9 & 18 \end{pmatrix}^{\mathrm{T}}$. To determine whether the model can understand and process this concept, we need to check if it lies within the model's representational capacity.

(c) By performing row operations, determine whether the word vector **v** can be represented using the model's current embedding subspace and state its implication. [3]

To speed up calculations, the model uses a linear transformation to compress 3-dimensional word vectors into 2 dimensions. The transformation is defined by the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}$.

(d) Determine the rank and the nullity of matrix **AE**. Obtain a basis of the null space of **AE**. [4]

5 **(a)** Sketch the locus of the complex number u satisfying $\text{Re}(u^6) < 0$. [3]

(b) A variable complex number v satisfies $\arg\left(\dfrac{v^3-1}{v^3+1}\right) = -\dfrac{\pi}{2}$. Show that the locus of v consists of three disconnected arcs on a unit circle with centre at the origin, stating the total length of the arcs. [4]

(c) **(i)** On an Argand diagram, sketch the locus of points representing the complex number z that satisfies the inequalities

$$|z+4-3\text{i}| \le 3 \quad \text{and} \quad \frac{\pi}{2} < \arg(z) \le \frac{\pi}{2} + \sin^{-1}\frac{4}{5}. \qquad [3]$$

(ii) Determine the exact maximum value of $|z+3|$. [2]

(iii) Find the range of values of $\arg(z-3-3\text{i})$ within the principal range $(-\pi, \pi]$. [3]

(iv) A variable complex number w satisfies $\text{Im}(w) = -3$. Determine the exact value of θ, where $0 < \theta < \dfrac{\pi}{2}$, such that the minimum possible value of $|w - \text{e}^{\text{i}\theta} z|$ is 2 as z and w varies. [2]

Section B: Probability and Statistics [50 marks]

6 A company intends to conduct a study on eco-friendly packaging for their products.

The first part of the study involves estimating the proportion of consumers who are willing to pay more for eco-friendly packaging.

(a) Show that, regardless of the sample proportion, a 95% confidence interval for a population proportion calculated using a large sample of size n must have a width less than $\dfrac{2}{\sqrt{n}}$. [2]

(b) The company surveyed n people and obtained a 95% confidence interval of (0.273669, 0.426331). Find the value of n. [2]

In the second part of the study, the company wants to evaluate the effectiveness of their new eco-friendly packaging design. They conducted a controlled experiment to measure the reduction in material waste achieved by the new eco-friendly design compared to their current standard packaging.

The company randomly selects 12 product packages and measures the material waste reduction for each. The waste reduction values (in grams) are:

13.88	14.21	14.53	13.11	15.29	14.00
14.73	13.60	14.32	15.07	12.94	14.15

(c) Construct a 99% confidence interval for the population mean waste reduction, giving your answer correct to 2 decimal places. [3]

(d) State what is meant by a '99% confidence interval'. [1]

7 A bird conservationist collected data on the number of Anna's Hummingbird, a species native to North America, foraging in three different habitats at three distinct times of the day as shown in the table below.

Habitat \ Time of day	Morning	Midday	Evening
Urban Gardens	148	54	103
Chaparral (Mediterranean Shrubland)	63	10	42
Oak Woodlands	49	14	17

(a) By using an appropriate test, determine if there is significant evidence at 5% significance level that the choice of habitat where the Anna's Hummingbird forage depends on the time of the day. [7]

(b) Based on the largest contribution to the test statistic, what can the bird conservationist infer about Anna's Hummingbird's preference for foraging habitats? [2]

8 It is given that b is a positive real constant where $2 < b < 4$.

The continuous random variable X has the probability density function

$$\mathrm{f}(x) = \begin{cases} \dfrac{4x^3}{b}, & 0 \le x \le 1 \\ \dfrac{1}{b}, & 1 < x < b \\ 0, & \text{otherwise.} \end{cases}$$

Another continuous random variable Y is such that $Y = \ln X$.

(a) Find the probability density function of Y in terms of b. [3]

(b) Find the upper quartile of Y in terms of b. [3]

It is now given that the interquartile range of Y is $\ln\left(\dfrac{375}{128\sqrt{2}}\right)$.

(c) Find the exact value of b. [5]

(d) The random variable V is the number of observations of Y until an observation of Y greater than the upper quartile is obtained. State the distribution of V and Var(V). [2]

9 Situated on the Pacific Ring of Fire, Japan experiences frequent seismic activity. Historical records show that, on average, one earthquake of magnitude 7.0+ occurs there each year. To predict the occurrence of such earthquakes, a scientist uses a Poisson model.

(a) State two assumptions necessary for the number of earthquakes in a year to be modelled using a Poisson distribution. [2]

(b) Find the exact probability that there will be more than 2 earthquakes of magnitude 7.0+ over two consecutive years, given that there are at most 2 earthquakes of magnitude 7.0+ in each individual year. [3]

It is known that there have not been any earthquakes of magnitude 7.0+ for one year since an earthquake of magnitude 7.0+ occurred at the end of August 2024.

(c) Find the probability that there will be another earthquake of magnitude 7.0+ before February 2027. [3]

10 A melittologist wanted to find out more about the time spent by bees on different coloured flowers. She hypothesised that bees spend more than 0.5 seconds longer on yellow flowers than on blue flowers. She observed 10 bees and collected the following data.

Bee	1	2	3	4	5	6	7	8	9	10
Time spent on a yellow flower (seconds)	4.6	4.7	5.1	4.4	4	5.0	4.3	4.2	5.0	4.3
Time spent on a blue flower (seconds)	3.0	3.2	3.0	4.4	4.3	3.0	4.1	3.1	3.3	4.0

(a) By conducting a Wilcoxon matched-pair signed rank test, determine if there is significant evidence to support the melittologist's hypothesis at 5% level of significance, stating clearly any necessary assumption. [5]

(b) Conduct a *t*-test to determine if there is significant evidence to support the melittologist's hypothesis at 5% level of significance, stating clearly any necessary assumption. [5]

(c) Another researcher suspects that the assumption in **(b)** may not be valid. A suitable χ^2-test is conducted and the test statistic, with 7 degrees of freedom, is 11.99. Explain whether this researcher's suspicion is supported by the χ^2-test. [2]

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