EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2025

General Certificate of Education Advanced Level

Higher 2

CANDIDATE NAME	

CIVICS GROUP		INDEX NO.	

FURTHER MATHEMATICS

9649/02

Paper 2

15 September 2025

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages and **2** blank pages.

Section A: Pure Mathematics [50 marks]

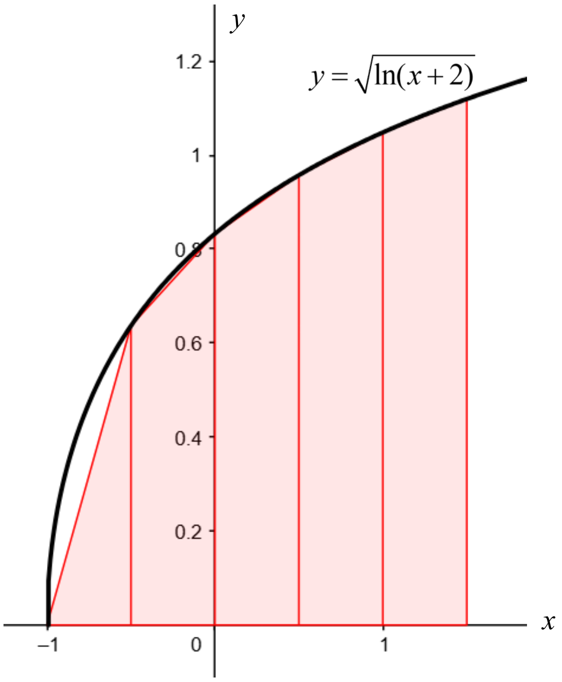
1 **Solution**

Let $\mathrm{f}(x)=\sqrt{\ln(x+2)}$

$$\int_{-1}^{1.5}\sqrt{\ln(x+2)}\,\mathrm{d}x$$

$$\approx\frac{1}{2}(0.5)\big(\mathrm{f}(-1)+2(\mathrm{f}(-0.5)+\mathrm{f}(0)+\mathrm{f}(0.5)+\mathrm{f}(1))+\mathrm{f}(1.5)\big)$$

$$=2.02\ (3\text{sf})$$



The graph is concave downwards over the interval $-1\le x\le 1.5$. Each The total area of the 5 rectangles is less than the actual area under the curve. Hence, the approximation by the trapezium rule is an underestimate of the area.

2 **Solution**

(a) $$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{z},\ \frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}=\frac{y}{z}$$

Let $z_0=\sqrt{{x_0}^2+{y_0}^2}$

Equation of tangent plane at (x_0,y_0):

$$z = z_0 + \frac{x_0}{z_0}(x - x_0) + \frac{y_0}{z_0}(y - y_0)$$

$$= z_0 + \frac{x_0}{z_0}x + \frac{y_0}{z_0}y - \frac{{x_0}^2 + {y_0}^2}{z_0}$$

$$= z_0 + \frac{x_0}{z_0}x + \frac{y_0}{z_0}y - z_0$$

$$= \frac{x_0}{z_0}x + \frac{y_0}{z_0}y$$

When $x = 0$ and $y = 0$, we have $z = 0$

So the tangent plane passes through the origin.

(b)

Normal of tangent plane $\begin{pmatrix} x_0 \\ y_0 \\ -z_0 \end{pmatrix}$

Angle between the normal and the *z*-axis

$$= \cos^{-1}\left(\frac{z_0}{\sqrt{{x_0}^2 + {y_0}^2 + {z_0}^2}}\right)$$

$$= \cos^{-1}\left(\frac{z_0}{\sqrt{2{z_0}^2}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 45°$$

Alternative

Let the angle between the tangent plane and *xy*-plane be θ, then

$$\tan\theta = \left|\nabla \mathrm{f}\right| = \sqrt{\frac{{x_0}^2}{{z_0}^2} + \frac{{y_0}^2}{{z_0}^2}} = \sqrt{\frac{{z_0}^2}{{z_0}^2}} = 1 .$$

Hence the angle between the tangent plane and *z*-axis is also $45°$

3 **Solution**

(a)

$$\mathrm{S}(x, y) = 0$$

$$3x^2 + y^2 - 2x = 0$$

$$y = \pm\sqrt{2x - 3x^2}$$

$$y = \pm\sqrt{x(2 - 3x)}$$

So points correspond to perfectly balanced spots must have

$$x(2 - 3x) \geq 0$$

$$0 \leq x \leq \frac{2}{3}$$

Alternatively,

	$3x^2+y^2-2x=0$ $3\left(x-\frac{1}{3}\right)^2+y^2=\frac{1}{3}$ $9\left(x-\frac{1}{3}\right)^2+3y^2=1$ Points correspond to perfectly balanced spots are on an ellipse centred $\left(\frac{1}{3},0\right)$ with semi-major $\frac{1}{3}$ and semi-minor $\frac{1}{\sqrt{3}}$.
(b)	$S_x=6x-2$, $S_y=2y$ $S_x=0\Rightarrow x=\frac{1}{3}$ $S_y=0\Rightarrow y=0$ $\therefore\left(\frac{1}{3},0\right)$ is a stationary point $S_{xx}=6$, $S_{yy}=2$, $S_{xy}=S_{yx}=0$ $S_{xx}S_{yy}-\left(S_{xy}\right)^2=12>0$ Since $S_{xx}>0$, S has a local minimum at $\left(\frac{1}{3},0\right)$
(c)	On the boundary, $x^2+y^2=4\Rightarrow y^2=4-x^2$ $S(x)=3x^2+4-x^2-2x$ $=2(x^2-x+2)$ $=2\left(x-\frac{1}{2}\right)^2+\frac{7}{2}$ $\qquad -2\le x\le 2$ $\therefore S(x)$ is a quadratic function and has a minimum at $x=\frac{1}{2}$ and $S\left(\frac{1}{2}\right)=\frac{7}{2}$ At endpoints $x=\pm 2$: $S(2)=8$ $S(-2)=16$ Minimum on the boundary: $\left(\frac{1}{2},\pm\frac{\sqrt{15}}{2},\frac{7}{2}\right)$ Maximum on the boundary: $(-2,0,16)$ Hence, absolute maximum is at $(-2,0,16)$ and absolute minimum is at $\left(\frac{1}{3},0,-\frac{1}{3}\right)$.

4	**Solution**

(a)	$\mathbf{A}=\begin{pmatrix}0 & 1\\ 10 & 3\end{pmatrix}$ Consider $\lvert\mathbf{A}-\lambda\mathbf{I}\rvert=0$ $\begin{vmatrix}-\lambda & 1\\ 10 & 3-\lambda\end{vmatrix}=0$ $-\lambda(3-\lambda)-10=0$ $\lambda^2-3\lambda-10=0$ $\lambda=-2$ or $\lambda=5$ For $\lambda=-2$: $\begin{pmatrix}2 & 1\\ 10 & 5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$ An eigenvector is $\begin{pmatrix}1\\ -2\end{pmatrix}$ For $\lambda=5$: $\begin{pmatrix}-5 & 1\\ 10 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$ An eigenvector is $\begin{pmatrix}1\\ 5\end{pmatrix}$
(b)	Let $\mathbf{v}_n=\begin{pmatrix}u_n\\ u_{n+1}\end{pmatrix}$ Then $\mathbf{v}_n=\begin{pmatrix}0 & 1\\ 10 & 3\end{pmatrix}\mathbf{v}_{n-1}=\mathbf{A}\mathbf{v}_{n-1}$ and $\mathbf{v}_0=\begin{pmatrix}u_0\\ u_1\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$ By **(a)**, $\mathbf{A}=\mathbf{QDQ}^{-1}=\begin{pmatrix}1 & 1\\ -2 & 5\end{pmatrix}\begin{pmatrix}-2 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}1 & 1\\ -2 & 5\end{pmatrix}^{-1}$

$$\begin{aligned}\mathbf{v}_n &= \mathbf{A}\mathbf{v}_{n-1}\\ &= \mathbf{A}^2\mathbf{v}_{n-2}\\ &\vdots\\ &= \mathbf{A}^n\mathbf{v}_0\\ &= \mathbf{Q}\mathbf{D}^n\mathbf{Q}^{-1}\mathbf{v}_0\\ &= \begin{pmatrix}1 & 1\\ -2 & 5\end{pmatrix}\begin{pmatrix}-2 & 0\\ 0 & 5\end{pmatrix}^n\begin{pmatrix}1 & 1\\ -2 & 5\end{pmatrix}^{-1}\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\\ -2 & 5\end{pmatrix}\begin{pmatrix}(-2)^n & 0\\ 0 & 5^n\end{pmatrix}\times\frac{1}{7}\begin{pmatrix}5 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \begin{pmatrix}(-2)^n & 5^n\\ (-2)^{n+1} & 5^{n+1}\end{pmatrix}\begin{pmatrix}\frac{4}{7}\\ \frac{3}{7}\end{pmatrix}\\ &= \begin{pmatrix}\frac{4}{7}(-2)^n+\frac{3}{7}(5)^n\\ \frac{4}{7}(-2)^{n+1}+\frac{3}{7}(5)^{n+1}\end{pmatrix}\end{aligned}$$

Hence $u_n = \frac{4}{7}(-2)^n + \frac{3}{7}(5)^n$

<u>Alternative</u>

Let $\begin{pmatrix}1\\ 1\end{pmatrix} = c_1\begin{pmatrix}1\\ -2\end{pmatrix} + c_2\begin{pmatrix}1\\ 5\end{pmatrix}$

By GC, $c_1 = \frac{4}{7}, c_2 = \frac{3}{7}$

$$\begin{aligned}\mathbf{v}_n &= \mathbf{A}\mathbf{v}_{n-1}\\ &= \mathbf{A}^n\begin{pmatrix}1\\ 1\end{pmatrix}\\ &= \mathbf{A}^n\left(\frac{4}{7}\begin{pmatrix}1\\ -2\end{pmatrix}+\frac{3}{7}\begin{pmatrix}1\\ 5\end{pmatrix}\right)\\ &= \frac{4}{7}\mathbf{A}^n\begin{pmatrix}1\\ -2\end{pmatrix}+\frac{3}{7}\mathbf{A}^n\begin{pmatrix}1\\ 5\end{pmatrix}\\ &= \frac{4}{7}\cdot(-2)^n\begin{pmatrix}1\\ -2\end{pmatrix}+\frac{3}{7}\cdot(5)^n\begin{pmatrix}1\\ 5\end{pmatrix}\\ &= \begin{pmatrix}\frac{4}{7}(-2)^n+\frac{3}{7}(5)^n\\ \frac{4}{7}(-2)^{n+1}+\frac{3}{7}(5)^{n+1}\end{pmatrix}\end{aligned}$$

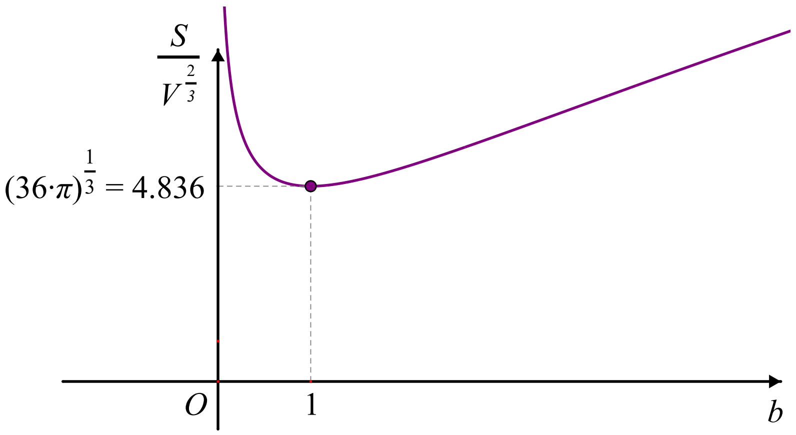
Hence $u_n = \frac{4}{7}(-2)^n + \frac{3}{7}(5)^n$

(c) Let λ be the corresponding eigenvalue for $\begin{pmatrix}u_0'\\ u_1'\end{pmatrix}$

	i.e. $\mathbf{A}\begin{pmatrix}u_0{}' \\ u_1{}'\end{pmatrix}=\lambda\begin{pmatrix}u_0{}' \\ u_1{}'\end{pmatrix}$ $\therefore \mathbf{v}_n=\mathbf{A}\mathbf{v}_{n-1}=\ldots=\mathbf{A}^n\begin{pmatrix}u_0{}' \\ u_1{}'\end{pmatrix}=\lambda^n\begin{pmatrix}u_0{}' \\ u_1{}'\end{pmatrix}$ and this means $u_n=\lambda^n u_0$ which a geometric progression with common ratio the eigenvalue λ.

5	**Solution**
(a)	Volume of required solid of revolution, $V=\int_{-a}^{a}\pi y^2\ \mathrm{d}x$. $x^2+\frac{y^2}{b^2}=1 \qquad \Rightarrow y^2=b^2\left(1-x^2\right)$ $V=\int_{-1}^{1}\pi b^2\left(1-x^2\right)\ \mathrm{d}x$ $=2\pi b^2\int_0^1\left(1-x^2\right)\ \mathrm{d}x$ $=2\pi b^2\left[x-\frac{x^3}{3}\right]_0^1 \ =2\pi b^2\left[\left(1-\frac{1^3}{3}\right)-0\right] \ =\frac{4}{3}\pi b^2$
(b)	$x^2+\frac{y^2}{b^2}=1$ $2x+\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0,\ \ \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{b^2x}{y}$ Curved surface area of required solid of revolution, $S=2\pi\int_{-1}^{1}y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$, available from MF27 $=4\pi\int_0^1 y\sqrt{1+\frac{b^4x^2}{y^2}}\ \mathrm{d}x$, due to reflective symmetry about $x=0$ $=4\pi\int_0^1\sqrt{y^2+b^4x^2}\ \mathrm{d}x$ $=4\pi\int_0^1\sqrt{b^2\left(1-x^2\right)+b^4x^2}\ \mathrm{d}x$ $=4\pi b\int_0^1\sqrt{1-x^2+b^2x^2}\ \mathrm{d}x$ $\therefore\ S=4\pi b\int_0^1\sqrt{1+\left(b^2-1\right)x^2}\ \mathrm{d}x$ (shown)
(c) (i)	When $b=1$, $b^2-1=0$.

	$S=4\pi b\int_0^1\sqrt{1+\left(b^2-1\right)x^2}\,\mathrm{d}x$ $=4\pi(1)\int_0^1 1\,\mathrm{d}x$ $=4\pi\,\left[x\right]_{x=0}^{x=1}$ $\therefore\ S=4\pi$
(c) (ii)	When $b<1,\ 1-b^2>0$. $x=\dfrac{\sin\theta}{\sqrt{1-b^2}}\ \Rightarrow\ \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{\cos\theta}{\sqrt{1-b^2}}$, and $(b^2-1)x^2=(b^2-1)\left(\dfrac{\sin\theta}{\sqrt{1-b^2}}\right)^2$ $=-\sin^2\theta$ $S=4\pi b\int_0^1\sqrt{1+\underline{\left(b^2-1\right)x^2}}\,\mathrm{d}x$ $=4\pi b\displaystyle\int_0^{\sin^{-1}\sqrt{1-b^2}}\sqrt{1\underline{-\sin^2\theta}}\,\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta$ $=4\pi b\displaystyle\int_0^{\sin^{-1}\sqrt{1-b^2}}\cos\theta\dfrac{\cos\theta}{\sqrt{1-b^2}}\mathrm{d}\theta$ $=\dfrac{4\pi b}{\sqrt{1-b^2}}\displaystyle\int_0^{\sin^{-1}\sqrt{1-b^2}}\dfrac{1+\cos2\theta}{2}\mathrm{d}\theta$ $=\dfrac{2\pi b}{\sqrt{1-b^2}}\left[\theta+\dfrac{1}{2}\sin2\theta\right]_0^{\sin^{-1}\sqrt{1-b^2}}$ $=\dfrac{2\pi b}{\sqrt{1-b^2}}\left[\theta+\sin\theta\cos\theta\right]_0^{\sin^{-1}\sqrt{1-b^2}}$ $=\dfrac{2\pi b}{\sqrt{1-b^2}}\left(\sin^{-1}\sqrt{1-b^2}+\sqrt{1-b^2}\cdot b\right)\qquad\because\ \cos\theta=b$ $\therefore\ S=\dfrac{2\pi b}{\sqrt{1-b^2}}\sin^{-1}\sqrt{1-b^2}+2\pi b^2$
(d) (i)	Consolidating from the results of parts **(a)** and **(c)**, $S=\begin{cases}2\pi b^2+\dfrac{2\pi b}{\sqrt{1-b^2}}\sin^{-1}\sqrt{1-b^2}, & \text{if } b<1\\ 4\pi\qquad\qquad\qquad\qquad\ , & \text{if } b=1\\ 2\pi b^2+\dfrac{2\pi b}{\sqrt{b^2-1}}\ln\left\|b+\sqrt{b^2-1}\right\|, & \text{if } b>1\end{cases}$ and $V=\dfrac{4}{3}\pi b^2$,

	$\therefore$ Required ratio $\frac{S}{V^{\frac{2}{3}}} = \left\{ \begin{array}{ll} \sqrt[3]{\frac{9\pi}{2b}}\left[b+\frac{\sin^{-1}\sqrt{1-b^2}}{\sqrt{1-b^2}} \right], & \text{if } b<1 \\ \sqrt[3]{36\pi} \quad , & \text{if } b=1 \\ \sqrt[3]{\frac{9\pi}{2b}}\left[b+\frac{\ln\left\| b+\sqrt{b^2-1} \right\|}{\sqrt{b^2-1}} \right], & \text{if } b>1 \end{array} \right\}$ Obtaining the graph of $\frac{S}{V^{\frac{2}{3}}}$ versus b (using the GC) : 
(d) (ii)	When this (isoperimetric) ratio is minimised, the shape of the solid of revolution is a sphere.

Section B: Probability and Statistics [50 marks]

6 **Solution**

(a) H_0 : The number of absentees is independent of the day of the week.

If the number of absentees is independent of the day of the week, then we'd expect the total of 500 to be uniformly spread throughout the week.

$\therefore$ Expected number of absentees for any day is $\frac{500}{5} = 100$.

(b) Alternative hypothesis H_1 :

The number of absentees is NOT independent of the day of the week.

Under H_0,

No. of absentees	Day of the week					Total
	Mon	Tues	Wed	Thurs	Fri	
Observed (O)	121	87	87	91	114	500
Expected (E)	100	100	100	100	100	500
$\frac{(O-E)^2}{E}$	4.41	1.69	1.69	0.81	1.96	10.56

$$\text{Test statistic } \chi^2 := \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} \sim \underbrace{\chi^2(4)}_{\substack{\chi^2 \text{ distribution} \\ \text{with } 5-1=4 \\ \text{degrees of freedom} \\ \text{(5 classes, 1 restriction)} \\ \text{where total freq.}=500}} .$$

$$\begin{pmatrix} \text{Observed value of} \\ \text{test statistic } \chi^2 \end{pmatrix} = 10.56$$

$$p\text{-value} = \text{P}(\ \chi^2 \geq \underbrace{10.56}_{\substack{\text{observed value} \\ \text{of test statistic}}}\ ) = 0.031980... < \underbrace{0.05 = 5\%}_{\text{significance level}}$$

$\therefore$ At 5% significance level, there is sufficient statistical evidence to reject the null hypothesis H_0, in favour of the alternative hypothesis H_1 that the number of absentees is NOT independent of the day of the week.

7 **Solution**

(a)

Tin		1	2	3	4	5	6	7	8	9	10	11	12
Safe lifetime	Treated (E999)	35	37	23	38	36	36	27	32	35	41	29	33
	Untreated	34	35	26	34	35	33	25	31	31	37	30	29
d, Difference (Treated – untreated)		1	2	-3	4	1	3	2	1	4	4	-1	4

From the sample of 12 tins taken,

- Sample mean paired difference in safe lifetime, $\bar{d} = \frac{\Sigma d}{12} = \frac{22}{12} = 1.8333...$

- Unbiased est. of population variance for the difference

$$s_d{}^2 = \frac{1}{11}\left(\Sigma d^2 - \frac{(\Sigma d)^2}{12}\right) = 4.8787...$$

- Standard error of the mean difference

$$\text{SE} = \frac{s}{\sqrt{12}} = \frac{\sqrt{4.8787...}}{\sqrt{12}} = 0.63762...$$

Assuming that the paired differences are normally distributed,

$$T := \frac{\bar{D} - \mu_d}{\frac{S_D}{\sqrt{12}}} \sim t(11)$$

For a 95% confidence interval,

$$\text{P}(-t \le T \le t) = 0.95 \quad \Rightarrow \quad \text{P}(T \le -t) = \frac{1-0.95}{2} = 0.025$$

$$\Rightarrow \quad t = 2.200985...$$

Required confidence interval for population mean difference in safe lifetime μ_d,

$$= \bar{d} \ \pm \ t_{11,\ 0.975}\,(\text{SE})$$

$$= 1.8333... \pm 2.200985... \times 0.63762...$$

$$= (0.42993...\ ,\ 3.2367...)$$

$$= (0.430,\ 3.24)$$

(b) Let D be a random variable denoting the difference in safe lifetime, in coded units, between treated and untreated portions for a randomly selected tin of catfood ($D :=$ Treated lifetime − Untreated lifetime).

Let μ_d be the population mean paired difference in safe lifetime, in coded units.

$$\text{H}_0 : \mu_d = 0$$
$$\text{H}_1 : \mu_d > 0$$

Under H_0, the test statistic $T := \dfrac{\bar{D} - \overbrace{\mu_d}^{=0}}{\frac{S_D}{\sqrt{12}}} \sim t(11)$

Observed value for test statistic, $t = \dfrac{\bar{d}}{\frac{s_d}{\sqrt{12}}} = \dfrac{\bar{d}}{\text{SE}} = \dfrac{1.8333...}{0.63732...}$

$$= 2.8752...$$

Against H_1, $p = \text{P}(T \ge t)$

$$= \text{P}(T \ge 2.8752...) = 0.0075494... < \underbrace{0.01 = 1\%}_{\text{significance level}}$$

	$\therefore$ At 1% significance level, there is sufficient statistical evidence to reject the null hypothesis H_0, in favour of the alternative hypothesis H_1 that there is an increase in the mean safe lifetime as a result of the addition of E999.

8 **Solution**

(a) Let X and Y be measurement error before and after respectively

Let D be the difference $D = Y - X$

Let m be the median difference

H_0 : $m = 0$

H_1 : $m < 0$

Let S be the number of negative signs out of 15

Under H_0, $S \sim \mathrm{B}\left(15, \frac{1}{2}\right)$

Bef	10.2	10.5	9.9	11.0	10.8	9.8	10.4	10.0
Aft	12.2	9.5	10.0	10.8	9.7	7.6	9.0	10.7
d	2.0	−1	0.1	−0.2	−1.1	−2.2	−1.4	0.7

Bef	10.2	10.3	10.6	10.0	10.7	9.8	10.0
Aft	7.8	9.9	10.0	8.7	13.2	8.1	9.2
d	−2.4	−0.4	−0.6	−1.3	2.6	−1.7	−0.8

From data, $s = 11$

$P -$ value $\mathrm{P}(S \geq 11) = 0.0592$

Since $p -$ value > 0.05, we do not reject H_0 and conclude that at 5% level of significance, there is insufficient evidence to support the claim that the new calibration procedure leads to a reduction in measurement error.

(b) Under H_0, the differences are symmetric around 0

So $\mathrm{P}(X_i = 1) = \mathrm{P}(X_i = 0) = \frac{1}{2}$

$$\begin{aligned}\mathrm{E}(W) &= \mathrm{E}\left(\sum_{i=1}^{n} X_i R_i\right)\\ &= \sum_{i=1}^{n} R_i \mathrm{E}(X_i)\\ &= \frac{1}{2}\sum_{i=1}^{n} R_i\\ &= \frac{1}{2}\cdot\frac{1}{2}n(n+1)\\ &= \frac{1}{4}n(n+1)\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X_i) &= \mathrm{E}(\mathrm{X}_i^{2}) - \left(\mathrm{E}(X_i)\right)^2\\ &= \frac{1}{2} - \frac{1}{4}\\ &= \frac{1}{4}\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(W) &= \mathrm{Var}\left(\sum_{i=1}^{n} X_i R_i\right)\\ &= \sum_{i=1}^{n} R_i^{\,2}\,\mathrm{Var}(X_i)\\ &= \frac{1}{4}\sum_{i=1}^{n} R_i^{\,2}\\ &= \frac{1}{4}\cdot\frac{1}{6}n(n+1)(2n+1)\\ &= \frac{1}{24}n(n+1)(2n+1)\end{aligned}$$

(c) $\mathrm{H}_0: \quad m = 0$

$\mathrm{H}_1: \quad m < 0$

Bef	10.2	10.5	9.9	11.0	10.8	9.8	10.4	10.0
Aft	12.2	9.5	10.0	10.8	9.7	7.6	9.0	10.7
d	2.0	−1	0.1	−0.2	−1.1	−2.2	−1.4	0.7
rank	20	10	1	2	11	22	14	7

Bef	10.2	10.3	10.6	10.0	10.7	9.8	10.0
Aft	7.8	9.9	10.0	8.7	13.2	8.1	9.2
d	−2.4	−0.4	−0.6	−1.3	2.5	−1.7	−0.8
rank	24	4	6	13	25	17	8

Bef	10.3	9.9	10.0	10.2	9.8	10.1	9.8	10.4	10.1	9.9
Aft	10.8	8.4	11.6	9.9	10.7	8.0	8.0	9.2	8.2	7.6
d	0.5	−1.5	1.6	−0.3	0.9	−2.1	−1.8	−1.2	−1.9	−2.3
rank	5	15	16	3	9	21	18	12	19	23

From table, sum of ranks for positive $= 83$ and sum of ranks for negative $= 242$

Using the results in **(b)**, $\mathrm{E}(W) = \frac{1}{4}(25)(26) = 162.5$, $\mathrm{Var}(W) = \frac{1}{24}(25)(26)(51) = 1381.25$

$W \sim \mathrm{N}(162.5, 1381.25)$ approximately

From data, $w = 83$

$p-$ value $= \mathrm{P}(W < 83) = 0.0162$ (3sf)

Since $p-$ value < 0.05, we reject H_0 and conclude that at 5% level of significance, there is sufficient evidence to support the claim that the new calibration procedure leads to a reduction in measurement error.

(d) The result in **(c)** is more reliable because it is based on a larger sample size and the Wilcoxon signed rank test considers both the sign and the magnitude of differences, whereas the sign test only considers the sign.

9	**Solution**
(a)	Casualties arrive independently of other casualties Average number of casualties arriving at the hospital per day is constant over time
(b)	Let S be a random variable denoting the number of casualties that require surgery, out of N casualties that arrive on a day. $S \sim \mathrm{B}\left(N,\ \frac{1}{4}\right)$ Required probability when n casualties arrive, $\mathrm{P}(S=r \mid N=n)=\binom{n}{r}\left(\frac{1}{4}\right)^{r}\left(\frac{3}{4}\right)^{n-r}$, where $r=0,\ 1,\ 2,\ \ldots,\ n$.
(c)	Given that $N \sim \mathrm{Po}(8)$, i.e. $\mathrm{P}(N=n)=\frac{\mathrm{e}^{-8}8^{n}}{n!},\ n=0,\ 1,\ 2,\ \ldots$ $\mathrm{P}(S=r)=\sum_{n=r}^{\infty}\mathrm{P}(S=r \mid N=n)\times\mathrm{P}(N=n)$ $=\sum_{n=r}^{\infty}\binom{n}{r}\left(\frac{1}{4}\right)^{r}\left(\frac{3}{4}\right)^{n-r}\times\frac{\mathrm{e}^{-8}8^{n}}{n!}$ $=\sum_{n=r}^{\infty}\frac{n!}{r!(n-r)!}\frac{3^{n-r}}{4^{n}}\frac{\mathrm{e}^{-8}8^{n}}{n!}$ $=\frac{\mathrm{e}^{-8}}{r!}\sum_{n=r}^{n=\infty}\frac{\cancel{n!}}{(n-r)!}\frac{3^{n-r}}{\cancel{n!}}\left(\frac{8}{4}\right)^{n}$ Let $m=n-r$, so $n=m+r$ $=\frac{\mathrm{e}^{-8}}{r!}\sum_{m=0}^{m=\infty}\frac{3^{m}2^{m+r}}{m!}$ $=\frac{\mathrm{e}^{-8}2^{r}}{r!}\underbrace{\sum_{m=0}^{m=\infty}\frac{6^{m}}{m!}}_{=\mathrm{e}^{6}}$ $\therefore\ \mathrm{P}(S=r)=\frac{\mathrm{e}^{-8}2^{r}}{r!}\mathrm{e}^{6}$ $=\frac{\mathrm{e}^{-2}2^{r}}{r!}$, i.e. $S \sim \mathrm{Po}(2)$ (shown) The number casualties of requiring surgery on a day also follows a Poisson distribution with mean 2.
(d)	Let M and T respectively denote the number of casualties requiring surgery on Monday and Tuesday of a particular randomly chosen week. Then, $M \sim \mathrm{Po}(2)$, $T \sim \mathrm{Po}(2)$, $M+T \sim \mathrm{Po}(\underbrace{4}_{2+2})$.

Required conditional probability,

$$\mathrm{P}(M=8 \mid M+T=12)$$

$$=\frac{\mathrm{P}(M=8 \ \cap \ M+T=12)}{\mathrm{P}(M+T=12)}$$

$$=\frac{\mathrm{P}(M=8 \ \cap \ T=4)}{\mathrm{P}(M+T=12)}$$

$$=\frac{\mathrm{P}(M=8\ ) \ \times \ \mathrm{P}(T=4)}{\mathrm{P}(M+T=12)}$$

$$=\frac{0.00085927\ldots \times 0.090223\ldots}{0.00064151\ldots} \quad \text{(use poissonpdf on GC)}$$

$$=0.12084\ldots \ =0.121 \quad \text{(3 s.f.)}$$

Alternative approach :

Let M and T respectively denote the number of casualties requiring surgery on Monday and Tuesday of a particular randomly chosen week.

So, $M \sim \mathrm{Po}(2)$, $T \sim \mathrm{Po}(2)$.

Given that 12 casualties require surgery across both Monday and Tuesday, i.e. $M+T=12$, each of these 12 casualties have a $\frac{2}{2+2}$ probability of requiring surgery on Monday.

$$M \mid (M+T=12) \sim \mathrm{B}\left(12, \ \underbrace{\frac{2}{2+2}}_{=\frac{1}{2}}\right), \quad \text{i.e. } \mathrm{B}\left(12, \ \frac{1}{2}\right)$$

Required conditional probability, $\mathrm{P}(M=8 \mid M+T=12)$

$$=\binom{12}{8}\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{4}=\frac{495}{4096}$$

10	**Solution**
(a)	$\because\ 1=\int_{-\infty}^{\infty}\mathrm{f}(x)\,\mathrm{d}x$, $=\int_0^1 kx^n(1-x)\,\mathrm{d}x$ $=k\int_0^1 (x^n-x^{n+1})\,\mathrm{d}x$ $=k\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_{x=0}^{x=1}$ $=k\left[\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-0\right]$ $\Rightarrow\ 1=k\ \frac{1}{(n+1)(n+2)}$ $\therefore\ k=(n+1)(n+2)$ (shown).
(b)	$\mathrm{E}(X)=\int_{-\infty}^{\infty}x\,\mathrm{f}(x)\,\mathrm{d}x$ $=\int_0^1 x\,kx^n(1-x)\,\mathrm{d}x$ $=k\int_0^1 (x^{n+1}-x^{n+2})\,\mathrm{d}x$ $=k\left[\frac{x^{n+2}}{n+2}-\frac{x^{n+3}}{n+3}\right]_{x=0}^{x=1}$ $=k\ \left(\frac{1}{n+2}-\frac{1}{n+3}\right)$ $=(n+1)\,\cancel{(n+2)}\ \frac{1}{\cancel{(n+2)}\,(n+3)}\ =\frac{n+1}{n+3}$
(c)	Given that m denotes the mode of X, Then $\left.\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x)\right\vert_{x=m}=\left.\frac{\mathrm{d}}{\mathrm{d}x}kx^n(1-x)\right\vert_{x=m}$ $=\left.k\frac{\mathrm{d}}{\mathrm{d}x}(x^n-x^{n+1})\right\vert_{x=m}$ $=\left.k\left(nx^{n-1}-(n+1)x^n\right)\right\vert_{x=m}$ $=k\left(nm^{n-1}-(n+1)m^n\right)$ $=km^{n-1}\left(n-(n+1)m\right)=0$ $\therefore\ n-(n+1)m=0$, since $k,\ m\neq 0$ $\Rightarrow\ m=\frac{n}{n+1}$

(d)

$$\begin{aligned}
\mathrm{P}(X \le m) &= \int_{-\infty}^{m} \mathrm{f}(x)\,\mathrm{d}x \\
&= \int_{0}^{m} kx^n(1-x)\,\mathrm{d}x \\
&= k\int_{0}^{m} (x^n - x^{n+1})\,\mathrm{d}x \\
&= k\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_{x=0}^{x=m} \\
&= k\left[\left(\frac{m^{n+1}}{n+1} - \frac{m^{n+2}}{n+2}\right) - 0\right] \\
&= \cancel{k}\, m^{n+1}\frac{(n+2)-(n+1)m}{\cancel{(n+1)(n+2)}} \\
&= m^{n+1}\big((n+2)-(n+1)m\big) \\
&= \left(\frac{n}{n+1}\right)^{n+1}\left((n+2)-(n+1)\frac{n}{n+1}\right) \\
&= 2\left(\frac{n}{n+1}\right)^{n+1}
\end{aligned}$$

Since $\left(\dfrac{x}{x+1}\right)^{x+1}$ is an increasing function of x,

and $n > 1$

$$\left(\frac{n}{n+1}\right)^{n+1} > \left(\frac{1}{1+1}\right)^{1+1} = \frac{1}{4}$$

$$\begin{aligned}
\text{Then, } \int_{-\infty}^{m} \mathrm{f}(x)\,\mathrm{d}x &= 2\left(\frac{n}{n+1}\right)^{n+1} \\
&> 2\left(\frac{1}{4}\right) = \frac{1}{2} = \int_{-\infty}^{\text{median of } X} \mathrm{f}(x)\,\mathrm{d}x \\
m &> \text{median of } X
\end{aligned}$$

i.e. The mode of X is greater than its median. (shown)