RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 6

H2 Math Year 6 Preliminary Examination Paper 1: Solutions with comments

1 A curve has equation $y = ax + b + \dfrac{c}{x^2 - 1}$, where a, b and c are real constants. It is given that the curve crosses the x-axis at $x = 2$. The normal to the curve at the point $(0, 3)$ meets the x-axis at $x = 7.5$. Find the values of a, b and c. [4]

	Solution	Comments
[4]	$y = ax + b + \dfrac{c}{x^2 - 1}$ Since $(2, 0)$ and $(0, 3)$ lies on the curve, $2a + b + \dfrac{1}{3}c = 0$ --- (1) $b - c = 3$ --- (2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = a - \dfrac{2cx}{\left(x^2 - 1\right)^2}$ Gradient of normal at $(0, 3) = \dfrac{3 - 0}{0 - 7.5} = -\dfrac{2}{5}$. Gradient of the tangent to the curve at $(0, 3)$ is $\dfrac{5}{2}$ and thus $a = \dfrac{5}{2}$ --- (3) Using GC, $a = \dfrac{5}{2}$, $b = -3$ and $c = -6$.	There are quite many sign/conceptual errors in the attempt to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Do note that the gradient of the normal to the curve at the point (0, 3) means that we need to substitute $x = 0$ in $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and get $-\dfrac{1}{a}$ as the gradient of the normal. The two points (0, 3) and (7.5, 0) gives the gradient of the normal as $-\dfrac{2}{5}$.

2 The point P travels along the curve C with equation $y = x\sin^{-1}x,\ -1 < x < 1$. Let the gradient of the curve C at the point P be m.

If the x-coordinate of P is increasing at the rate of 9 units per second when $x = \frac{1}{2}$, find the exact value of the rate at which m is changing at this instant. [4]

	Solution	**Comments**
[4]	$y = x\sin^{-1}x,\ -1 < x < 1$ Differentiate with respect to x: $m = \frac{\mathrm{d}y}{\mathrm{d}x} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$ Differentiate with respect to x again: $\frac{\mathrm{d}m}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2} - x\left(\frac{1}{2\sqrt{1-x^2}}\right)(-2x)}{\left(1-x^2\right)}$ $= \frac{1}{\sqrt{1-x^2}} + \frac{\left(1-x^2\right)+x^2}{\sqrt{1-x^2}\left(1-x^2\right)}$ $= \frac{2-x^2}{\sqrt{\left(1-x^2\right)^3}}$ When $x = \frac{1}{2}$, $\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2-\frac{1}{4}}{\sqrt{\left(1-\frac{1}{4}\right)^3}} \times 9 = 14\sqrt{3}$ Therefore, required rate is $14\sqrt{3}$.	Most students did well but many did not simplify the answer to the lowest terms. Answer mark was not awarded to non-exact answers and unsimplified answers. Some students substituted $x=p$, assuming the parameter x takes the value of p at point P, note that p will be taken as a constant in this case and we shall not differentiate with respect to p.

3 Relative to the origin O, the points A and C have position vectors $\mathbf{a}$ and $2\mathbf{b}$ such that

$$\mathbf{a} = 5p\mathbf{i} - 2p\mathbf{j} + 4p\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

where p is a positive constant. It is given that $|\mathbf{a}| = 2|\mathbf{b}|$.

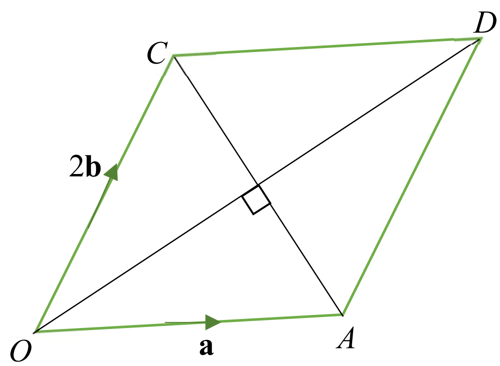
(a) Find the exact value of p. [2]

(b) Evaluate $(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})$. [1]

(c) Evaluate $\mathbf{a}\times\mathbf{b}$ and hence find the area of triangle OAC. [3]

(d) Use a geometrical reason to explain why $|\mathbf{a}\times\mathbf{b}| = \frac{1}{4}(|\mathbf{a}+2\mathbf{b}|)(|\mathbf{a}-2\mathbf{b}|)$. [2]

	Solution	**Comments**
(a) [2]	Since $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{b}\rvert$, $\sqrt{(5p)^2+(-2p)^2+(4p)^2} = 2\sqrt{1^2+(-2)^2+2^2}$ $3\sqrt{5}p = 6$ (given $p>0$) $p = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$	Most students did well but many did not simplify the answer or made careless mistakes, which affected the rest of the parts, too.
(b) [1]	$(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})$ $=\mathbf{a.a}-2\mathbf{a.b}+2\mathbf{b.a}-4\mathbf{b.b}$ $=\lvert\mathbf{a}\rvert^2-4\lvert\mathbf{b}\rvert^2$ since $\lvert\mathbf{a}\rvert=2\lvert\mathbf{b}\rvert$ $=0$	Most students did well.
(c) [3]	$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}5p\\-2p\\4p\end{pmatrix}\times\begin{pmatrix}1\\-2\\2\end{pmatrix} = p\begin{pmatrix}5\\-2\\4\end{pmatrix}\times\begin{pmatrix}1\\-2\\2\end{pmatrix} = p\begin{pmatrix}4\\-6\\-8\end{pmatrix} = \frac{4\sqrt{5}}{5}\begin{pmatrix}2\\-3\\-4\end{pmatrix}$ Area of triangle OAC $=\frac{1}{2}\lvert\mathbf{a}\times2\mathbf{b}\rvert$ $=\lvert\mathbf{a}\times\mathbf{b}\rvert$ $=\frac{4\sqrt{5}}{5}\sqrt{2^2+(-3)^2+(-4)^2} = \frac{4\sqrt{145}}{5}$	Some students did not evaluate $\mathbf{a}\times\mathbf{b}$ to answer the first part of the question. Many students did not rationalise the denominator or simplify the answer.

(d) **[2]**	Observe that $\overrightarrow{OD} = \mathbf{a} + 2\mathbf{b}$ and $\overrightarrow{CA} = \mathbf{a} - 2\mathbf{b}$. From **(b)**, OD is perpendicular to CA. In addition, CA and OD, which are the diagonals of the parallelogram $OADC$, bisect each other.  From **(c)**, area of triangle OAC $= \lvert \mathbf{a} \times \mathbf{b} \rvert$. But area of triangle OAC is also $= \frac{1}{2}(CA)\left(\frac{1}{2}OD\right)$ $= \frac{1}{4}(\lvert \mathbf{a} - 2\mathbf{b} \rvert)(\lvert \mathbf{a} + 2\mathbf{b} \rvert)$ Hence $\lvert \mathbf{a} \times \mathbf{b} \rvert = \frac{1}{4}(\lvert \mathbf{a} + 2\mathbf{b} \rvert)(\lvert \mathbf{a} - 2\mathbf{b} \rvert)$.	Marks are awarded only if there are: 1. correct discussion on the perpendicular property of $\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} - 2\mathbf{b}$. 2. correct discussion on the relationship between the areas, building upon Point 1.

4 **(a)** Find $\displaystyle\int \frac{9x}{(2x-1)(x+1)^2}\,\mathrm{d}x$. [4]

(b) **(i)** Differentiate $\dfrac{1}{x^2+1}$ with respect to x. [1]

(ii) Differentiate $\ln\sqrt{x^2+1}$ with respect to x. [1]

(iii) Hence find $\displaystyle\int \frac{x\ln\sqrt{x^2+1}}{(x^2+1)^2}\,\mathrm{d}x$. [4]

	Solution	**Comments**
(a) **[4]**	Let $\dfrac{9x}{(2x-1)(x+1)^2}=\dfrac{A}{2x-1}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$. $9x\equiv A(x+1)^2+B(2x-1)(x+1)+C(2x-1)$ Sub. $x=\dfrac{1}{2}$, $A=2$. Sub. $x=-1$, $C=3$. Compare the coefficient of x^2, $A+2B=0\Rightarrow B=-1$. $\displaystyle\int\frac{9x}{(2x-1)(x+1)^2}\,\mathrm{d}x=\int\frac{2}{2x-1}-\frac{1}{x+1}+\frac{3}{(x+1)^2}\,\mathrm{d}x$ $=\ln\lvert 2x-1\rvert-\ln\lvert x+1\rvert-\dfrac{3}{x+1}+c,\quad c\in\mathbb{R}$ $=\ln\left\lvert\dfrac{2x-1}{x+1}\right\rvert-\dfrac{3}{x+1}+c$	Most students did well but many made careless mistakes, which affected the rest of the parts. A handful is still not familiar with splitting into appropriate partial fractions in order to integrate.
(b) **(i)** **[1]**	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x^2+1}\right)=\dfrac{-2x}{(x^2+1)^2}$	Most students did well but some made careless mistake.s
(b) **(ii)** **[1]**	$\dfrac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{x^2+1}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{2}\ln\left(x^2+1\right)\right]=\dfrac{x}{x^2+1}$	Most students did well but again some made careless mistakes particularly for

<table>
<tr>
<td></td>
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<td>those who did not simplify first using the property of logarithm.</td>
</tr>
<tr>
<td>(b)
(iii)
[4]</td>
<td>
<table>
<tr>
<td>$u = \ln\sqrt{x^2+1}$
$= \frac{1}{2}\ln\left(x^2+1\right)$</td>
<td>$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-2x}{(x^2+1)^2}$</td>
</tr>
<tr>
<td>$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{x^2+1}$</td>
<td>$v = \frac{1}{x^2+1}$</td>
</tr>
</table>
$\int \frac{x\ln\sqrt{x^2+1}}{\left(x^2+1\right)^2}\,\mathrm{d}x$
$= -\frac{1}{2}\int \frac{-2x}{(x^2+1)^2}\left(\ln\sqrt{x^2+1}\right)\mathrm{d}x$
$= -\frac{1}{2}\left[\left(\frac{1}{x^2+1}\right)\ln\sqrt{x^2+1} - \int\left(\frac{1}{x^2+1}\right)\frac{x}{x^2+1}\,\mathrm{d}x\right]$ ---integrate by parts
$= -\frac{\ln\sqrt{x^2+1}}{2\left(x^2+1\right)} + \frac{1}{2}\int \frac{x}{\left(x^2+1\right)^2}\,\mathrm{d}x$
$= -\frac{\ln\sqrt{x^2+1}}{2\left(x^2+1\right)} - \frac{1}{4\left(x^2+1\right)} + c$
$= -\frac{\ln\left(x^2+1\right)+1}{4\left(x^2+1\right)} + c$
</td>
<td>Those students who could not do this part fail to observe the role of the previous two parts and the word “hence”. As a result, they could not select the correct terms for integration by parts. Still there are many who made careless mistakes in the integrations.</td>
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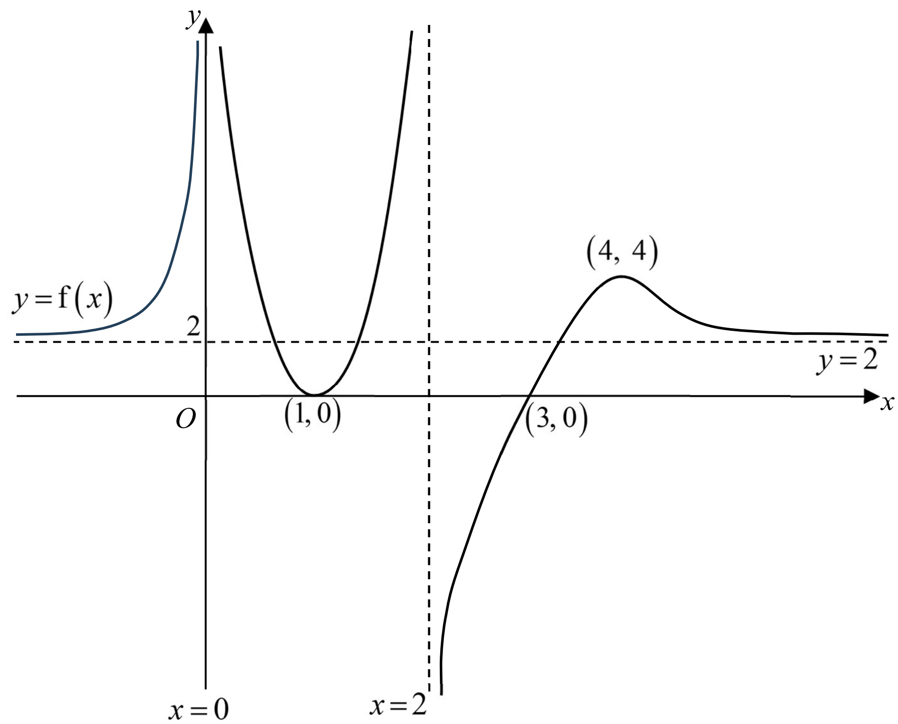
</table>

5 **(a)** The diagram below shows the sketch of the graph of $y = \mathrm{f}(x)$. The curve passes through the points with coordinates $(1, 0)$ and $(3, 0)$, and has turning points at $(1, 0)$ and $(4, 4)$. The asymptotes are $x = 0$, $x = 2$ and $y = 2$.



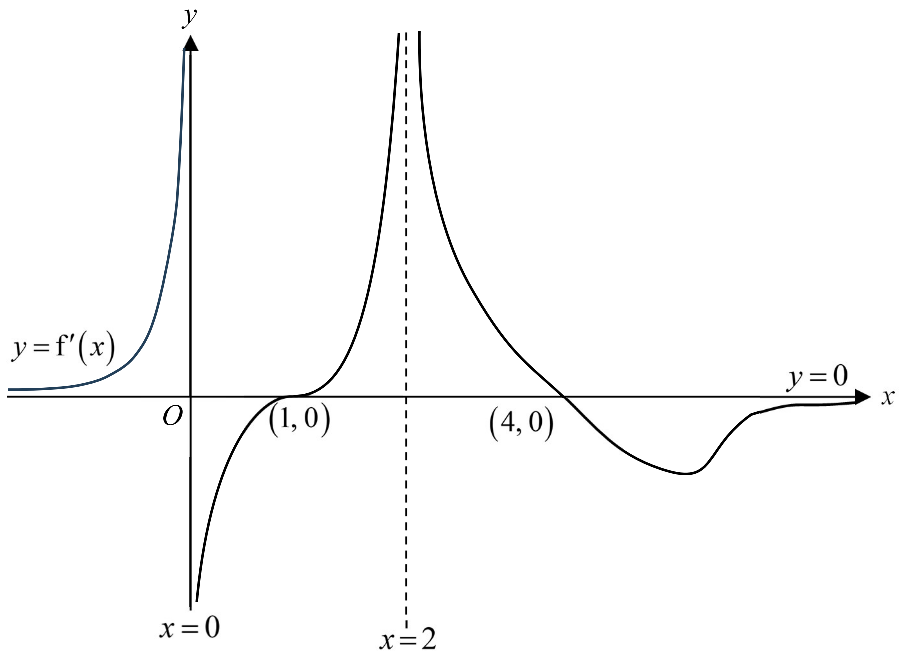
Sketch on separate diagrams, the graphs of

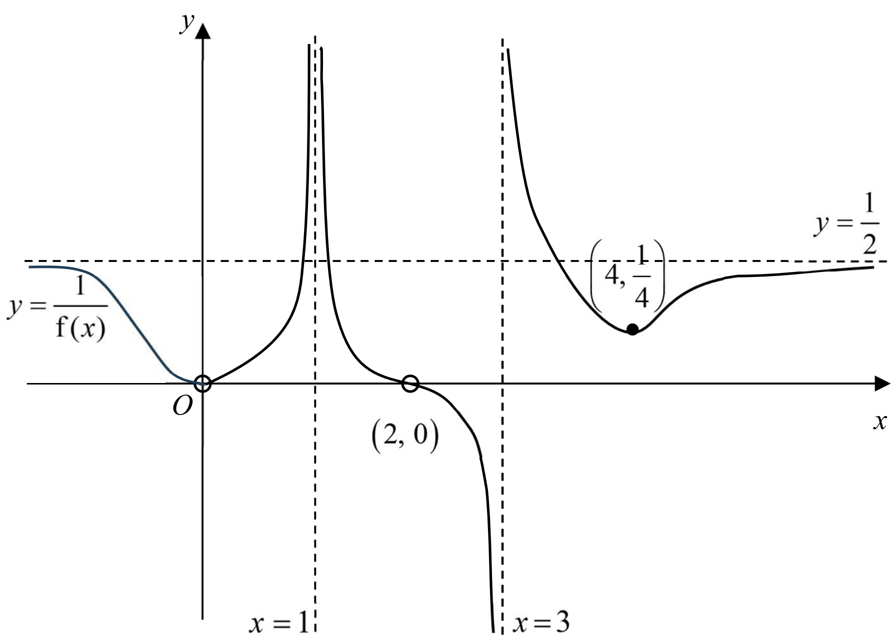
(i) $y = \mathrm{f}'(x)$, [3]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

showing clearly the main features of the graphs.

(b) Describe a sequence of transformations which transforms the graph of $y = \dfrac{x^2+1}{x}$ to the graph of $y = \dfrac{2x^2-5x+4}{x-2}$. [3]

<table>
<tr><th></th><th>Solution</th><th>Comments</th></tr>
<tr><td>(a)(i)
[3]</td><td>


</td><td rowspan="2">Students need to ensure the graphs are drawn proportionally along the axis according to their defined scale. Furthermore, the graphs should visibly demonstrate asymptotic behavior (sketches should tend to the asymptotes when applicable).</td></tr>
<tr><td>(a)(ii)
[3]</td><td>


</td></tr>
</table>

(b) **[3]**	$y=\dfrac{x^2+1}{x}$ Replace x by $(x-2)$, $y=\dfrac{(x-2)^2+1}{(x-2)}=\dfrac{x^2-4x+5}{x-2}$ **(A)** Translation in the positive x-direction by 2 units Replace y by $\dfrac{y}{2}$, $\dfrac{y}{2}=\dfrac{x^2-4x+5}{x-2}$ $\Rightarrow y=\dfrac{2x^2-8x+10}{x-2}=\dfrac{2x^2-5x+4-3(x-2)}{x-2}$ $\Rightarrow y=\dfrac{2x^2-5x+4}{x-2}-3$ **(B)** Scaling parallel to the y-axis by a factor of 2 Replace y by $(y-3)$, $y-3=\dfrac{2x^2-5x+4}{x-2}-3\Rightarrow y=\dfrac{2x^2-5x+4}{x-2}$ **(C)** Translation in the positive y-direction by 3 units **OR** (B) (C) (A) **OR** **(A')** Translation in the positive x-direction by 2 units **(B')** Translation in the positive y-direction by $\dfrac{3}{2}$ units **(C')** Scaling parallel to the y-axis by a factor of 2 **OR** (B')(A')(C') **Alternative** $y=\dfrac{x^2+1}{x}=x+\dfrac{1}{x}$ and $y=\dfrac{2x^2-5x+4}{x-2}=2x-1+\dfrac{2}{x-2}$ $y=\dfrac{x^2+1}{x}=x+\dfrac{1}{x}$	Students should use proper and accurate phrasing as taught in lectures, such as "translate" instead of "shift/move".

	Replace x by $(x-2)$, $y = x - 2 + \dfrac{1}{x-2}$ Replace y by $\dfrac{y}{2}$, $y = 2(x-2) + \dfrac{2}{x-2}$ Replace y by $(y-3)$, $y = 2(x-2) + \dfrac{2}{x-2} + 3 = 2x - 1 + \dfrac{2}{x-2}$. **(A)** Translation in the positive x-direction by 2 units **(B)** Scaling parallel to the y-axis by a factor of 2 **(C)** Translation in the positive y-direction by 3 units	

6 **(a)** A geometric series has first term 2 and common ratio 0.8. Find algebraically, the least value of m for the sum of the first $4m$ terms of the series to be greater than 99% of the sum to infinity. [3]

(b) A finite arithmetic progression A has n terms, first term a and common difference d.
In another progression B, the kth term is obtained by adding the kth positive odd integer to the corresponding term of A. That is, the first term of B is obtained by adding 1 to the first term of A, the second term of B is obtained by adding 3 to the second term of A and so on.

It is given that the twelfth term of A is 25, the sum of all the terms of A is 676 and the sum of all the terms of B is twice the sum of all the terms of A.

(i) Find the values of n, a and d. [4]

(ii) Obtain the sum of the first ten terms of B. [2]

	Solution	**Comments**
(a) **[3]**	$S_{4m}=\dfrac{2\left(1-0.8^{4m}\right)}{0.2}=10\left(1-0.8^{4m}\right)$ Sum to infinity, $S=\dfrac{2}{1-0.8}=10$ $S_{4m}>0.99S$ $10\left(1-0.8^{4m}\right)>0.99(10)$ $0.8^{4m}<0.01$ $4m>\dfrac{\ln 0.01}{\ln 0.8}=20.637$ Since $4m$ is a positive integer, $4m\geq 21$ $\therefore m\geq\dfrac{21}{4}$ or 5.25 Least $m=\dfrac{21}{4}$.	Candidates often overlooked that the question required you to solve **algebraically**. Using a table to solve for values of m is not sufficient. In addition, the question does not state that m itself must be an integer—only that $4m$ must be an integer, since it refers to the "first $4m$ terms". Hence, answer like m = 6 is not accepted. Unless otherwise specified, we should work with the largest possible set of values of m that satisfy the given conditions.

(b)(i) **[4]**	Sum of all the terms of the arithmetic progression *A*, $\frac{n}{2}\left[2a+(n-1)d\right]=676$ --- (1) Sum of all the terms of the arithmetic progression *B*, $\frac{n}{2}\left[2a+(n-1)d\right]+1+3+\cdots+2n-1=2(676)$ Applying (1), $\frac{n}{2}\left[1+2n-1\right]=676$ $\Rightarrow n=26$ Alternatively, since the *k*th term of *B* is obtained by adding the *k*th positive odd integer to the corresponding term of *A* and this results in the sum of the first *n* terms of *B* to be twice the sum of the first *n* terms of *A*, the sum of the increases is 676. Hence $\frac{n}{2}\left[2(1)+(n-1)2\right]=676 \Rightarrow n=26$ Substitute into (1), $\frac{26}{2}(2a+25d)=676$ $\Rightarrow 2a+25d=52$ --- (2) $u_{12}=a+11d=25$ --- (3) From GC, $a=\frac{53}{3}$ and $d=\frac{2}{3}$.	Most candidates started well by correctly formulating the equations for the sums of the two progressions. However, many who tried to equate the two directly ended up with messy algebra and were not successful. Candidates who used **subtraction or substitution** tended to be more successful, though a significant number still made algebraic slips and did not reach the correct answers.
(b)(ii) **[2]**	*B* is an arithmetic progression with first term $(a+1)$ and common difference $(d+2)$. Sum of the first ten terms of *B* $=\frac{10}{2}\left[2\left(\frac{56}{3}\right)+9\left(\frac{8}{3}\right)\right]=\frac{920}{3}$ Or Sum of the first ten terms of *B* = Sum of the first 10 terms of *A* + sum of the first 10 positive odd integers	A significant number of candidates who obtained the correct answers in Part (b)(i) went on to make mistakes in computing the sum here. Common errors included: 1. Mixing up the values of *a* and *d*, 2. Reusing 676 as the sum even

	$=\frac{10}{2}\left[2\left(\frac{53}{3}\right)+9\left(\frac{2}{3}\right)\right]+\frac{10}{2}\left[2(1)+9(2)\right]=\frac{920}{3}$	though this part asked for the sum of the **first 10 terms**, not all the terms, 3. Substituting n = 26 instead of n = 10.

7 The function f is defined by $\mathrm{f}: x \mapsto 1+\dfrac{3}{\mathrm{e}^x-2}$ for $x \in \mathbb{R},\ x > \ln 2$.

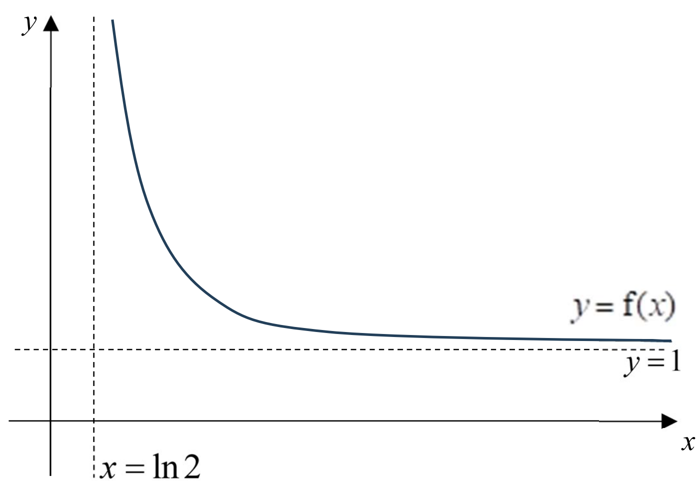
(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

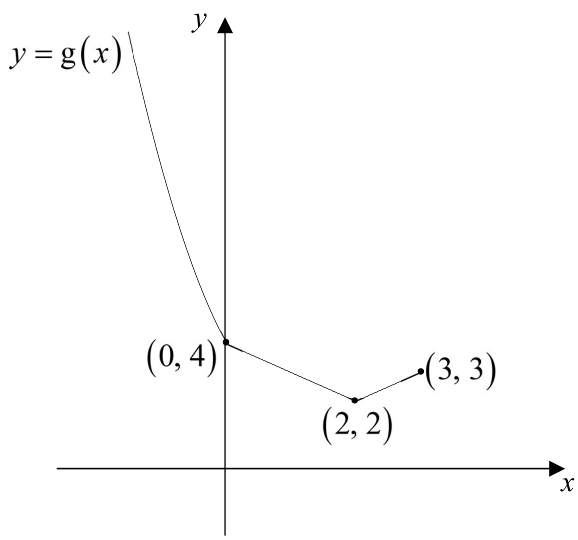
Another function g is defined by

$$\mathrm{g}: x \mapsto \begin{cases} 2(x-1)^2+2 & \text{for} \quad x \le 0, \\ 2+|2-x| & \text{for} \quad 0 < x \le 3. \end{cases}$$

(b) Sketch the graph of $y=\mathrm{g}(x)$ and explain why the composite function $\mathrm{f}^{-1}\mathrm{g}$ exists. [4]

(c) Hence find the exact value of k such that $\mathrm{f}^{-1}\mathrm{g}(k) = \ln\dfrac{5}{2}$. [3]

	Solution	Comments
(a) **[3]**	 $\mathrm{R_f} = (1, \infty)$	Graph is not required but presented here to show how the domain of f^{-1} (which is the range of f) is obtained.
	Let $y = 1+\dfrac{3}{\mathrm{e}^x-2}$ $\Rightarrow \mathrm{e}^x - 2 = \dfrac{3}{y-1}$ $\Rightarrow x = \ln\left(2+\dfrac{3}{y-1}\right)$ $\mathrm{f}^{-1}(x) = \ln\left(2+\dfrac{3}{x-1}\right),\quad x \in \mathbb{R},\ x>1$	Do not write $x = \ln\left\|2+\dfrac{3}{y-1}\right\|$ as the modulus sign should be removed, since $\mathrm{R_f} = (1, \infty)$

<table>
<tr><td></td><td>$D_{f^{-1}} = (1, \infty)$</td><td>$\Rightarrow y > 1 \Rightarrow y - 1 > 0$
$\Rightarrow 2 + \dfrac{3}{y-1} > 2 > 0$.
Students who left their answer with the modulus sign do not obtain full credit.</td></tr>
<tr><td>(b)
[4]</td><td>

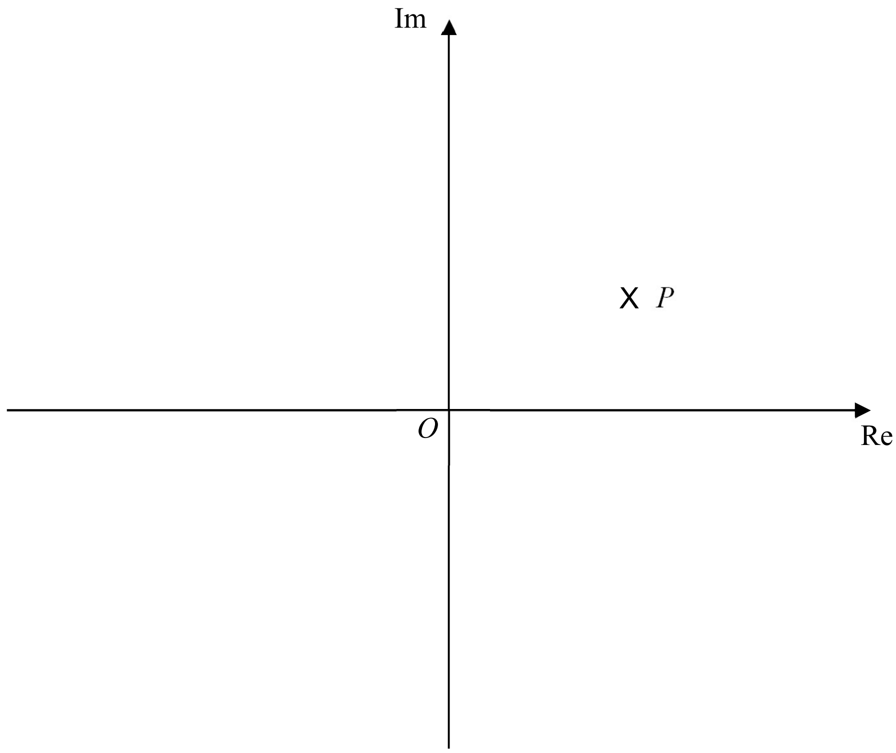
Since $R_g = [2, \infty) \subseteq D_{f^{-1}} = (1, \infty)$, $f^{-1}g$ exists.</td><td>At $(0, 4)$, the gradient is undefined as $\lim_{x \to 0^-} g'(x) = -4$ while $\lim_{x \to 0^+} g'(x) = -1$.
Hence the curve has a sharp point at $(0, 4)$.

Use mathematical terms like ‘subset’ and not phrases like ‘lies within’.</td></tr>
<tr><td>(c)
[3]</td><td>$f^{-1}g(k) = \ln\dfrac{5}{2}$
$\Rightarrow g(k) = f\left(\ln\dfrac{5}{2}\right) = 1 + \dfrac{3}{\frac{5}{2} - 2} = 7 > 4$
Therefore,
$g(k) = 2(k-1)^2 + 2 = 7$
$\Rightarrow (k-1)^2 = \dfrac{5}{2}$
$\Rightarrow k = 1 - \sqrt{\dfrac{5}{2}}$ since $k \le 0$
Hence $k = 1 - \sqrt{\dfrac{5}{2}} = 1 - \dfrac{\sqrt{10}}{2}$.</td><td>Need to state explicitly, the value of k.</td></tr>
</table>

	Alternative method: $$\mathrm{f}^{-1}\mathrm{g}: x \mapsto \begin{cases} \ln\left(2+\dfrac{3}{2(x-1)^2+1}\right) & \text{for} \quad x \le 0, \\ \ln\left(2+\dfrac{3}{1+\lvert 2-x\rvert}\right) & \text{for} \quad 0 < x \le 3. \end{cases}$$ For $k \le 0$, $$\mathrm{f}^{-1}\mathrm{g}(k) = \ln\frac{5}{2}$$ $$\ln\left(2+\frac{3}{2(k-1)^2+1}\right) = \ln\frac{5}{2}$$ $$\frac{3}{2(k-1)^2+1} = \frac{1}{2}$$ $$2(k-1)^2+1 = 6$$ $$(k-1)^2 = \frac{5}{2}$$ $$k = 1-\sqrt{\frac{5}{2}} \quad \left(\text{reject } k = 1+\sqrt{\frac{5}{2}} \text{ since } k \le 0\right)$$ For $0 < k \le 3$, $$\mathrm{f}^{-1}\mathrm{g}(k) = \ln\frac{5}{2}$$ $$\ln\left(2+\frac{3}{1+\lvert 2-k\rvert}\right) = \ln\frac{5}{2}$$ $$\frac{3}{1+\lvert 2-k\rvert} = \frac{1}{2}$$ $$1+\lvert 2-k\rvert = 6$$ $$\lvert 2-k\rvert = 5$$ $$2-k = \pm 5$$ $k = -3$ or 7 (reject both as $0 < k \le 3$) Hence $k = 1-\sqrt{\dfrac{5}{2}} = 1-\dfrac{\sqrt{10}}{2}$.	The alternative method (not recommended) is presented here for students who attempted to solve this way.

8 **Do not use a calculator in answering this question.**

(a)



The point P on the Argand diagram represents the complex number w given by $w = u + \mathrm{i}v$, where u and v are real and positive.

(i) Explain algebraically why $\arg(kw) = \arg(w)$ for any real constant $k > 1$. [1]

Points Q and R represent the complex numbers kw and $\mathrm{i}kw$, where k is a real constant, and $k > 1$.

(ii) On the same Argand diagram in the Printed Answer Booklet, plot the points Q and R. Show clearly the geometrical relationship between the points P, Q and R. [2]

(b) In another Argand diagram, the points A, B and C represent the complex numbers z, $\mathrm{f}(z)$ and $\mathrm{f}(\mathrm{f}(z))$ respectively, where $\mathrm{f}(z) = z^2 - 2z$.

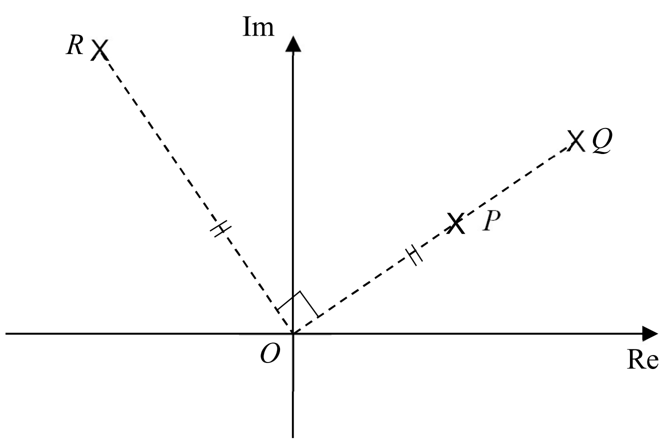
(i) Show that $\mathrm{f}(\mathrm{f}(z)) - \mathrm{f}(z) = z(z-2)(z-3)(z+1)$. [2]

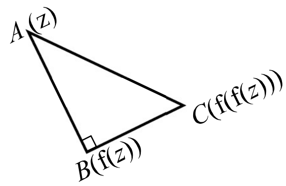
It is given that ABC is a right-angled triangle, described in an anticlockwise sense, with a right angle at B, and $BC = mBA$, where m is a positive real constant.

(ii) By considering $\dfrac{\mathrm{f}(\mathrm{f}(z)) - \mathrm{f}(z)}{\mathrm{f}(z) - z}$, or otherwise, show that

$$(z-2)(z+1) = m\mathrm{i}\,. \qquad [2]$$

(iii) In the case where $z = x + 2\mathrm{i}$, where x is a positive real number, find x and m. Hence obtain the complex number represented by the point B. [5]

	Solution	Comments
(a)(i) **[1]**	Since $w = u + \mathrm{i}v$, $u > 0$ and $v > 0$, we have, $kw = ku + \mathrm{i}kv$, $ku > 0$ and $kv > 0$. Therefore, $\arg(w) = \tan^{-1}\dfrac{v}{u}$ $\arg(kw) = \tan^{-1}\dfrac{kv}{ku} = \tan^{-1}\dfrac{v}{u} = \arg(w)$	Note that algebraically means that the definition of $\arg(w)$ needs to be given. Those who use the property $\arg(kw)$ $= \arg(k) + \arg(w)$ need to state that $\arg(k) = 0$ since $k > 0$.
(a)(ii) **[2]**		The geometrical relationship is that OR is perpendicular to OQ. This needs to be shown on the diagram. Also, since $k > 1$, $OQ > OP$. Since OR is the rotation of OQ 90^o anticlockwise about the origin, then $OR = OQ$.
(b)(i) **[2]**	$\mathrm{f}(\mathrm{f}(z)) - \mathrm{f}(z) = (z^2-2z)^2 - 2(z^2-2z) - (z^2-2z)$ $= (z^2-2z)(z^2-2z-3)$ $= z(z-2)(z-3)(z+1)$	Note that this question does not allow the use of a calculator, so

		factorization needs to be shown clearly.
(b)(ii) **[2]**	 Rotate $\overrightarrow{BC}$ anticlockwise by $\frac{\pi}{2}$ about B to obtain $m\overrightarrow{BA}$: $\mathrm{i}\left[\mathrm{f}\left(\mathrm{f}\left(z\right)\right)-\mathrm{f}\left(z\right)\right]=m\left(z-\mathrm{f}\left(z\right)\right)$ $\frac{\mathrm{f}\left(\mathrm{f}\left(z\right)\right)-\mathrm{f}\left(z\right)}{\mathrm{f}\left(z\right)-z}=-\frac{m}{\mathrm{i}}$ $\frac{z\left(z-2\right)\left(z-3\right)\left(z+1\right)}{z\left(z-3\right)}=m\mathrm{i}$ $\left(z-2\right)\left(z+1\right)=m\mathrm{i}$	The explanation of rotating $\overrightarrow{BC}$ 90^o anticlockwise about B to get $m\overrightarrow{BA}$ needs to be given. The directions of the "vectors" need to be noted – just stating that BC is perpendicular to AB is not sufficient to explain for the "i" factor. No marks are awarded for the simplification of $\frac{z\left(z-2\right)\left(z-3\right)\left(z+1\right)}{z\left(z-3\right)}=\left(z-2\right)\left(z+1\right)$ Note that $\frac{BC}{AB}=m\mathrm{i}$ is incorrect: LHS is a ratio of lengths, RHS is a complex number.
(b) **(iii)** **[5]**	Since $z=x+2\mathrm{i}$, $(z-2)(z+1)=m\mathrm{i}$ $z^2-z-2=m\mathrm{i}$ $\left(x+2\mathrm{i}\right)^2-(x+2\mathrm{i})-2=m\mathrm{i}$ $x^2-x-6+\mathrm{i}\left(4x-2\right)=m\mathrm{i}$ Compare real and imaginary parts, $x^2-x-6=0 \quad ---(1)$ $4x-2=m \quad ---(2)$ From (1), $\left(x-3\right)\left(x+2\right)=0\Rightarrow x=3$ or -2 (rej. since x is positive).	There are quite many sign errors in the expansion of the 4th line. Students are reminded to check their working and write clearly so that careless mistakes would not lead to subsequent deductions being incorrect.

	Sub. (2), $m=10$. $\mathrm{f}(3+2\mathrm{i})=(3+2\mathrm{i})^2-2(3+2\mathrm{i})$ $=9+12\mathrm{i}-4-6-4\mathrm{i}$ $=-1+8\mathrm{i}$ B represents the complex number $-1+8\mathrm{i}$.	

9 Decontamination is a water treatment method to remove a certain undesirable chemical from contaminated water. As the liquid agent is continuously added to the contaminated water in a large tank, the undesirable chemical in the water is removed gradually.

The rate of decrease of the concentration of the undesirable chemical is proportional to the product of its current concentration and the rate at which the liquid agent flows into the tank. At time t minutes after the start of the treatment process, the concentration of the undesirable chemical is C mg/L and the liquid agent flows into the tank at a rate of $\frac{1}{1+t^2}$ L/min. It is known that the initial concentration of the undesirable chemical in the tank is 50 mg/L and after 10 minutes, its concentration is measured to be 20 mg/L.

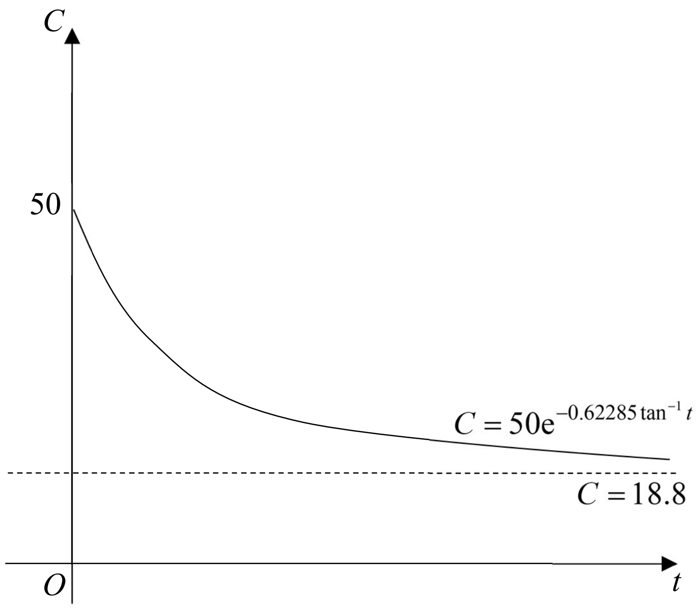
(a) By setting up and solving a differential equation relating C and t, show that $C = 50\mathrm{e}^{q\tan^{-1}t}$, giving the value of q correct to 5 decimal places. [7]

(b) Determine the value of C when $t = 50$. [1]

(c) Sketch the graph of C against t and state what happens to C in the tank for large values of t. [3]

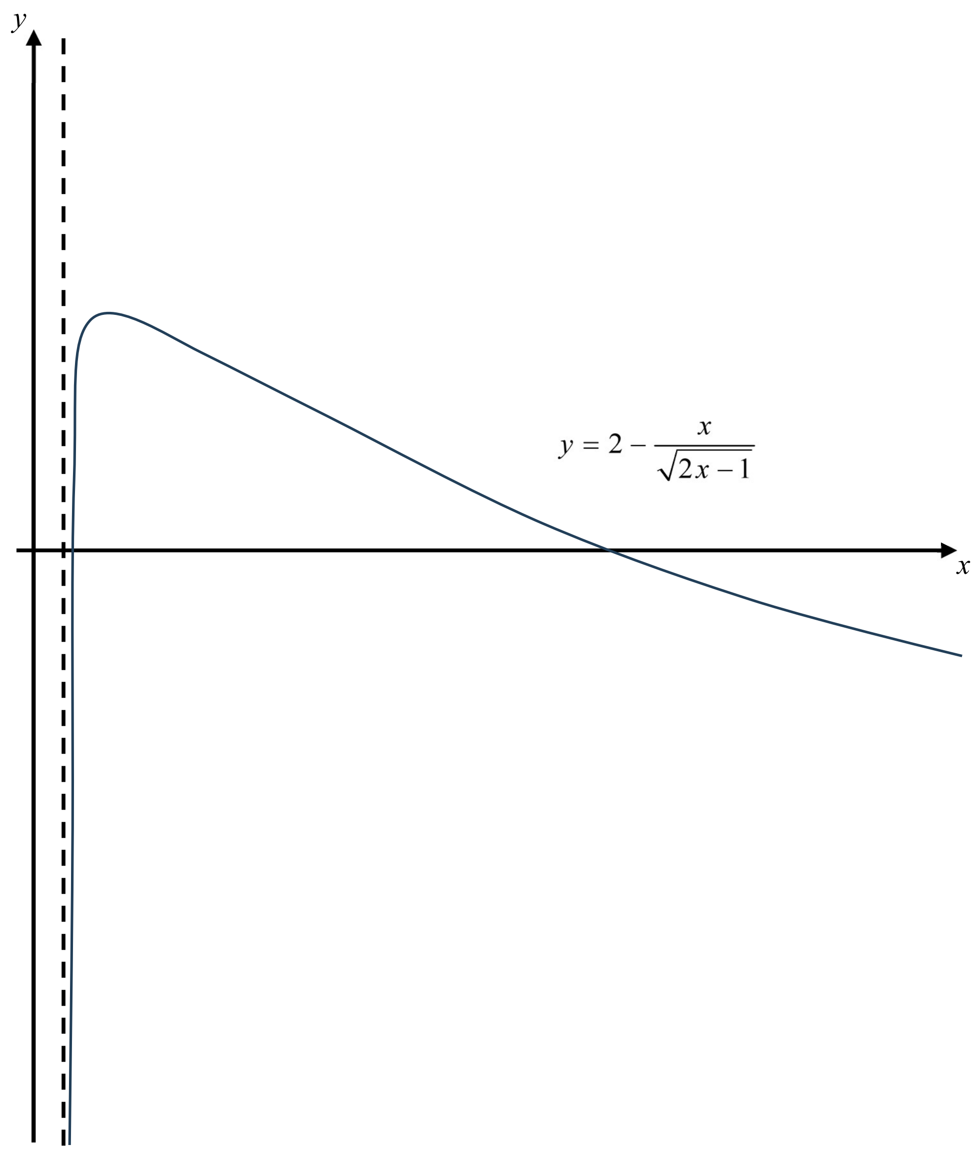
	Solution	Comments
(a) **[7]**	$\frac{\mathrm{d}C}{\mathrm{d}t} = -kC\left(\frac{1}{1+t^2}\right)$ where $k>0$. (Note: Since $C=50$ when $t=0$, $C\neq0$.) $\int \frac{1}{C}\,\mathrm{d}C = -k\int \frac{1}{1+t^2}\,\mathrm{d}t$ $\ln\lvert C\rvert = -k\tan^{-1}t + c_1$, where $c_1 \in \mathbb{R}$ $\lvert C\rvert = \mathrm{e}^{-k\tan^{-1}t+c_1}$ As $C>0$, $C = c_2\mathrm{e}^{-k\tan^{-1}t}$, where $c_2 = \mathrm{e}^{c_1}$ (Alternatively, $C=50$ when $t=0 \Rightarrow \lvert C\rvert = C$) When $t=0$ and $C=50$, $50 = c_2\mathrm{e}^0$ $c_2 = 50$ $C = 50\mathrm{e}^{-k\tan^{-1}t}$	Choice of ***constant of proportionality*** : It is more convenient to write this as "$-k$" and work with a positive k, noting that the required value of $q=-k$. $\int \frac{1}{C}\mathrm{d}C = \ln\lvert C\rvert + d$. A reminder that "$\lvert C\rvert$" is expected in general, unless the student explicitly discussed that $C>0$ based on contextual clue. Algebraic manipulation leading to $C = c_2\mathrm{e}^{-k\tan^{-1}t}$ ought to be clearly shown. Choice of ***constant of integration*** :

	When $t=10$ and $C=20$, $20=50\mathrm{e}^{-k\tan^{-1}10}$ $\mathrm{e}^{-k\tan^{-1}10}=\dfrac{2}{5}$ $-k\tan^{-1}10=\ln\dfrac{2}{5}$ $k=-\dfrac{1}{\tan^{-1}10}\ln\dfrac{2}{5}\approx 0.62285$ $C=50\mathrm{e}^{-0.62285\tan^{-1}t}$ where $q=-0.62285$ (5 d.p.) (shown).	Since the letter C is already used in the question as a variable, it is preferable to choose another letter. The value of q should be stated to 5 dp. Many errors arose from using insufficient accuracy in intermediate steps. *** In Calculus, all angles must be in **radians**, not degrees. Hence, $\tan^{-1}(10)=1.471127674$ (and **<u>not</u>** 84.28940686). Answer like $C=50\mathrm{e}^{-0.01087\tan^{-1}t}$ is a clear sign that the wrong unit was being used for angle measurement.
(b) **[1]**	When $t=50$, $C=50\mathrm{e}^{-0.62285\tan^{-1}50}$ $=19.0317=19.0$ (correct to 3sf)	This is a good example of how incorrect working can still appear to give the "correct" final answer. Working such as $C=50\mathrm{e}^{-0.01087\tan^{-1}(50)}$ $=19.0\ \ (3\text{ sf})$ **<u>cannot</u>** be accepted. The value "–0.01087" arose because the wrong unit was used in earlier steps. The student must have repeated the calculation using the same incorrect unit (for $\tan^{-1}(50)$), which happened to produce the expected numerical value.

(c) **[3]**	 Therefore C decreases and **approaches** 18.8 mg/L for large values of t. **Note (**for reference only): When $t \to \infty$, $\tan^{-1}t \to \dfrac{\pi}{2}$ and $C \to 50\mathrm{e}^{-0.62285\left(\pi/2\right)} = 18.8$.	The sketch should be drawn only for the range $t \geq 0$, as required by the context of the question. Axial intercept and the horizontal asymptote ($y = m$, with m explicitly evaluated.) should be labelled. For questions that ask *"what happens ..."* or *"explain ..."*, the response should be given in **words**, taking the context into account, rather than only using mathematical expressions. In describing trend, it is important to mention whether the function is increasing or decreasing, and, where a limiting value exists, to state that value clearly.

10 **(a)** Using the substitution $u=\sqrt{2x-1}$, show that $\int \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x=\int \frac{u^2+1}{2}\,\mathrm{d}u$. [2]

(b) **(i)** The diagram below shows the graph of $y=2-\frac{x}{\sqrt{2x-1}}$. Indicate on the same diagram in the Printed Answer Booklet, the equation of the asymptote of the curve and the coordinates of its turning point and points of intersection with the x-axis. [2]



(ii) The region R is bounded by the curve $y=2-\dfrac{x}{\sqrt{2x-1}}$ and the lines $x=1$, $x=3$ and $y=\dfrac{1}{2}$. Find the exact area of R. [5]

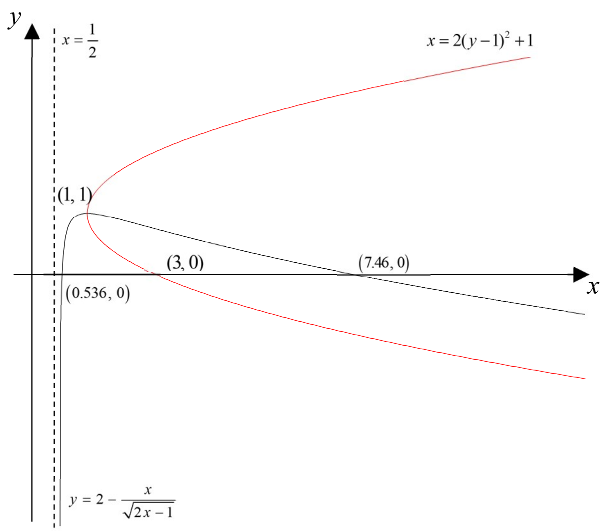
(c) The region S is bounded by the curves $y=2-\dfrac{x}{\sqrt{2x-1}}$, $x=2(y-1)^2+1$, the lines $x=1$, $x=4$ and the x-axis.

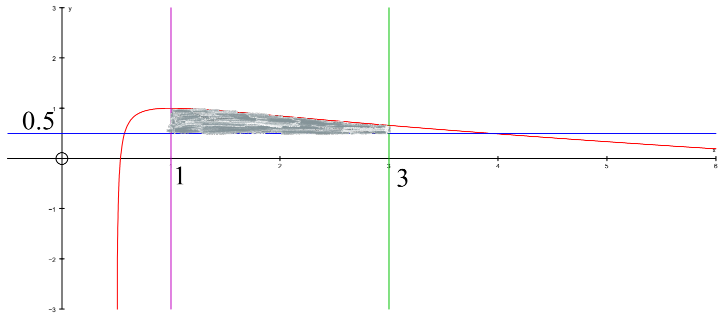
(i) Express $x=2(y-1)^2+1$ in the form $y=\mathrm{f}(x)$. [1]

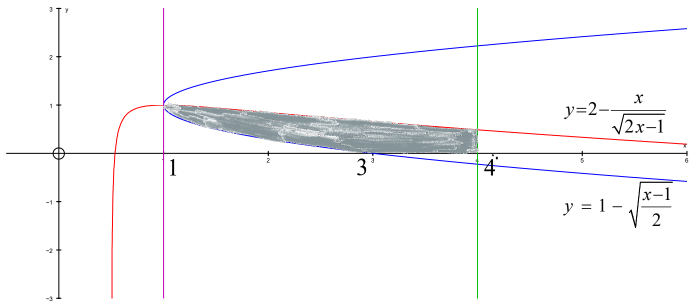
(ii) On the same diagram as in part **(b)(i)**, sketch the graph of $x=2(y-1)^2+1$. [2]

(iii) Find the volume of the solid generated when S is rotated through 2π radians about the x-axis. Give your answer correct to 3 decimal places. [3]

	Solution	Comments
(a) **[2]**	$u=\sqrt{2x-1}\Rightarrow u^2=2x-1$ Hence $u\dfrac{\mathrm{d}u}{\mathrm{d}x}=1\Rightarrow\dfrac{\mathrm{d}x}{\mathrm{d}u}=u$. $\displaystyle\int\frac{x}{\sqrt{2x-1}}\,\mathrm{d}x=\int\frac{\frac{u^2+1}{2}}{u}\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)\mathrm{d}u=\int\frac{u^2+1}{2u}(u)\,\mathrm{d}u=\int\frac{u^2+1}{2}\,\mathrm{d}u$ (shown) Alternatively, $u=\sqrt{2x-1}\Rightarrow\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{\sqrt{2x-1}}$ and therefore $\displaystyle\int\frac{x}{\sqrt{2x-1}}\,\mathrm{d}x=\int x\cdot\frac{1}{\sqrt{2x-1}}\,\mathrm{d}x=\int\frac{u^2+1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x=\int\frac{u^2+1}{2}\,\mathrm{d}u$	As this is a show question, simply differentiating the substitution given and quoting what you are asked to show $\displaystyle\int\frac{x}{\sqrt{2x-1}}\,\mathrm{d}x$ $\displaystyle=\int\frac{u^2+1}{2}\,\mathrm{d}u$ is insufficient.

(b)(i) **[2]** **(c)(ii)** **[2]**	 **Note:** Detailed workings for intercepts and turning points (Optional, since more efficient to use GC) When $y = 0$, $\frac{x}{\sqrt{2x-1}} = 2$ $\Rightarrow x^2 = 4(2x-1)$ $\Rightarrow x = 4 \pm 2\sqrt{3}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sqrt{2x-1} - \frac{x}{\sqrt{2x-1}}}{2x-1} = \frac{1-x}{\sqrt{(2x-1)^3}} = 0 \Rightarrow x = 1 \Rightarrow y = 1$	**(b)(i)** Do take note that there are 2 intercepts, it is quite common for students to miss out 0.536 as it might not be shown clearly on the GC (but it is clearly there in the sketch in the answer booklet!) **(c)(ii)** For the sketch of $x = 2(y-1)^2 + 1$, to obtain any credit, you should recognize that there are 2 branches, with the vertex at (1, 1). Students who do so should also pay attention that the position of the x-intercept 3 should be considered carefully in relation to the origin and the x-intercept of the first curve which is 7.46.

(b)(ii) **[5]**	 Required area of R $= \int_1^3 \left(2 - \frac{x}{\sqrt{2x-1}} - \frac{1}{2}\right) \mathrm{d}x$ $= \int_1^3 \frac{3}{2} \mathrm{d}x - \int_1^3 \frac{x}{\sqrt{2x-1}} \mathrm{d}x$ $= \frac{3}{2}[x]_1^3 - \int_1^{\sqrt{5}} \frac{u^2+1}{2} \mathrm{d}u$ $u = \sqrt{2x-1}$ When $x = 3$, $u = \sqrt{5}$ When $x = 1$, $u = \sqrt{1} = 1$ $= 3 - \frac{1}{2}\left[\frac{u^3}{3} + u\right]_1^{\sqrt{5}}$ $= 3 - \frac{1}{2}\left[\frac{5\sqrt{5}}{3} + \sqrt{5} - \frac{1}{3} - 1\right]$ $= 3 - \left[\frac{4\sqrt{5}}{3} - \frac{2}{3}\right]$ $= \frac{11 - 4\sqrt{5}}{3}$	It is easiest to use the GC to sketch the graph and the line $y = \frac{1}{2}$ to ascertain the desired region correctly. Be careful that the result in (a) only applies to $\int \frac{x}{\sqrt{2x-1}} \mathrm{d}x = \int \frac{u^2+1}{2} \mathrm{d}u$ Hence the constants should not be affected by the change in the limits. The final answer should be simplified whenever possible if an exact answer is required, in particular the surds in this case are all of the form $\sqrt{5}$.
(c) **(i)** **[1]**	$x = 2(y-1)^2 + 1 \Rightarrow y = 1 \pm \sqrt{\frac{x-1}{2}}$	This is a quadratic in y. Hence when expressing y in terms of x, there are 2 possible values.

(c) (ii) [2]	Please refer to part (b)(i)	
(c) (iii) [3]	 Required volume $= \pi\int_1^4\left(2-\frac{x}{\sqrt{2x-1}}\right)^2 \mathrm{d}x - \pi\int_1^3\left(1-\sqrt{\frac{x-1}{2}}\right)^2 \mathrm{d}x$ $= 4.518334$ $= 4.518 \text{ units}^3$ (3 d.p.)	A very common mistake is that the difference between two volumes is evaluated as $\pi\int_a^b (y_1-y_2)^2 \mathrm{d}x$ This is a bad conceptual error. The correct way to understand is it is the difference of 2 volumes and thus $\pi\int_a^b (y_1)^2 \mathrm{d}x$ $-\pi\int_c^d (y_2)^2 \mathrm{d}x$ As the question does not require an exact answer, you should immediately use the GC to evaluate the definite integral and not waste time evaluating it algebraically.

11 A curve C has parametric equations

$$x = 2\cos\theta + \cos 2\theta\,,\ y = 2\sin\theta - \sin 2\theta\,,$$

where $0 < \theta < 2\pi$ and $\theta \neq \frac{2\pi}{3}, \frac{4\pi}{3}$.

The point P with parameter p lies on C, and the lines M_p and N_p are the tangent and normal to C at P respectively.

(a) It is given that

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \text{ and}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Using these two results, show that the gradient of M_p is $-\tan\left(\frac{p}{2}\right)$. [2]

(b) For $p = \frac{\pi}{3}$, M_p and N_p cut the x-axis at points Q and R respectively. Find the area of ΔPQR. [4]

(c) The point S with parameter s lies on C, and the line N_s is the normal to C at S. It is given that M_p and N_s are parallel.
Given $p \neq \frac{\pi}{3}$, π and $p < s$, by using the result $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$, find an equation relating p and s. [2]

	Solution	Comments
(a) **[2]**	$x = 2\cos\theta + \cos 2\theta \Rightarrow \frac{dx}{d\theta} = -2\sin\theta - 2\sin 2\theta$ $y = 2\sin\theta - \sin 2\theta \Rightarrow \frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$	**Note 1** Since this is a 'SHOW' question, working presented needs to be complete and with clear explanation. No steps should be skipped or assumed, and in doing so it may result in losing marks. The most glaring 'error' was writing

	$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-2(\cos 2\theta - \cos\theta)}{-2(\sin 2\theta + \sin\theta)}$ $= \dfrac{-2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$ $\left(\begin{array}{l}\text{factor formula for}\\ \cos 2\theta - \cos\theta \text{ and}\\ \sin 2\theta + \sin\theta\end{array}\right)$ $= -\tan\left(\frac{\theta}{2}\right)$ Alternatively, $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2(\cos\theta - \cos 2\theta)}{-2(\sin 2\theta + \sin\theta)}$ $= \dfrac{-2\sin\left(\frac{3\theta}{2}\right)\sin\left(-\frac{\theta}{2}\right)}{-2\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$ $\left(\begin{array}{l}\text{factor formula for}\\ \cos\theta - \cos 2\theta \text{ and}\\ \sin 2\theta + \sin\theta\end{array}\right)$ $= \dfrac{\sin\left(-\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = -\dfrac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$ $\left(\begin{array}{l}\sin\left(-\frac{\theta}{2}\right) = -\sin\left(\frac{\theta}{2}\right)\\ \text{as the sine function is}\\ \text{an odd function}\end{array}\right)$ $= -\tan\left(\frac{\theta}{2}\right)$ When $\theta = p$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\tan\left(\dfrac{p}{2}\right)$	$\cos\theta - \cos 2\theta$ $= 2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$ without any explanation **Note 2** In this question θ is the variable and p is a constant that θ takes on at different points on C. As such, any differentiation should be done in terms of θ $\left(\frac{\mathrm{d}x}{\mathrm{d}\theta} \text{and} \frac{\mathrm{d}y}{\mathrm{d}\theta}\right)$, and not p $\left(\frac{\mathrm{d}x}{\mathrm{d}p} \text{and} \frac{\mathrm{d}y}{\mathrm{d}p}\right)$. **Note 3** Remember to answer the question and find an expression for the gradient of the tangent line at $\theta = p$.
(b) **[4]**	At $p = \frac{\pi}{3}: x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}, \frac{\mathrm{d}y}{\mathrm{d}x} = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ Equation of M_P : $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right) \Rightarrow y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}} \Rightarrow Q(2,0)$ Equation of N_P : $y - \frac{\sqrt{3}}{2} = \sqrt{3}\left(x - \frac{1}{2}\right) \Rightarrow y = \sqrt{3}x \Rightarrow R(0,0)$ Area of $\Delta PQR = \frac{1}{2}(2)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$ units^2	**Note 1** In this part, $p = \frac{\pi}{3}$, and so the gradient and point P should be evaluated instead of being expressed as the trigo expressions, making the working very cumbersome.

		Note 2 Always read the question carefully. In this case, Q and R lie on the x-axis and hence should be found by letting $y=0$, instead of $x=0$. **Note 3** It may be useful to draw a diagram relating the 3 points P, Q and R to identify how to find the area of the triangle.
(c) **[2]**	Gradient of $M_p = -\tan\left(\frac{p}{2}\right)$ Gradient of $N_s = \dfrac{1}{\tan\left(\frac{s}{2}\right)}$ Since M_p and N_s are parallel, $\dfrac{1}{\tan\left(\frac{s}{2}\right)} = -\tan\left(\frac{p}{2}\right) \quad \Rightarrow 1+\tan\left(\frac{s}{2}\right)\tan\left(\frac{p}{2}\right)=0$ $\Rightarrow \tan\left(\frac{s}{2}-\frac{p}{2}\right)$ is undefined $\Rightarrow \frac{s}{2}-\frac{p}{2}=\frac{\pi}{2}$ $\Rightarrow s-p=\pi$ **Note 2** By considering $1+\tan\left(\frac{s}{2}\right)\tan\left(\frac{p}{2}\right)=\dfrac{\tan\left(\frac{s}{2}\right)-\tan\left(\frac{p}{2}\right)}{\tan\left(\frac{s}{2}-\frac{p}{2}\right)}=0$, note that $\tan\left(\frac{s}{2}\right)-\tan\left(\frac{p}{2}\right)=0 \Rightarrow \frac{s}{2}=\frac{p}{2}+\pi \Rightarrow \tan\left(\frac{s}{2}-\frac{p}{2}\right)=0$, which	**Note 1** When 2 **lines** are **parallel**, their **gradients** are **equal**. Some mistakenly thought that the gradients would be a multiple of each other, possibly confusing it with the idea of parallel lines when the direction of one line is a multiple of the other line. **Note 2** See note on the left

	makes the term $\dfrac{\tan\left(\frac{s}{2}\right)-\tan\left(\frac{p}{2}\right)}{\tan\left(\frac{s}{2}-\frac{p}{2}\right)}$ undefined instead of 0. Hence we need only to consider $\tan\left(\frac{s}{2}-\frac{p}{2}\right)$ to be undefined as stated in the solution above.	