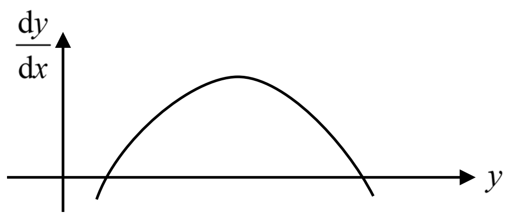
<table>
<tr><th>1</th><th>Suggested Answers</th></tr>
<tr><td>(a)</td><td>

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{175}y\left(180-y\right)-h$$

$$=\frac{1}{175}\left(-\left(y-90\right)^2+8100\right)-h$$

$$=-\frac{\left(y-90\right)^2}{175}+\frac{324}{7}-h$$



For $\frac{\mathrm{d}y}{\mathrm{d}x}\geq 0$ for some values of y, $\frac{324}{7}-h\geq 0$

$$h\leq\frac{324}{7}$$

$$\therefore 0<h\leq\frac{324}{7}$$

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<tr><td>(b)</td><td>

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\left(y-90\right)^2}{175}+\frac{324}{7}-25$$

Let $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$-\frac{\left(y-90\right)^2}{175}+\frac{324}{7}-25=0$$

$$\left(y-90\right)^2=3725$$

$$y=90\pm\sqrt{3725}$$

$90-\sqrt{3725}$ is the unstable equilibrium soln

$90+\sqrt{3725}$ is the stable equilibrium soln

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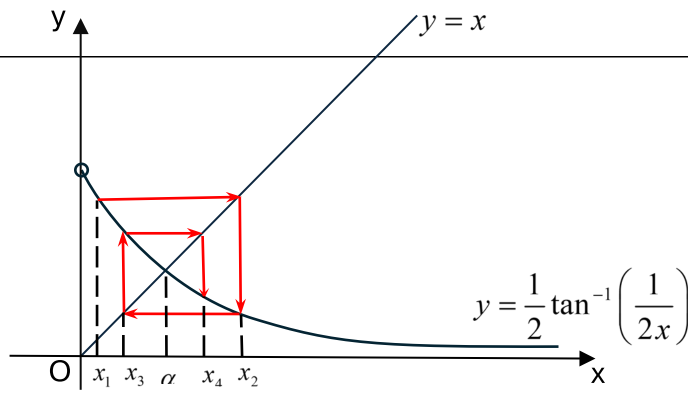
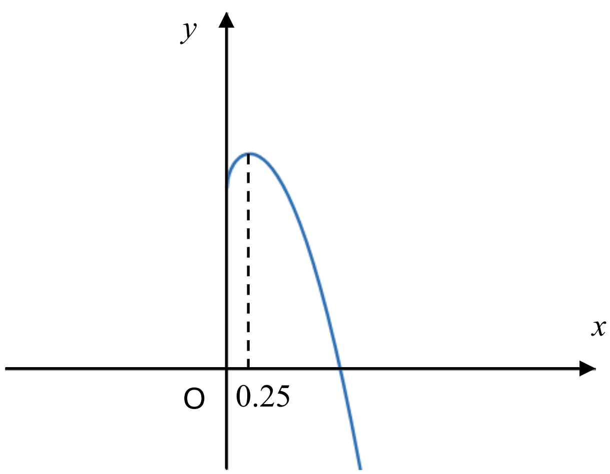
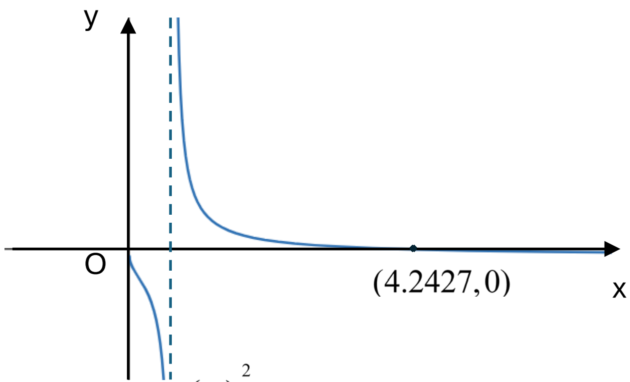
2	Suggested Solution
(a)	$y = x\cos 2x \quad \Rightarrow \quad \dfrac{dy}{dx} = -2x\sin 2x + \cos 2x$ At P, $2x\sin 2x = \cos 2x \quad \Rightarrow \quad 2\tan 2x = \dfrac{1}{x} \quad \Rightarrow \quad x = \dfrac{1}{2}\tan^{-1}\left(\dfrac{1}{2x}\right)$ $\therefore \quad x - \dfrac{1}{2}\tan^{-1}\left(\dfrac{1}{2x}\right) = 0$
(b)	Let $f(x) = x - \dfrac{1}{2}\tan^{-1}\left(\dfrac{1}{2x}\right)$ $f(0.3) = -0.21518 < 0$ and $f(0.5) = 0.10730 > 0$ Since f is continuous in the interval [0.3, 0.5] and there is a sign change, there must be at least one root in the interval [0.3, 0.5].
(c)	$x_2 = 0.51518$ $x_3 = 0.38521$ $x_4 = 0.45717$
(d)	

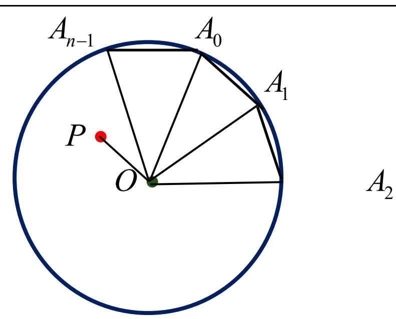


3	**Solutions**
(a)	$x_0 = \dfrac{0\lvert f(4)\rvert + 4\lvert f(0)\rvert}{\lvert f(4)\rvert + \lvert f(0)\rvert} = \dfrac{4}{15}$
(b)	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $f'(x) = \dfrac{1}{2\sqrt{x}} - 4x$ $x_1 = 24.38936$ $x_2 = 12.25320$ $x_3 = 6.22125$ $x_4 = 3.26743$ $x_5 = 1.89602$ $x_6 = 1.36799$ $x_7 = 1.25436$ $x_8 = 1.24848 = 1.248$ (3 d.p) $x_9 = 1.24846 = 1.248$ (3 d.p) Thus $\beta = 1.248$ (3 d.p.) Since $f(1.2475) = 0.00440 > 0$, $f(1.2485) = -0.000141 < 0$ and f is continuous over (1.2475, 1.2485), the result obtained is correct to 3 decimal places.
(c)	 $f'(x) = \dfrac{1}{2\sqrt{x}} - 4x$ For $f'(x) = 0$: $\dfrac{1}{2\sqrt{x}} - 4x = 0$ $x = \dfrac{1}{4}$ Newton-Raphson method would fail if $x_0 \le \dfrac{1}{4}$

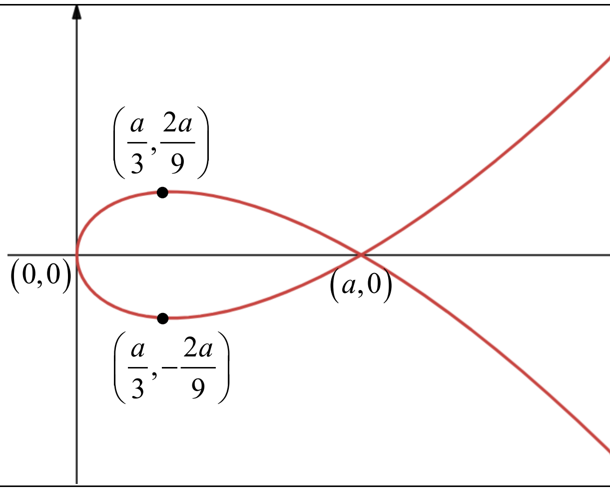
	i.e. $x_0 = \dfrac{0\lvert f(k)\rvert + k\lvert f(0)\rvert}{\lvert f(k)\rvert + \lvert f(0)\rvert} \leq \dfrac{1}{4}$ $\lvert f(0)\rvert = 2$, thus $\lvert f(k)\rvert = 2k^2 - \sqrt{k} - 2$ $\dfrac{k(2)}{2k^2 - \sqrt{k} - 2 + 2} \leq \dfrac{1}{4}$ $\dfrac{2k}{2k^2 - \sqrt{k}} \leq \dfrac{1}{4}$ $\dfrac{2k}{2k^2 - \sqrt{k}} - \dfrac{1}{4} \leq 0$  $x = \left(\dfrac{1}{2}\right)^{\frac{2}{3}} = 0.62996$ Since $k > \beta$, ($\beta = 1.248$ from part (ii)), $k \geq 4.24$)

4	**Suggested Solution**
(a)	As $n \to \infty$, $C_n \to C^*$, $C_{n-1} \to C^*$, $T_n \to T^*$ and $T_{n-1} \to T^*$ $C^* = 0.8C^* + 0.2T^* + 5 \Rightarrow 0.2C^* - 0.2T^* - 5 = 0$ --------(1) $T^* = 0.1C^* + 0.7T^* + 3 \Rightarrow 0.1C^* - 0.3T^* + 3 = 0$ ---------(2) Using GC, $C^* = 52.5$ and $T^* = 27.5$
(b)	Using $X_n = C_n - C^* \Rightarrow C_n = X_n + 52.5$ $Y_n = T_n - T^* \Rightarrow T_n = Y_n + 27.5$ Therefore $X_n + 52.5 = 0.8\left(X_{n-1} + 52.5\right) + 0.2\left(Y_{n-1} + 27.5\right) + 5$ $\Rightarrow X_n = 0.8X_{n-1} + 42 + 0.2Y_{n-1} + 5.5 + 5 - 52.5$ $\Rightarrow X_n = 0.8X_{n-1} + 0.2Y_{n-1}$ ----------- (3) Similarly $Y_n + 27.5 = 0.1\left(X_{n-1} + 52.5\right) + 0.7\left(Y_{n-1} + 27.5\right) + 3$ $\Rightarrow Y_n = 0.1X_{n-1} + 0.7Y_{n-1}$ ------------(4) From (3): $Y_{n-1} = \dfrac{X_n - 0.8X_{n-1}}{0.2} = 5X_n - 4X_{n-1}$ ----------- (5) Sub (5) into (4): $\Rightarrow 5X_{n+1} - 4X_n = 0.1X_{n-1} + 0.7\left(5X_n - 4X_{n-1}\right)$ $\Rightarrow 5X_{n+1} - 4X_n = 0.1X_{n-1} + 3.5X_n - 2.8X_{n-1}$ $\Rightarrow 5X_{n+1} = 7.5X_n - 2.7X_{n-1}$ $\Rightarrow X_n = 1.5X_{n-1} - 0.54X_{n-2}$
(c)	$X_n = 1.5X_{n-1} - 0.54X_{n-2}$ Auxiliary equation: $\lambda^2 - 1.5\lambda + 0.54 = 0$ Using GC: $\lambda = 0.6$ or $\lambda = 0.9$ Therefore $X_n = A\left(0.6\right)^n + B\left(0.9\right)^n$ for $n \geq 0$ $\Rightarrow C_n = A\left(0.6\right)^n + B\left(0.9\right)^n + 52.5$ Given $C_0 = 20$ and $T_0 = 15$. Using the recurrence from the question $C_1 = 0.8C_0 + 0.2T_0 + 5 = 24$ Hence $C_0 = A + B + 52.5 = 20 \Rightarrow A + B = -32.5$ $C_1 = A\left(0.6\right) + B\left(0.9\right) + 52.5 = 24 \Rightarrow 0.6A + 0.9B = -28.5$ Using GC $A = -2.5$ and $B = -30$ Therefore $C_n = 52.5 - 2.5\left(0.6\right)^n - 30\left(0.9\right)^n$

5	**Suggested Solution**
(a)	$(x-R)^2+y^2=r^2$ $y=\pm\sqrt{r^2-(x-R)^2}$ $V=2\pi\int_{R-r}^{R+r} x\left[\sqrt{r^2-(x-R)^2}-\left(-\sqrt{r^2-(x-R)^2}\right)\right]\mathrm{d}x$ $=2\pi\int_{R-r}^{R+r} x\left[2\sqrt{r^2-(x-R)^2}\right]\mathrm{d}x$ $=4\pi\int_{R-r}^{R+r} x\sqrt{r^2-(x-R)^2}\ \mathrm{d}x$ $x=R-r,\ \ u=(R-r)-R=-r$ $x=R+r,\ \ u=(R+r)-R=r$ And $u=x-R\Rightarrow x=R+u$ $V=4\pi\int_{-r}^{r}(R+u)\sqrt{r^2-u^2}\ \mathrm{d}u$ (shown)
(b)(i)	Let $\mathrm{f}(u)=(R+u)\sqrt{r^2-u^2}$ $u_0=-r$: $\mathrm{f}(-r)=0$ $u_1=-\frac{r}{2}$: $\mathrm{f}\left(-\frac{r}{2}\right)=\left(R-\frac{r}{2}\right)\sqrt{\frac{3r^2}{4}}=\left(R-\frac{r}{2}\right)\frac{\sqrt{3}}{2}r$ $u_2=0$: $\mathrm{f}(0)=Rr$ $u_3=\frac{r}{2}$: $\mathrm{f}\left(\frac{r}{2}\right)=\left(R+\frac{r}{2}\right)\frac{\sqrt{3}}{2}r$ $u_4=r$: $\mathrm{f}(r)=0$ $h=\frac{r-(-r)}{4}=\frac{r}{2}$ By Simpson's rule, $V\approx 4\pi\times\frac{\frac{r}{2}}{3}\left[0+4\left(R-\frac{r}{2}\right)\frac{\sqrt{3}}{2}r+2Rr+4\left(R+\frac{r}{2}\right)\frac{\sqrt{3}}{2}r+0\right]$ $=\frac{2\pi r}{3}\left[4Rr\sqrt{3}+2Rr\right]$ $=\frac{4\pi Rr^2}{3}\left(2\sqrt{3}+1\right)$ $\therefore a=2,\ b=3,\ c=1$
(b)(ii)	If r is halved $\Rightarrow\frac{1}{2}r$ and R is doubled $\Rightarrow 2R$, and since $V\propto Rr^2$ $(2R)\left(\frac{1}{2}r\right)^2=\frac{1}{2}Rr^2$ The answer in (b)(i) would be halved.

6	**Suggested Solution**
(a)	$\sum_{k=0}^{n-1} e^{i\left(\theta+\frac{2k\pi}{n}\right)} = e^{i\theta}\sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}}$ $= e^{i\theta}\left[\frac{1-\left(e^{i\frac{2\pi}{n}}\right)^n}{1-e^{i\frac{2\pi}{n}}}\right]$ $= e^{i\theta}\left(\frac{1-e^{i2\pi}}{1-e^{i\frac{2\pi}{n}}}\right)$ $=0$ since $e^{i2\pi}=1$ But on the other hand, by Euler's formula, $0=\sum_{k=0}^{n-1} e^{i\left(\theta+\frac{2k\pi}{n}\right)}$ $=\sum_{k=0}^{n-1}\left[\cos\left(\theta+\frac{2k\pi}{n}\right)+i\sin\left(\theta+\frac{2k\pi}{n}\right)\right]$ $=\sum_{k=0}^{n-1}\cos\left(\theta+\frac{2k\pi}{n}\right)+i\sum_{k=0}^{n-1}\sin\left(\theta+\frac{2k\pi}{n}\right)$ Equating real parts gives $\sum_{k=0}^{n-1}\cos\left(\theta+\frac{2k\pi}{n}\right)=0$ where $n>1$.
(b)	 In general, in triangle POA_k, cosine rule gives $PA_k^{\ 2}=OP^2+OA_k^{\ 2}-2(OP)(OA_k)\cos\left(\theta+\frac{2k\pi}{n}\right)$ $=p^2+r^2-2pr\cos\left(\theta+\frac{2k\pi}{n}\right), k=0,1,2,\ldots,n-1$ Adding gives $PA_0^{\ 2}+PA_1^{\ 2}+\ldots+PA_{n-1}^{\ 2}$

	$=\sum_{k=0}^{n-1} PA_k^{\ 2}$ $=\sum_{k=0}^{n-1}\left[p^2+r^2-2pr\cos\left(\theta+\frac{2k\pi}{n}\right)\right]$ $=n\left(r^2+p^2\right)-2pr\underbrace{\sum_{k=0}^{n-1}\cos\left(\theta+\frac{2k\pi}{n}\right)}_{0}$ $=n\left(r^2+p^2\right)$
(c)	When P is the centre O of the circle, $OP=p=0$. $X=nr^2$. When P lies on the circumference of the circle, $OP=p=r$. $Y=2nr^2$. $\frac{Y}{X}=\frac{2nr^2}{nr^2}=2$.

7	Suggested Solution
(a)	$3ay^2 = x(a-x)^2$ $6ay\frac{\mathrm{d}y}{\mathrm{d}x} = (a-x)^2 + x\cdot 2(a-x)(-1)$ $= (a-x)^2 - 2x(a-x)$ $= (a-x)[(a-x)-2x]$ $= (a-x)(a-3x)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(a-x)(a-3x)}{6ay}$
(b)	
(c)	For region R, $0 \le x \le a$. Considering the upper half of region R where $y \ge 0$, we have $y^2 = \frac{x(a-x)^2}{3a} \Rightarrow y = \frac{\sqrt{x}(a-x)}{\sqrt{3a}}$. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(a-x)(a-3x)}{6a\left(\frac{\sqrt{x}(a-x)}{\sqrt{3a}}\right)} = \frac{(a-3x)}{6a\left(\frac{\sqrt{x}}{\sqrt{3a}}\right)} = \frac{(a-3x)}{2\sqrt{3ax}}$ $1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1+\left(\frac{a-3x}{2\sqrt{3ax}}\right)^2$ $= 1+\frac{(a-3x)^2}{12ax}$ $= \frac{12ax+a^2-6ax+9x^2}{12ax}$ $= \frac{a^2+6ax+9x^2}{12ax}$ $= \frac{(a+3x)^2}{12ax}$ Required arc length

$$= 2 \times \int_0^a \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$$

$$= 2 \times \int_0^a \sqrt{\frac{(a+3x)^2}{12ax}}\ \mathrm{d}x$$

$$= \int_0^a \frac{a+3x}{\sqrt{3ax}}\ \mathrm{d}x$$

$$= \frac{1}{\sqrt{3a}} \int_0^a \frac{a+3x}{\sqrt{x}}\ \mathrm{d}x$$

$$= \frac{1}{\sqrt{3a}} \int_0^a ax^{-\frac{1}{2}} + 3x^{\frac{1}{2}}\ \mathrm{d}x$$

$$= \frac{1}{\sqrt{3a}} \left[2ax^{\frac{1}{2}} + 2x^{\frac{3}{2}} \right]_0^a$$

$$= \frac{1}{\sqrt{3}} \left(2a\sqrt{a} + 2a\sqrt{a} \right)$$

$$= \frac{1}{\sqrt{3}} (4a) = \frac{4\sqrt{3}}{3} a$$

(d) Surface area generated

$$= 2\pi \int_0^a x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$$

$$= 2\pi \int_0^a x\sqrt{\frac{(a+3x)^2}{12ax}}\ \mathrm{d}x$$

$$= \pi \int_0^a x \cdot \frac{a+3x}{\sqrt{3ax}}\ \mathrm{d}x$$

$$= \pi \int_0^a \frac{ax+3x^2}{\sqrt{3ax}}\ \mathrm{d}x$$

$$= \frac{\pi}{\sqrt{3a}} \int_0^a a\sqrt{x} + 3x^{\frac{3}{2}}\ \mathrm{d}x$$

$$= \frac{\pi}{\sqrt{3a}} \left[\frac{2a}{3} x^{\frac{3}{2}} + \frac{6}{5} x^{\frac{5}{2}} \right]_0^a$$

$$= \frac{\pi}{\sqrt{3a}} \left[\frac{2}{3} a^{\frac{5}{2}} + \frac{6}{5} a^{\frac{5}{2}} \right]$$

$$= \frac{\pi}{\sqrt{3a}} \left(\frac{28}{15} a^{\frac{5}{2}} \right) = \frac{28}{15\sqrt{3}} a^2 \pi \text{ or } \frac{28\sqrt{3}}{45} a^2 \pi$$

Surface Area by the entire loop about y-axis

$$= 2 \times \frac{28\sqrt{3}}{45} a^2 \pi = \frac{56\sqrt{3}}{45} a^2 \pi$$

8	**Suggested Solution**
(a)	$y = ux \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x} + \left(\frac{x}{y}\right)^2$ $x\frac{\mathrm{d}u}{\mathrm{d}x} + u = 2u + \left(\frac{1}{u}\right)^2$ $x\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u^3+1}{u^2}$ $\int \frac{u^2}{u^3+1}\,\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ $\frac{1}{3}\ln(u^3+1) = \ln x + c$ since $u > 0$ $u^3 + 1 = Ax^3, \quad A = \mathrm{e}^{3c}$ $\therefore \quad y^3 = x^3(Ax^3 - 1)$ [General Solution] When $x = 1$, $y = 1$ and so $A = 2$ $\therefore \quad y^3 = 2x^6 - x^3$ [Particular Solution]
(b)	$y_{n+1} = y_n + \frac{1}{2}\left[\frac{2y_n}{x_n} + \left(\frac{x_n}{y_n}\right)^2\right]$ $y_1 = y_0 + \frac{1}{2}\left[\frac{2y_0}{x_0} + \left(\frac{x_0}{y_0}\right)^2\right] = 1 + \frac{1}{2}(2+1) = 2.5$ $y_2 = y_1 + \frac{1}{2}\left[\frac{2y_1}{x_1} + \left(\frac{x_1}{y_1}\right)^2\right] = 2.5 + \frac{1}{2}\left[\frac{5}{1.5} + \left(\frac{1.5}{2.5}\right)^2\right] = \frac{326}{75}$ Therefore, the estimated value of y when $x = 2$ is $\frac{326}{75}$ or 4.3467
(c)	When $x = 2$, $y = \sqrt[3]{120}$ Percentage error $= \left\lvert \frac{\sqrt[3]{120} - \frac{326}{75}}{\sqrt[3]{120}} \right\rvert \times 100\% = 11.876\%$
(d)	1. Use Improved Euler's Method 2. Use a smaller step size

9	**Suggested Solution**
(a) (i)	$\begin{pmatrix} 5 & -a & a \\ 12 & -11 & 12 \\ a & -a & 5 \end{pmatrix} \xrightarrow[5R_3-aR_1]{5R_2-12R_1} \begin{pmatrix} 5 & -a & a \\ 0 & -55+12a & 60-12a \\ 0 & a^2-5a & 25-a^2 \end{pmatrix}$ Observe that $a \neq \frac{55}{12}$ otherwise $\dim(T) \neq 2$. $\xrightarrow{(-55+12a)R_3-(a^2-5a)R_2} \begin{pmatrix} 5 & -a & a \\ 0 & -55+12a & 60-12a \\ 0 & 0 & (-55+12a)(25-a^2)-(a^2-5a)(60-12a) \end{pmatrix}$ $\dim(T)=2 \Rightarrow (-55+12a)(25-a^2)-(a^2-5a)(60-12a)=0$ $(-55+12a)(5-a)(5+a)-a(a-5)(60-12a)=0$ $(5-a)[(-55+12a)(5+a)+a(60-12a)]=0$ $(5-a)[-275+65a]=0$ $\Rightarrow a=5$ or $\frac{55}{13}$ $a=5$ since a is an integer
	When $a=5, \mathbf{A} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ using GC. So, a basis for the range space R of T = $\left\{ \begin{pmatrix} 5 \\ 12 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 11 \\ 5 \end{pmatrix} \right\}$.
(a)(ii)	R = $\left\{ \mathbf{x} = \alpha \begin{pmatrix} 5 \\ 12 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 11 \\ 5 \end{pmatrix}, \alpha, \lambda \in \mathbb{R} \right\}$ When $S \cap R$, $\left[\alpha \begin{pmatrix} 5 \\ 12 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 11 \\ 5 \end{pmatrix} \right] \bullet \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} = 1$ $21\alpha + 18\lambda = 1 \Rightarrow \lambda = \frac{1}{18}(1-21\alpha)$ $\mathbf{x} = \alpha \begin{pmatrix} 5 \\ 12 \\ 5 \end{pmatrix} + \frac{1}{18}(1-21\alpha) \begin{pmatrix} 5 \\ 11 \\ 5 \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 5 \\ 11 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} -\frac{5}{6} \\ -\frac{5}{6} \\ -\frac{5}{6} \end{pmatrix}$

	Thus $S \cap R$ represents the set of points lying on a line parallel to $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and passing through the point $\left(\frac{5}{18},\frac{11}{18},\frac{5}{18}\right)$. Since $\mathbf{0} \notin S \cap R$, it is not a subspace of $\mathbb{R}^3$.
(b)	$\mathbf{Ax} = -3\mathbf{x} \Rightarrow (\mathbf{A}+3\mathbf{I})\mathbf{x} = \mathbf{0}$ Since there exists non-zero eigenvector **x**, $\det(\mathbf{A}+3\mathbf{I}) = 0$ $\begin{vmatrix}8 & -a & a\\12 & -8 & 12\\a & -a & 8\end{vmatrix} = 0$ $8(-64+12a)+a(96-12a)+a(-12a+8a)=0$ $a^2-12a+32=0$
(b)(i)	$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \begin{pmatrix}5-\lambda & -4 & 4\\12 & -11-\lambda & 12\\4 & -4 & 5-\lambda\end{pmatrix}\mathbf{x} = \mathbf{0}$ $\begin{vmatrix}5-\lambda & -4 & 4\\12 & -11-\lambda & 12\\4 & -4 & 5-\lambda\end{vmatrix} = 0$ Using GC, $\lambda = 1, 1, -3$ When $\lambda = 1$, $\begin{pmatrix}4 & -4 & 4\\12 & -12 & 12\\4 & -4 & 4\end{pmatrix}\mathbf{x} = \mathbf{0} \Rightarrow \begin{pmatrix}1 & -1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\mathbf{x} = \mathbf{0}$ $x-y+z=0 \Rightarrow x = y-z$ $\mathbf{x} = \begin{pmatrix}y-z\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}y + \begin{pmatrix}-1\\0\\1\end{pmatrix}z$ Full set of eigenvectors is $\left\{\mathbf{x}\in\mathbb{R}^3 : \mathbf{x} = \beta\begin{pmatrix}1\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}-1\\0\\1\end{pmatrix} \text{ where } \beta,\gamma \in \mathbb{R}\setminus\{0\}\right\}$ When $\lambda = -3$, $\begin{pmatrix}8 & -4 & 4\\12 & -8 & 12\\4 & -4 & 8\end{pmatrix}\mathbf{x} = \mathbf{0} \Rightarrow \begin{pmatrix}0 & 1 & -3\\3 & -2 & 3\\1 & -1 & 2\end{pmatrix}\mathbf{x} = \mathbf{0}$

	$$\mathbf{x}=\begin{pmatrix}0\\1\\-3\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}-1\\-3\\-1\end{pmatrix}=-\begin{pmatrix}1\\3\\1\end{pmatrix}$$ Full set of eigenvectors is $\left\{\mathbf{x}\in\mathbb{R}^3:\mathbf{x}=\mu\begin{pmatrix}1\\3\\1\end{pmatrix},\mu\in\mathbb{R}\backslash\{0\}\right\}$
(b)(ii)	When $\lambda=1$, we have a **plane** (punctured (or excluded) at the **origin**) of **invariant points**. When $\lambda=-3$, we have an **invariant line** (punctured (or excluded) at the **origin**).
(b)(iii)	Since $\det\begin{pmatrix}1&-1&1\\1&0&3\\0&1&1\end{pmatrix}=-1\neq 0$ **,** thus when $$c_1\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\begin{pmatrix}-1\\0\\1\end{pmatrix}+c_3\begin{pmatrix}1\\3\\1\end{pmatrix}=\mathbf{0}\Rightarrow c_1=c_2=c_3=0$$ The eigenvectors are linearly independent and forms a basis for the space for $\mathbb{R}^3$.
(b)(iv)	$$\mathbf{A}=\begin{pmatrix}1&-1&1\\1&0&3\\0&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-3\end{pmatrix}\begin{pmatrix}1&-1&1\\1&0&3\\0&1&1\end{pmatrix}^{-1}$$ $$\mathbf{A}^3=\begin{pmatrix}1&-1&1\\1&0&3\\0&1&1\end{pmatrix}\begin{pmatrix}1^3&0&0\\0&1^3&0\\0&0&(-3)^3\end{pmatrix}\begin{pmatrix}1&-1&1\\1&0&3\\0&1&1\end{pmatrix}^{-1}$$ $$=\begin{pmatrix}1&-1&1\\1&0&3\\0&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-27\end{pmatrix}\begin{pmatrix}1&-1&1\\1&0&3\\0&1&1\end{pmatrix}^{-1}$$

10	**Suggested Solution**
(a) **(i)**	$W_n = \left(k_d\sqrt{W_{n-1}} + k_p T\right)^2$ When $k_p = 0$, $W_n = \left(k_d\sqrt{W_{n-1}}\right)^2 = \left(k_d\right)^2 W_{n-1}$, $n \geq 1$ For $0 < k_d < 1$, $\dfrac{W_n}{W_{n-1}} = \left(k_d\right)^2$ this implies that W_n is a GP with common ratio $\left(k_d\right)^2$. Since $0 < \left(k_d\right)^2 < 1 \Rightarrow 0 < \dfrac{W_n}{W_{n-1}} < 1 \Rightarrow W_n < W_{n-1}$ for all $n \geq 1$. Therefore, the sequence W_n is decreasing and converges to 0. This means that without probiotic treatment ($k_p = 0$), the waste naturally breakdown due to microbial activity with the waste concentration decreasing each day and approaches 0 in the long run.
(ii)	For $k_d = 1$, $W_n = W_{n-1}$ Therefore, the sequence W_n is a constant sequence that converges to W_0. This means that without probiotic treatment ($k_p = 0$), the microbial activity alone is enough to balance the waste concentration which remain stable at the initial level W_0.
(b)	As $n \to \infty$, $W_n \to 4$, $W_{n-1} \to 4$ Therefore $4 = \left(0.5\sqrt{4} + 0.2T\right)^2$ $\Rightarrow 4 = \left(1 + 0.2T\right)^2$ $\Rightarrow 1 + 0.2T = \pm 2$ $\Rightarrow T = 5$ or $T = -15$ (reject, dosage cannot be negative)
(c)	$W_n = \left(0.5\sqrt{W_{n-1}} + 0.2T\right)^2$, $n \geq 1$. Given that $W_0 = 16$, $I_n = \sqrt{W_n}$ $\left(I_n\right)^2 = \left(0.5 I_{n-1} + 0.2T\right)^2$, $I_0 = \sqrt{16} = 4$ $I_n = 0.5 I_{n-1} + 0.2T$
	Method 1: General Solution General solution: $I_n = A\left(0.5\right)^n + B$ B is a particular solution: $B = 0.5B + 0.2T$ $\Rightarrow B = 0.4T$ $\therefore I_n = A\left(0.5\right)^n + 0.4T$ $\sqrt{W_n} = A\left(0.5\right)^n + 0.4T$

	$n=0, \sqrt{16}=A+0.4T$ $\Rightarrow A=4-0.4T$ $\sqrt{W_n}=(4-0.4T)(0.5)^n+0.4T$ $W_n=\left[(4-0.4T)(0.5)^n+0.4T\right]^2$
	Method 2: Back Substitution $I_n=0.5(0.5I_{n-2}+0.2T)+0.2T$ $=(0.5)^2 I_{n-2}+0.5(0.2T)+0.2T$ $=(0.5)^2(0.5I_{n-3}+0.2T)+0.5(0.2T)+0.2T$ $\vdots$ $\vdots$ $=(0.5)^n I_0+0.2T\left[(0.5)^{n-1}+\cdots+(0.5)^2+0.5+1\right]$ $=(0.5)^n(4)+\dfrac{0.2T\left(1-(0.5)^n\right)}{0.5}$ $=4(0.5)^n+0.4T\left(1-(0.5)^n\right)$ $=0.4T+(4-0.4T)(0.5)^n$ Since $I_n=\sqrt{W_n}$, $\sqrt{W_n}=0.4T+(4-0.4T)(0.5)^n$ $W_n=\left[0.4T+(4-0.4T)(0.5)^n\right]^2$
(d)	$W_n=\left(0.4T+(4-0.4T)(0.5)^n\right)^2$, $W_0=16$ As $n\to\infty$, $(0.5)^n\to 0$ $\therefore W_n\to(0.4T)^2$ $\therefore (0.4T)^2>16$ $\Rightarrow 0.4T>4$ (since $T>0$) $\Rightarrow T>10$ Therefore when $T>10$ mL/L, the long-term waste concentration exceeds $W_0=16$
(e)	When $T>10$, the probiotics overwhelm the microbes causing higher waste accumulations. This may deplete oxygen and threaten aquatic life like fishes Or It may promote algae bloom due to the extra nutrients from the waste accumulations.