

Uppper Sec Amath

Revision Notes

Name: _____

MATHEMATICAL FORMULAE

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2}bc \sin A$$

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1 Quadratic Functions

Functions

Type	Name	Degree	Example
Constant	f	0	$f(x) = 3$
Linear	g	1	$g(x) = 2x + 5,$
Quadratic	h	2	$h(x) = x^2 - 8x + 4,$
Cubic	p	3	$p(x) = 2x^3 - 5x^2 + x + 10$

A function is a relationship between two variables. The above table summarises some of the different functions we'll be dealing with in A-Math.

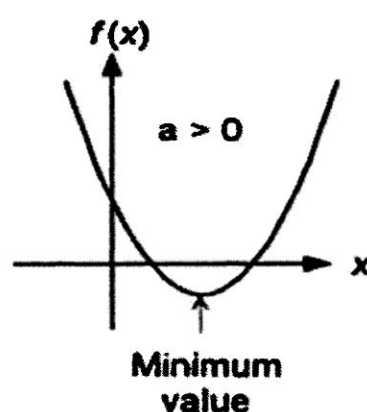
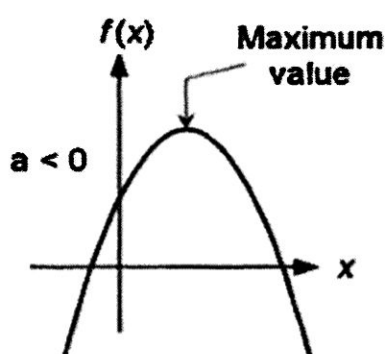
This chapter, we'll be dealing with quadratic polynomials, which are functions of the **second degree**. In general, quadratic polynomials are of the form

$$f(x) = ax^2 + bx + c$$

Properties of Quadratic Functions (RECAP)

1. Turning point / vertex

Coefficient of a	Type of value
$a > 0$	Minimum value
$a < 0$	Maximum value



2. Line of symmetry - the vertical line that cuts through the turning point
3. x -intercepts - the points where the graph cuts the x -axis
4. y -intercept - the point where the graph cuts the y -axis

Methods to determine turning points (RECAP)

1. Completing the square

- Rearrange the quadratic function $f(x) = ax^2 + bx + c$ in the form of

$$f(x) = a(x - h)^2 + k$$

- Steps

Factor out the quadratic coefficient a in the quadratic function and then rearrange the function as shown:

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left[c - a\left(\frac{b}{2a}\right)^2\right] \\ &= a(x - h)^2 + k \end{aligned}$$

Manipulation Trick!

$$x + kx = \left(x + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2$$

- Important Tips!

We need to learn how to deal with the following tricky cases in the exam:

- Coefficient of x^2 is negative

$$-x^2 + bx + c = -(x^2 - bx - c)$$

- Coefficient of x^2 is not 1

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

- **Maximum/Minimum value of quadratic function is k , when $x = h$.**

2. Line of symmetry

- Can be obtained by finding the midpoint of the two x -intercepts. This is also the x -coordinate of the turning point.
- y - coordinate of the turning point can be found by substituting the x - coordinate into $f(x)$.

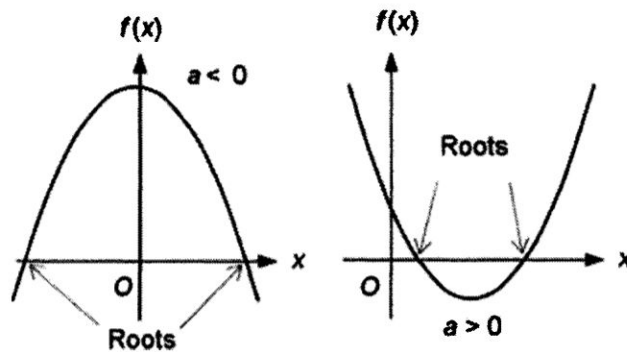
3. Differentiation (will learn in future!)

- Differentiate the quadratic function $f(x) = ax^2 + bx + c$ to obtain the differential function $f'(x) = 2ax + b$.
- Maximum / Minimum points, solve for x when $f'(x) = 0$.

Roots of quadratic equations

The solution to the equation $ax^2 + bx + c = 0$ are also known as the roots. We have learnt previously that the roots can be found by completing the square, or by quadratic formula, namely:

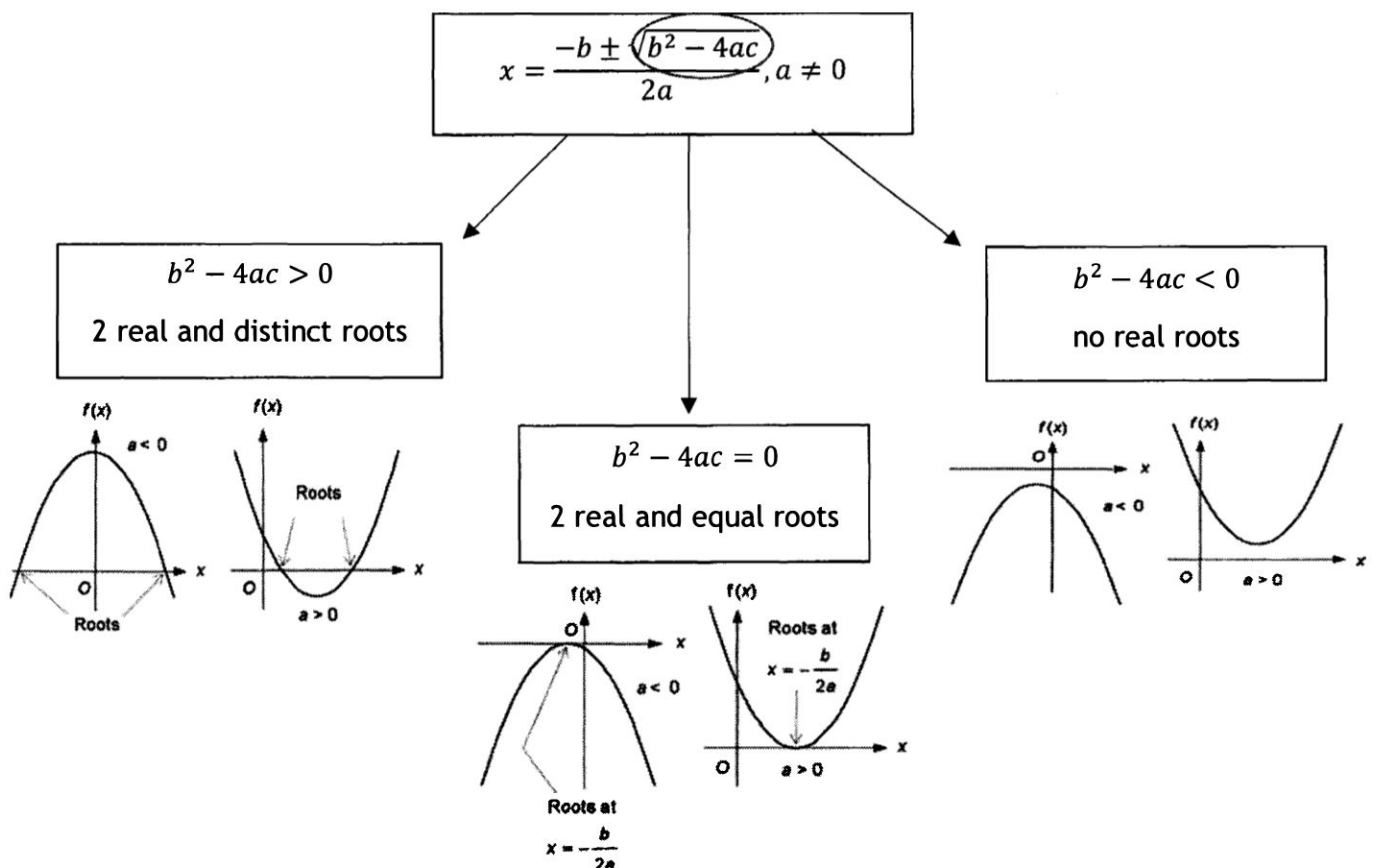
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



The portion within the square root of the quadratic formula is known as the discriminant. The discriminant takes the form of

$$D = b^2 - 4ac$$

By examining the discriminant, we are able to determine the nature of the roots.



The following are some ways we can be tested on discriminant:

- **Always positive / always lie above the x -axis:** $D < 0$, $a > 0$
- **Always negative / always lie below the x -axis:** $D < 0$, $a < 0$
- **Non-negative / never negative:** $D \leq 0$, $a > 0$
- **Non-positive / never positive:** $D \leq 0$, $a < 0$
- **Two real roots:** $D \geq 0$
- **Two distinct real roots:** $D > 0$
- **Two equal real roots / One real root / Tangent to x -axis:** $D = 0$
- **No real roots:** $D < 0$

Conditions for quadratic function to always be positive (or negative)

1. Completing the square

To show that a quadratic function is always positive (or negative), we can first complete the square, to obtain one of the following:

Form	Conclusion
$a(x \pm h)^2 + k$	$\because (x \pm h)^2 \geq 0$ and $a > 0$ $a(x \pm h)^2 \geq 0$ $a(x \pm h)^2 + k > 0$ Hence, the quadratic function is always positive
$-a(x \pm h)^2 - k$	$\because -(x \pm h)^2 \leq 0$ and $a > 0$ $-a(x \pm h)^2 \leq 0$ $-a(x \pm h)^2 - k < 0$ Hence, the quadratic function is always negative

Assumption: $a > 0$

2. Discriminant < 0

If $a > 0$, quadratic function is always positive

If $a < 0$, quadratic function is always negative

Common Mistake!

Many students will use $b^2 - 4ac > 0$ when they see the question phrase:

“curve is always positive”

This is INCORRECT! 😞 If the discriminant is positive, there are 2 real and distinct solutions.

However, if the curve is always positive, does the curve ever cut the x -axis? NO!

Since there is no intersection, the **discriminant value should be negative**: $b^2 - 4ac < 0$.

Note!

We will start with more straightforward questions, before increasing the difficulty and including other topics.

Most questions on the concept of discriminant will combine with the concept of quadratic inequalities. If you're unfamiliar with that concept, read ahead and strengthen that first!

Worked Example 1.1

State the range of values of p for which the roots of the equation are real.

$$(x - 1)^2 = p - 3$$

[S4 CGSS P2/2011 PRELIM Q1]

[Solution]

$$(x - 1)^2 = p - 3$$

$$x^2 - 2x + 1 - p + 3 = 0$$

$$x^2 - 2x + (4 - p) = 0$$

Since the roots of the equation are real, $b^2 - 4ac \geq 0$

$$(-2)^2 - 4(1)(4 - p) \geq 0$$

$$4 - 16 + 4p \geq 0$$

$$4p \geq 12$$

$$p \geq 3$$

Worked Example 1.2

Given that the quadratic equation $kx^2 + 2(k - 1)x + k + 3 = 0$ has equal roots, calculate the value of k .

[Solution]

$$kx^2 + 2(k - 1)x + k + 3 = 0$$

Since the roots are equal, $b^2 - 4ac = 0$

$$[2(k - 1)]^2 - 4(k)(k + 3) = 0$$

$$[2k - 2]^2 - 4(k)(k + 3) = 0$$

$$4(k^2 - 2k + 1) - 4k^2 - 12k = 0$$

$$4k^2 - 8k + 4 - 4k^2 - 12k = 0$$

$$20k = 4$$

$$k = \frac{1}{5}$$

Worked Example 1.3

A quadratic equation $x^2 + px + q = 3$ has roots -1 and 2 .

- (a) Shows that the value of $p = -1$ and the value of $q = 1$.
(b) Find the range of values of m for which the following equation has no real roots.

$$x^2 + px + q = mx - 3$$

[Solution]

(a) $x^2 + px + q = 3$

Since we know that the roots are -1 and 2 ,

$$(x + 1)(x - 2) = 0$$

$$x^2 - 2x + x - 2 = 0$$

$$x^2 - x + 1 = 3$$

Hence, by comparing coefficients,

$$p = -1 \text{ and } q = 1 \text{ (shown)}$$

(b) $x^2 - x + 1 = mx - 3$

$$x^2 - x + 1 - mx + 3 = 0$$

$$x^2 + (-1 - m)x + 4 = 0$$

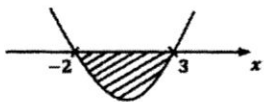
Since the equation has no real roots, $b^2 - 4ac < 0$

$$(-1 - m)^2 - 4(1)(4) < 0$$

$$1 + 2m + m^2 - 16 < 0$$

$$m^2 + 2m - 15 < 0$$

$$(m + 5)(m - 3) < 0$$



$$-5 < m < 3$$

Tips!

Part (a) is a very common examination question where they will start from the solutions/roots of the equation and ask for the starting equation. Always remember that from the roots, you can work backwards to find the equation.

Worked Example 1.4

Find the range of values of k for which the expression $x^2 + 5k(2k - x) + 2k - 3$ is never negative.

[S4 CHS P1/2011 PRELIM Q3]

[Solution]

$$\text{Let } x^2 + 5k(2k - x) + 2k - 3 = 0$$

$$x^2 + 10k^2 - 5kx + 2k - 3 = 0$$

$$x^2 + (-5k)x + (10k^2 + 2k - 3) = 0$$

This is important! The discriminant only exists for a quadratic equation. There are no roots to find in a quadratic expression.

For the equation to never be negative, $b^2 - 4ac \leq 0$

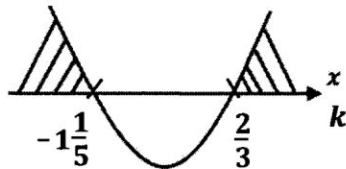
$$(-5k)^2 - 4(1)(10k^2 + 2k - 3) \leq 0$$

$$25k^2 - 40k^2 - 8k + 12 \leq 0$$

$$-15k^2 - 8k + 12 \leq 0$$

$$15k^2 + 8k - 12 \geq 0$$

$$(3k - 2)(5k + 6) \geq 0$$



$$\therefore k \leq -1\frac{1}{5} \quad \text{or} \quad k \geq \frac{2}{3}$$

Practice: How can we solve the above question by completing the square?

Worked Example 1.5

Show that the expression, where p is a constant, it is always positive for all real value of

$$x^2 + px - x + p^2 + 2$$

Hence, find the range of values of x for which

$$\frac{x^2 - 3x - 28}{x^2 + px - x + p^2 + 2} < 0$$

[S4 TSS P2/2015 PRELIM Q2(a)]

[Solution]

$$\text{Let } x^2 + px - x + p^2 + 2 = 0$$

$$\Rightarrow x^2 + (p - 1)x + (p^2 + 2) = 0$$

For the equation to always be positive, $b^2 - 4ac < 0$

$$\text{Discriminant} = (p - 1)^2 - 4(1)(p^2 + 2)$$

$$= p^2 - 2p + 1 - 4p^2 - 8$$

$$= -3p^2 - 2p - 7$$

$$= -(3p^2 + 2p + 7)$$

$$= -3\left(p^2 + \frac{2}{3}p + 2\frac{1}{3}\right)$$

$$= -3\left[\left(p + \frac{1}{3}\right)^2 - \frac{1}{9} + 2\frac{1}{3}\right]$$

$$= -3\left[\left(p + \frac{1}{3}\right)^2 + 2\frac{2}{9}\right]$$

$$= -3\left(p + \frac{1}{3}\right)^2 - 6\frac{2}{3}$$

We now need to show that $-3\left(p + \frac{1}{3}\right)^2 - 6\frac{2}{3}$ is always negative.

$$\text{Since } \left(p + \frac{1}{3}\right)^2 \geq 0,$$

$$-3\left(p + \frac{1}{3}\right)^2 \leq 0,$$

$$\therefore -3\left(p + \frac{1}{3}\right)^2 - 6\frac{2}{3} < 0$$

Since the coefficient of x^2 is positive and the discriminant is strictly less than 0,

$x^2 + px - x + p^2 + 2$ is always positive for all real values of x .

Hence, want to find the range of

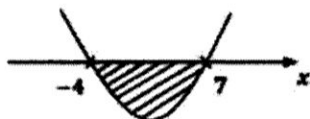
$$\frac{x^2 - 3x - 28}{x^2 + px - x + p^2 + 2} < 0$$

$$\because x^2 + px - x + p^2 + 2 > 0,$$

this question is equivalent to solving $x^2 - 3x - 28 < 0$.

$$x^2 - 3x - 28 < 0$$

$$(x - 7)(x + 4) < 0$$



$$-4 < x < 7$$

Tips!

This is a classic question that always comes out for examinations. Look out for keywords like “never positive” / “lies above the x -axis”.

When proving a discriminant is always positive or always negative, remember to complete the square. Completing the square will always make the proof very visual and obvious.

Do not find the discriminant of the discriminant!

Worked Example 1.6

Given a line with equation $y = -k - 3$, and a curve with equation $y = x^2 + kx$, find the range of values of k such that the line does not intersect the curve.

[Solution]

Consider the intersection points of the two graphs.

$$x^2 + kx = -k - 3$$

$$x^2 + kx + k + 3 = 0$$

Since the 2 graphs do not intersect, $b^2 - 4ac < 0$

$$(k)^2 - 4(1)(k + 3) < 0$$

$$k^2 - 4k - 12 < 0$$

$$(k + 2)(k - 6) < 0$$



$$-2 < k < 6$$

Tips!

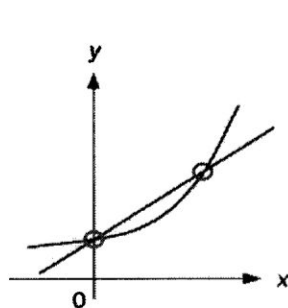
Do note the similarity and differences of this question to the previous questions.

Similarity: Both uses the concept of discriminant.

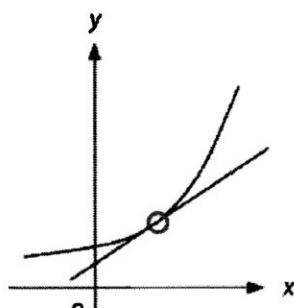
Differences: This question includes the context of finding the intersection between a line and curve, before we get the quadratic equation that equates to 0.

In general, we can extend the question to the following cases:

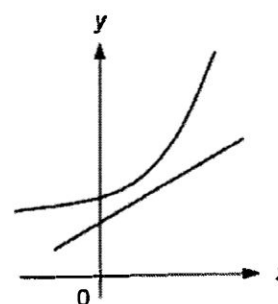
- Cuts at two points: $D > 0$
- Tangential to one another: $D = 0$
- Do not intersect: $D < 0$
- Intersects one another: $D \geq 0$



Intersection at two different points



Tangency



Non-Intersection

Worked Example 1.7

Find the range of values of k for which the line $y = x - 3$ and the curve $y^2 = k(1 - x^2)$ do not meet.

[Solution]

$$y = x - 3 \dots \dots \dots (1)$$

$$y^2 = k(1 - x^2) \dots \dots \dots (2)$$

Substitute Equation (1) into Equation (2)

$$(x - 3)^2 = k(1 - x^2)$$

$$x^2 - 6x + 9 = k - kx^2$$

$$x^2 - 6x + 9 - k + kx^2 = 0$$

$$(1 + k)x^2 - 6x + (9 - k) = 0$$

Since the line and the curve do not meet, $b^2 - 4ac < 0$

$$\therefore (-6)^2 - 4(1 + k)(9 - k) < 0$$

$$36 - 4[9 + 8k - k^2] < 0$$

$$36 - 36 - 32k + 4k^2 < 0$$

$$4k^2 - 32k < 0$$

$$4k(k - 8) < 0$$



$$0 < k < 8$$

Worked Example 1.8

Find the exact values of c for which the line is a tangent to the curve

$$y = 3x + c$$

$$x^2 - y^2 = 4x + 5$$

[S4 MGS P1/2011 PRELIM Q8(b)]

[Solution]

$$y = 3x + c \dots \dots (1)$$

$$x^2 - y^2 = 4x + 5 \dots \dots (2)$$

Substitute Equation (1) into Equation (2),

$$x^2 - (3x + c)^2 = 4x + 5$$

$$x^2 - (9x^2 + 6xc + c^2) - 4x - 5 = 0$$

$$x^2 - 9x^2 - 6xc - c^2 - 4x - 5 = 0$$

$$-8x^2 - (6c + 4)x - (c^2 + 5) = 0$$

Since the line is a tangent to the curve, $b^2 - 4ac = 0$

$$(-(6c + 4))^2 - 4(-8)(-(c^2 + 5)) = 0$$

$$36c^2 + 48c + 16 - 32c^2 - 160 = 0$$

$$4c^2 + 48 - 144 = 0$$

$$c^2 + 12c - 36 = 0$$

$$c = \frac{-(12) \pm \sqrt{(12)^2 - 4(1)(-36)}}{2(1)}$$

$$= \frac{-12 \pm \sqrt{288}}{2}$$

$$= \frac{-12 \pm 12\sqrt{2}}{2}$$

$$= -6 \pm 6\sqrt{2}$$

Worked Example 1.9

Find the range of value of p for which the line

$$y = (4p + 2)x$$

cuts the curve

$$y = (p + 2)x^2 + 4p$$

at two distinct points.

[S4 CCHS(M) P1/2011 PRELIM Q4]

[Solution]

$$y = (4p + 2)x \dots \dots (1)$$

$$y = (p + 2)x^2 + 4p \dots \dots (2)$$

Equation (1) = Equation (2)

$$(4p + 2)x = (p + 2)x^2 + 4p$$

$$(p + 2)x^2 - (4p + 2)x + 4p = 0$$

Since the line and the curve cuts at 2 distinct points, $b^2 - 4ac > 0$

$$(-(4p + 2))^2 - 4(p + 2)(4p) > 0$$

$$(-4p - 2)^2 - 16p^2 - 32p > 0$$

$$16p^2 + 16p + 4 - 16p^2 - 32p > 0$$

$$-16p + 4 > 0$$

$$16p - 4 < 0$$

$$p < \frac{1}{4}$$

Tips!

All 3 examples have the same tips/ thinking process so it will be discussed here instead.

Always look out in the question for key words: “Real”, “Real & Distinct”, “Equal”. These key words will signify the discriminant value.

These questions are very popular during examinations. Do note that the methods are exactly the same! Practice makes perfect for these kind of questions.

Worked Example 1.10

Show that the line meets the curve for all real values of p

$$y = 1 - 2x$$

$$x(4px - 5p) = y - p$$

[Solution]

$$y = 1 - 2x \dots \dots (1)$$

$$x(4px - 5p) = y - p \dots \dots (2)$$

Substitute Equation (1) into Equation (2),

$$x(4px - 5p) = (1 - 2x) - p$$

$$4px^2 - 5px - 1 + 2x + p = 0$$

$$4px^2 + (2 - 5p)x + (p - 1) = 0$$

Since we need to show that the line meets the curve for all real values of p , we need to show that $D > 0$

$$\begin{aligned} D &= b^2 - 4ac = (2 - 5p)^2 - 4(4p)(p - 1) \\ &= 4 - 20p + 25p^2 - 16p^2 + 16p \\ &= 9p^2 - 4p + 4 \\ &= 9\left(p^2 - \frac{4}{9}p\right) + 4 \\ &= 9\left(\left(p - \frac{2}{9}\right)^2 - \left(\frac{2}{9}\right)^2\right) + 4 \\ &= 9\left(p - \frac{2}{9}\right)^2 + 3\frac{5}{9} \end{aligned}$$

$$\text{Since } \left(p - \frac{2}{9}\right)^2 \geq 0, \quad 9\left(p - \frac{2}{9}\right)^2 \geq 0,$$

$$\therefore 9\left(p - \frac{2}{9}\right)^2 + 3\frac{5}{9} > 0$$

Since the discriminant is strictly greater than 0, this implies that the curve will meet the line for all real values of p .

Tips!

This is a show question, hence the answering style is different from the previous worked examples.

Students are not allowed to straight away assume that the discriminant of this case is greater than and equals to 0 even though it is mentioned in the question that the line will meet the curve for all real values of p .

2 Quadratic Inequalities

Properties of inequalities (RECAP)

Let a, b and c be any 3 real numbers.

Property/Conditions	Result
$a - b > 0$	$a > b$
$a - b < 0$	$a < b$
$a - b = 0$	$a = b$
$a > b$ and $b > c$	$a > c$
$a > b$	$a + c > b + c$ or $a - c > b - c$
$a < b$	$a + c < b + c$ or $a - c < b - c$
$a > b$ and $c > 0$	$ac > bc$ or $\frac{a}{c} > \frac{b}{c}$
$a > b$ and $c < 0$	$ac < bc$ or $\frac{a}{c} < \frac{b}{c}$

Solving quadratic inequalities

1. Finding the roots

- Let the quadratic inequality $f(x) > 0$ or $f(x) < 0$ be a quadratic equation

$$f(x) = 0$$

- Factorise the equation to get the form

$$a(x - \alpha)(x - \beta) = 0$$

OR

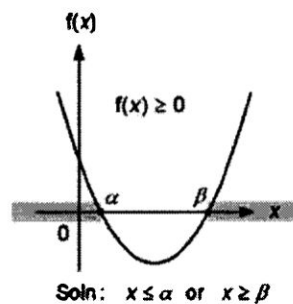
Solve the equation using quadratic formula, to give the roots

$$x = \alpha \text{ or } \beta.$$

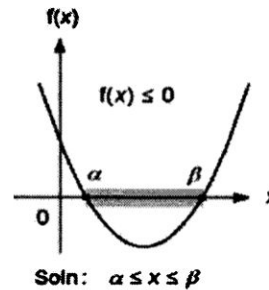
2. Critical Region

- Draw a number line and identify the critical regions.

Pro tip! Use the direction of the inequality to help you identify the critical regions. [Take into account happy face curves]



If the inequality symbol of the quadratic inequality is pointing to the right, then the critical region will be outside of the curve



If the inequality symbol of the quadratic inequality is pointing to the left, then the critical region will be on the inside of the curve

Common Mistake

Always check if the coefficient of x^2 is positive or not. In the event the coefficient of the quadratic term is not positive, multiply both sides by -1 first.

This allows us to always draw a “happy face” graph!

Note the flip in the inequality symbol. Since you are multiplying by a negative value, the inequality symbol should change direction

$$-x^2 + bx + c < 0$$

$$x^2 - bx - c > 0$$

or

$$-x^2 + bx + c > 0$$

$$x^2 - bx - c < 0$$

Worked Example 2.1

Find the range of values of x for which

$$(x - 3)^2 < x + 27$$

[S4 CGS P1/2009 PRELIM Q1]

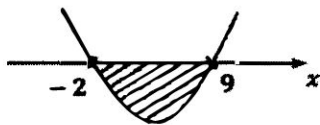
[Solution]

$$(x - 3)^2 < x + 27$$

$$x^2 - 6x + 9 < x + 27$$

$$x^2 - 6x - x + 9 - 27 < 0$$

$$x^2 - 7x - 18 < 0$$



$$\therefore -2 < k < 9$$

Tips!

This is a rather simple question. Expand and group all the terms like how you would for a quadratic equation.

Common Mistake!

Many students will write out the solutions of the quadratic like this:

$$k = -2, \quad k = 9$$

This is incorrect as this is an inequality

Worked Example 2.2

Find the range of x values that satisfy the inequality

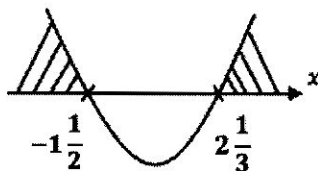
$$x(6x - 5) > 21$$

[Solution]

$$x(6x - 5) > 21$$

$$6x^2 - 5x - 21 > 0$$

$$(3x - 7)(2x + 3) > 0$$



$$\therefore x < -1\frac{1}{2} \quad \text{or} \quad x > 2\frac{1}{3}$$

Worked Example 2.3

Find the value of m and n where

$$x^2 - x + 1 = (x + m)^2 + n$$

Hence or otherwise, for all real values of x , find the largest value of k such that

$$x^2 - x + 1 \geq k$$

[Solution]

$$\begin{aligned}x^2 - x + 1 &= (x + m)^2 + n \\&= x^2 + 2mx + m^2 + n\end{aligned}$$

Comparing coefficients,

$$2m = -1$$

$$m = -\frac{1}{2}$$

$$\left(\frac{1}{2}\right)^2 + n = 1$$

$$n = \frac{3}{4}$$

Hence,

$$x^2 - x + 1 \geq k$$

$$\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq k$$

$$\left(x - \frac{1}{2}\right)^2 \geq k - \frac{3}{4}$$

Since $\left(x - \frac{1}{2}\right)^2 \geq 0$, the largest value for $k - \frac{3}{4}$ is 0, since any other negative value would also satisfy the inequality.

$$\therefore k = \frac{3}{4}$$

3 Surds

A surd is an **irrational number** containing a root and cannot be expressed as a fraction.

Given real numbers a, b, m and n , a surd can be written in the form

$$a\sqrt{m} + b\sqrt{n}$$

The conjugate pair of a surd is changing the sign that separates the two roots. For example, $1 + \sqrt{2}$ is the conjugate pair of the surd $1 - \sqrt{2}$.

Laws of Surds

Operations	Formulae	Example
Addition	$a\sqrt{m} + b\sqrt{m} = (a + b)\sqrt{m}$	$2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$
Subtraction	$a\sqrt{m} - b\sqrt{m} = (a - b)\sqrt{m}$	$2\sqrt{3} - 5\sqrt{3} = -3\sqrt{3}$
Multiplication	$a\sqrt{m} \times b\sqrt{n} = ab\sqrt{mn}$	$2\sqrt{3} \times 5\sqrt{3} = 10(3) = 30$ $2\sqrt{3} \times 5\sqrt{7} = 10\sqrt{21}$
Division	$a\sqrt{m} \div b\sqrt{n} = \frac{a}{b}\sqrt{\frac{m}{n}}$	$2\sqrt{3} \div 5\sqrt{3} = \frac{2}{5}\sqrt{\frac{3}{3}} = \frac{2}{5}$ $2\sqrt{3} \div 5\sqrt{7} = \frac{2}{5}\sqrt{\frac{3}{7}}$

Using the above multiplication law, we can also reverse it to simplify surds.

$$\sqrt{294} = \sqrt{2 \times 3 \times 7^2} = (\sqrt{2})(\sqrt{3})(\sqrt{7^2}) = 7\sqrt{6}$$

In general, we can get

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

and simplify if either a or b are squares.

Important property!

Recall the quadratic identity $(a + b)(a - b) = a^2 - b^2$. Applying this to surds, we can see that the multiplication of conjugate pairs always gives a real number! For example:

$$(4 + \sqrt{5})(4 - \sqrt{5}) = (4)^2 - (\sqrt{5})^2 = 16 - 5 = 11$$

Rationalising surds

Rationalisation is the act of removing surds from the denominator. We do this by utilising the above property, to turn the denominator into a real number.

Broadly speaking there are two cases for rationalisation.

1. Denominator is a single-term surd

- Rationalise by multiplying the numerator and denominator by $\sqrt{a}$ to get a .

Example Question (Case 1)

Rationalise the following surd.

$$\frac{9}{\sqrt{12}}$$

[Method A]

$$\frac{9}{\sqrt{12}} = \left(\frac{9}{\sqrt{12}}\right)\left(\frac{\sqrt{12}}{\sqrt{12}}\right) = \frac{9\sqrt{2^2 \times 3}}{12} = \frac{9(2)(\sqrt{3})}{12} = \frac{3\sqrt{3}}{2}$$

However, we can make our lives easier by first simplifying the denominator.

[Method B]

$$\frac{9}{\sqrt{12}} = \frac{9}{\sqrt{4 \times 3}} = \frac{9}{2\sqrt{3}} = \left(\frac{9}{2\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{9\sqrt{3}}{2(3)} = \frac{3\sqrt{3}}{2}$$

2. Denominator is a sum/difference of surds

- Multiply the denominator by the conjugate surd to turn the denominator into a real number. In general, multiply $a\sqrt{m} + b\sqrt{n}$ by $a\sqrt{m} - b\sqrt{n}$ to get

$$(a\sqrt{m})^2 - (b\sqrt{n})^2$$

Example Question (Case 2)

Rationalise the following surd.

$$\frac{3}{\sqrt{5} - 2}$$

[Solution]

$$\frac{3}{\sqrt{5} - 2} = \left(\frac{3}{\sqrt{5} - 2} \right) \left(\frac{\sqrt{5} + 2}{\sqrt{5} + 2} \right) = \frac{3(\sqrt{5} + 2)}{(\sqrt{5})^2 - (2)^2} = \frac{3(\sqrt{5} + 2)}{5 - 4} = 3(\sqrt{5} + 2)$$

Worked Example 3.1

Given that $2a - 8\sqrt{5} = 20 - 4\sqrt{b}$, where a and b are integers, find the value of a and of b .

[Solution]

By comparing coefficients,

$$2a = 20$$

$$\therefore a = 10$$

$$8\sqrt{5} = \sqrt{64 \times 5} = \sqrt{320}$$

$$4\sqrt{b} = \sqrt{16b}$$

$$\therefore 16b = 320$$

$$b = 20$$

Therefore, $a = 10$ and $b = 20$

Note!

We are unable to compare coefficients for the roots directly and say that $b = 5$ as the coefficients are different. ($8 \neq 4$)

With that being said, if the coefficients are the same, we can compare them directly 😊

Worked Example 3.2

Triangle ABC is such that AB is $(6 - \sqrt{11})$ cm and its area is given by $(5 + \sqrt{44})$ cm². Express the perpendicular distance from C to AB in the form $\frac{m+n\sqrt{11}}{25}$, where m and n are integers.

[S4 ANDSS P1/2009 PRELIM Q4]

[Solution]

$$\text{Area of triangle} = \frac{1}{2} (\text{base area})(\text{perpendicular distance})$$

Let the perpendicular distance be h

$$(5 + \sqrt{44}) = \frac{1}{2} (6 - \sqrt{11})(h)$$

$$\therefore h = \frac{2(5 + \sqrt{44})}{6 - \sqrt{11}} \times \frac{6 + \sqrt{11}}{6 + \sqrt{11}}$$

$$= \frac{(10 + 2\sqrt{44})(6 + \sqrt{11})}{36 - 11}$$

$$= \frac{60 + 10\sqrt{11} + 12\sqrt{44} + 2\sqrt{44 \times 11}}{25}$$

$$= \frac{60 + 10\sqrt{11} + 24\sqrt{11} + 44}{25}$$

$$= \frac{104 + 34\sqrt{11}}{25}$$

$$m = 104, n = 34$$

Tips!

This is a typical question for surds. Usually in such scenario-based questions students will be tasked to form an equation on their own.

Most of the time, these scenario-based questions will be geometry questions (very simple). Use the known formulas to form the equation.

Worked Example 3.3

Without using a calculator, find the value of m such that

$$\frac{\sqrt{2}}{3} \left(\frac{3}{\sqrt{6}} - \frac{64}{\sqrt{384}} + \frac{\sqrt{486}}{6} \right) = m\sqrt{3}$$

[S4 CCHS(M) P1/2011 PRELIM Q6]

[Solution]

By simplifying the LHS:

$$\begin{aligned} & \frac{\sqrt{2}}{3} \left(\frac{3}{\sqrt{6}} - \frac{64}{\sqrt{384}} + \frac{\sqrt{486}}{6} \right) \\ &= \frac{\sqrt{2}}{\sqrt{6}} - \frac{64\sqrt{2}}{3\sqrt{384}} + \frac{(\sqrt{2})(\sqrt{486})}{18} \\ &= \frac{\sqrt{2}}{\sqrt{2 \times 3}} - \frac{64\sqrt{2}}{3\sqrt{2^7 \times 3}} + \frac{(\sqrt{2})(\sqrt{2 \times 3^5})}{18} \\ &= \frac{\sqrt{2}}{(\sqrt{2})(\sqrt{3})} - \frac{64\sqrt{2}}{24\sqrt{2 \times 3}} + \frac{(9\sqrt{2})(\sqrt{2 \times 3})}{18} \\ &= \frac{1}{\sqrt{3}} - \frac{8}{3\sqrt{3}} + \sqrt{3} \\ &= \left(\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \right) - \left(\frac{8}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \right) + \sqrt{3} \\ &= \frac{\sqrt{3}}{3} - \frac{8\sqrt{3}}{9} + \sqrt{3} \\ &= \frac{4}{9}\sqrt{3} \\ \therefore m &= \frac{4}{9} \end{aligned}$$

Tips!

This is a more challenging variant of surd questions.

Usually for such questions, always simplify the bigger surds into simpler form will help in the following steps. We can do this by prime factorising the integer within the surd, to identify the squares, then simplify from there.

Always start from the most difficult side and work towards the simpler side.

Worked Example 3.4

Solve the following equation

$$\sqrt{4 + \frac{3}{x}} = \frac{1}{\sqrt{x}} + 2$$

[S4 AHS P1/2018 PRELIM Q3]

[Solution]

$$\sqrt{4 + \frac{3}{x}} = \frac{1}{\sqrt{x}} + 2$$

$$4 + \frac{3}{x} = \left(\frac{1}{\sqrt{x}} + 2\right)^2$$

$$4 + \frac{3}{x} = \left(\frac{1 + 2\sqrt{x}}{\sqrt{x}}\right)^2$$

$$4 + \frac{3}{x} = \frac{1 + 4\sqrt{x} + 4x}{x}$$

$$4x + 3 = 1 + 4\sqrt{x} + 4x$$

$$4\sqrt{x} = 2$$

$$\sqrt{x} = \frac{1}{2}$$

$$x = \frac{1}{4}$$

Common Mistake

Many students will do this:

$$\sqrt{4 + \frac{3}{x}} = \frac{1}{\sqrt{x}} + 2$$

$$\left(\sqrt{4 + \frac{3}{x}}\right)^2 = \left(\frac{1}{\sqrt{x}}\right)^2 + (2)^2$$

$$4 + \frac{3}{x} = \frac{1}{x} + 4$$

...

This is mathematically inaccurate as if you want to remove the square roots on both sides, you must square every single term together as a whole, to not miss out any hidden terms.

Worked Example 3.5

Given that a and b are roots of the equation $x^2 + x - 1 = 0$ and $a < b$, prove that

$$\frac{1}{\sqrt{3}}a^2b = \frac{\sqrt{5} + 1}{2\sqrt{3}}$$

Hence, simplify the right-hand side of the equation above

[Solution]

Solving for the roots of the equation,

$$\begin{aligned}x &= \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)} \\&= \frac{-1 \pm \sqrt{5}}{2}\end{aligned}$$

Since $a < b$,

$$\begin{aligned}\therefore a &= \frac{-1 - \sqrt{5}}{2}, & b &= \frac{-1 + \sqrt{5}}{2} \\ \therefore a^2b &= \left(\frac{-1 - \sqrt{5}}{2}\right)^2 \cdot \left(\frac{-1 + \sqrt{5}}{2}\right) \\&= \left(\frac{(-1 - \sqrt{5})(-1 - \sqrt{5})}{4}\right) \cdot \left(\frac{-1 + \sqrt{5}}{2}\right) \\&= \frac{(-1 - \sqrt{5})(-1 - \sqrt{5})(-1 + \sqrt{5})}{8} \\&= \frac{(-1 - \sqrt{5})[(-1)^2 - (\sqrt{5})^2]}{8} \\&= \frac{(-1 - \sqrt{5})(-4)}{8} \\&= \frac{4(1 + \sqrt{5})}{8} \\&= \frac{1 + \sqrt{5}}{2} \\ \therefore \frac{1}{\sqrt{3}}a^2b &= \frac{\sqrt{5} + 1}{2\sqrt{3}}\end{aligned}$$

Thinking Process!

Since the question is asking for the roots of the equation, we need to use the quadratic formula to find the roots in the surd form. The remaining part of the question is pretty standard.

Worked Example 3.6

Find, without using tables or calculators, the value

$$\left(\sqrt{9 + \sqrt{45}} + \sqrt{9 - \sqrt{45}}\right)^2$$

[Solution]

$$\begin{aligned}\left(\sqrt{9 + \sqrt{45}} + \sqrt{9 - \sqrt{45}}\right)^2 &= \left(\sqrt{9 + \sqrt{45}}\right)^2 + 2\left(\sqrt{9 + \sqrt{45}}\right)\left(\sqrt{9 - \sqrt{45}}\right) + \left(\sqrt{9 - \sqrt{45}}\right)^2 \\&= (9 + \sqrt{45}) + 2\sqrt{(9 + \sqrt{45})(9 - \sqrt{45})} + (9 - \sqrt{45}) \\&= 18 + 2\sqrt{36} \\&= 30\end{aligned}$$

Tips/Thinking Process

Be careful when squaring terms like this, remember to use the quadratic identity:

$$(a + b)^2 = a^2 + 2ab + b^2$$

4 Polynomials & Partial Fractions

Polynomials

A polynomial function is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

where all powers of x must be **positive integers**.

$$\text{e.g.: } f(x) = x^3 + 2x^2 - 3x + 4$$

When substituting x into the function, for example $x = -1$, we replace all the x on both sides of the equation.

$$\begin{aligned} \text{e.g.: } f(-1) &= (-1)^3 + 2(-1)^2 - 3(-1) + 4 \\ &= 8 \end{aligned}$$

Division Algorithm

In the scenario where $f(x)$ and $g(x)$ are 2 polynomials,

$$f(x) = g(x) \cdot q(x) + r(x)$$

where $f(x)$ represents the dividend, $g(x)$ represents the divisor, $q(x)$ represents the quotient and $r(x)$ represents the remainder.

$$\begin{aligned} \text{e.g.: } f(x) &= x^3 + 2x^2 - 3x + 4 \\ &= (x + 1)(x^2 + x - 4) + 8 \end{aligned}$$

$$\begin{array}{r} q(x) \text{ [Quotient]} \\ g(x) \text{ [Divisor]} \overline{) f(x) \text{ [Dividend]}} \\ \hline r(x) \text{ [Remainder]} \end{array}$$

Note!

The degree of $g(x)$ is always smaller than the degree of $f(x)$

Operations of Polynomials

We use a combination of skill we have learnt earlier when we operate with polynomials.

- BODMAS
- Distributive law
- Combining like terms
- Comparing coefficients

Worked Example 4.1

It is given that $8x^3 + 4x^2 - 26x - 14 = 2A(x^2 - 4)(Bx + 1) + C(2x - 4) + 2D$ for all values of x . Find the values of A , B , C and of D .

[Solution]

By expanding RHS,

$$2A(Bx^3 + x^2 - 4Bx - 4) + 2Cx - 4C + 2D = 0$$

$$2ABx^3 + 2Ax^2 + (-8AB + 2C)x + (-8A - 4C + 2D) = 0$$

By comparing coefficients of:

$$x^2: 2A = 4 \Rightarrow A = 2$$

$$x^3: 2(2)B = 8 \Rightarrow B = 2$$

$$x: -8(2)(2) + 2C = -26 \Rightarrow C = 3$$

$$\text{constant: } -8(2) - 4(3) + 2D = -14 \Rightarrow D = 7$$

Hence, $A = 2$, $B = 2$, $C = 3$, $D = 7$.

Elimination method

For a polynomial equation, since $\text{LHS} = \text{RHS}$ for all values of x , we can substitute any value of x into the equation and it will still be true.

Tip! Always aim to eliminate an unknown with your substitution!

Worked Example 4.1 (redo)

It is given that $8x^3 + 4x^2 - 26x - 14 = 2A(x^2 - 4)(Bx + 1) + C(2x - 4) + 2D$ for all values of x . Find the values of A , B , C and of D .

[Solution]

Let $x = 2$

$$8(2)^3 + 4(2)^2 - 26(2) - 14 = 0 + 0 + 2D$$

$$14 = 2D$$

$$\therefore D = 7$$

Let $x = -2$

$$8(-2)^3 + 4(-2)^2 - 26(-2) - 14 = 0 + C(-8) + 2(7)$$

$$-24 = -8C$$

$$\therefore C = 3$$

Hence, we now have $8x^3 + 4x^2 - 26x - 14 = 2A(x^2 - 4)(Bx + 1) + 6x + 2$

By comparing coefficients of

$$x^2: 2A = 4 \Rightarrow A = 2$$

$$x^3: 2(2)B = 8 \Rightarrow B = 2$$

Hence, $A = 2$, $B = 2$, $C = 3$, $D = 7$.

Tips!

Notice that this method is a lot cleaner, as we deal with more elementary operations, as compared to tedious distribution of terms in the polynomial expansion.

Remainder Theorem

The Remainder Theorem states that when a polynomial function $f(x)$ is divided by $(x - a)$, the remainder is equal to $f(a)$, where a is a constant.

Mathematically, if you recall from the division algorithm above where $f(x)$ and $g(x)$ are 2 polynomials,

$$f(x) = g(x) \cdot q(x) + r(x)$$

When a polynomial $f(x)$ is divided by $(x - a)$, you substitute $x = a$, therefore making the divisor $g(x)$ to be equal to 0, hence $g(x) \cdot q(x)$ will be equal to 0 as well, leaving $f(a)$ to be equal to the remainder.

Factor Theorem

The Factor Theorem is closely related to the Remainder Theorem, where you repeat the same steps as in Remainder Theorem. The main difference is that, if $(x - a)$ is a factor of $f(x)$, $f(a) = 0$.

In other words, if substituting $x = a$ into the polynomial $f(x)$ results in zero, then $(x - a)$ is a factor of $f(x)$.

Worked Example 4.2

The function $f(x) = 2x^3 + mx^2 + nx - 12$ leaves a remainder of -14 when divided by $(2x + 1)$ and is divisible by $(x - 3)$, find the value of m and of n .

[Solution]

$$\text{Let } x = -\frac{1}{2}$$

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 + m\left(-\frac{1}{2}\right)^2 + n\left(-\frac{1}{2}\right) - 12 \\ &= -\frac{1}{4} + \frac{1}{4}m - \frac{1}{2}n - 12 \\ &= 14 \end{aligned}$$

$$\frac{1}{4}m - \frac{1}{2}n = -\frac{7}{4} \Rightarrow m - 2n = -7 \quad (\text{Equation 1})$$

$$\text{Let } x = 3$$

$$\begin{aligned} f(3) &= 2(3)^3 + m(3)^2 + n(3) - 12 \\ &= 54 + 9m + 3n - 12 \\ &= 0 \end{aligned}$$

$$3m + n = -14 \quad (\text{Equation 2})$$

From equation 1, $m = 2n - 7$ (Equation 3)

Sub (3) into (2)

$$3(2n - 7) + n = -14$$

$$6n - 21 + n = -14$$

$$7n = 7 \Rightarrow n = 1$$

Sub $n = 1$ into (3)

$$\begin{aligned} m &= 2(1) - 7 \\ &= -5 \end{aligned}$$

Factorisation and Solving of Cubic Polynomials

Step 0: Let that expression be $f(x)$, if it is not already defined.

Step 1: Using a calculator, we execute a “trial and error” method to identify one factor of the polynomial.

- For calculator model **CASIO FX-96SG**, press **MODE**, **3**, **4**.
- For calculator model **CASIO FX-97SG**, press **MENU**, **5**, **2**, **SELECT DEGREE** (usually **3**)

Step 2: Using the factor identified in Step 1, use Factor Theorem to prove that it is a factor. (“By factor theorem, $(x - a)$ is a factor of $f(x)$.”)

Step 3: To find the remaining factors (usually a quadratic equation),

- Long division **OR**
- Comparison of coefficients and constant : $f(x) = (x - a)(Ax^2 + Bx + C)$

Step 4: Factorise the quadratic equation found in previous step using calculator or Multiplication frame method.

Step 5: Conclude and present your answer as an expression of 3 linear factors.

If you are looking to **solve** the polynomial,

Step 6: Equate the 3 linear factors in Step 5 to be equal to 0 and solve the polynomial. Alternatively, if you are unable to factorise the quadratic equation in Step 4, you can solve the polynomial by using the quadratic formula to solve for the roots.

Worked Example 4.3

Factorise $f(x) = 2x^3 - 9x^2 + 3x + 4$ completely.

[Solution]

Using the trial and error method, we have identified that $(x - 1)$ is a factor.

Let $x = 1$

By factor theorem,

$$\begin{aligned}f(1) &= 2(1)^3 - 9(1)^2 + 3(1) + 4 \\&= 2 - 9 + 3 + 4 \\&= 0\end{aligned}$$

Hence, $(x - 1)$ is a factor of $f(x)$.

Method 1: Long Division

$$\begin{array}{r}2x^2 - 7x - 4 \\x - 1 \overline{) 2x^3 - 9x^2 + 3x + 4} \\ \underline{-(2x^3 - 2x^2)} \\ -7x^2 + 3x + 4 \\ \underline{-(-7x^2 + 7x)} \\ -4x + 4 \\ \underline{-(-4x + 4)} \\ 0\end{array}$$

$$\begin{aligned}\therefore f(x) &= (x - 1)(2x^2 - 7x - 4) \\&= (x - 1)(x - 4)(2x + 1)\end{aligned}$$

Method 2: Comparison

$$f(x) = 2x^3 - 9x^2 + 3x + 4 = (x - 1)(Ax^2 + Bx + C)$$

By comparing the coefficient of x^3 : $A = 2$

By comparing the constant: $4 = -C \Rightarrow C = -4$

By comparing the coefficient of x : $3 = C - B \Rightarrow B = -7$

$$\begin{aligned}\therefore f(x) &= (x - 1)(2x^2 - 7x - 4) \\&= (x - 1)(x - 4)(2x + 1)\end{aligned}$$

Tips!

It is always the easiest to compare the coefficient of the highest power and the constant, hence start with these two. The last comparison can be between x or x^2 . Comparison method is always recommended as it is easier and less time-consuming.

Worked Example 4.4

The cubic polynomial $f(x)$ is such that roots of the equation $f(x) = 0$ are $-1, 3$ and k . Given that $f(x)$ has a remainder of 20 when divided by $(x - 4)$, and a remainder of 72 when divided by $(x - 5)$,

(a) Show that $k = 2$

(b) Find the remainder when $f(x)$ is divided by $(2x - 3)$.

Tip! Never ever assume that the coefficient of x^3 is 1 unless stated in the question

[Solution]

(a) Let $f(x) = a(x + 1)(x - 3)(x - k)$

Let $x = 4$

$$\begin{aligned}f(4) &= a(4 + 1)(4 - 3)(4 - k) \\&= 5a(4 - k) = 20\end{aligned}$$

Let $x = 5$

$$\begin{aligned}f(5) &= a(5 + 1)(5 - 3)(5 - k) \\&= 12a(5 - k) = 72\end{aligned}$$

$$\frac{5a(4 - k)}{12a(5 - k)} = \frac{20}{72}$$

$$\frac{(4 - k)}{(5 - k)} = \frac{4}{6} = \frac{2}{3}$$

By applying cross-multiplication,

$$3(4 - k) = 2(5 - k)$$

$$12 - 3k = 10 - 2k$$

$$3k - 2k = 12 - 10$$

$$k = 2 \text{ (Shown)}$$

(b) To solve for a ,

$$5a(4 - 2) = 20$$

$$5a = 10$$

$$a = 2$$

$$f(x) = 2(x + 1)(x - 3)(x - 2)$$

$$\text{Let } x = \frac{3}{2}$$

$$\begin{aligned}\text{Remainder} &= f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2} + 1\right)\left(\frac{3}{2} - 3\right)\left(\frac{3}{2} - 2\right) \\&= \frac{15}{4}\end{aligned}$$

Tip! Even if you are not able to solve part (a), you can proceed with part (b) since the value of k is given in the question.

Worked Example 4.5

(a) Solve the equation $6 + 11x - 3x^2 - 2x^3 = 0$.

(b) Hence solve $-2 - 3y^2 + 11y^4 + 6y^6 = 0$

[Solution]

(a) Let $f(x) = 2x^3 + 3x^2 - 11x - 6$

Using the trial and error method, we can identify that $(x - 2)$ is a factor of $f(x)$.

Let $(x - 2)$ be a factor of $f(x)$.

$$\begin{aligned} f(2) &= 2(2)^3 + 3(2)^2 - 11(2) - 6 \\ &= 0 \end{aligned}$$

Hence, $(x - 2)$ is a factor of $f(x)$.

$$\text{By comparison, } 2x^3 + 3x^2 - 11x - 6 = (x - 2)(Ax^2 + Bx + C)$$

By comparing the coefficient of x^3 : $A = 2$

By comparing the constant: $-6 = -2C \Rightarrow C = 3$

By comparing the coefficient of x : $-11 = C - 2B \Rightarrow B = 7$

$$\begin{aligned} \therefore f(x) &= (x - 2)(2x^2 + 7x + 3) \\ &= (x - 2)(2x + 1)(x + 3) \end{aligned}$$

When $f(x) = 0$, $x = 2$ or $x = -\frac{1}{2}$ or $x = -3$

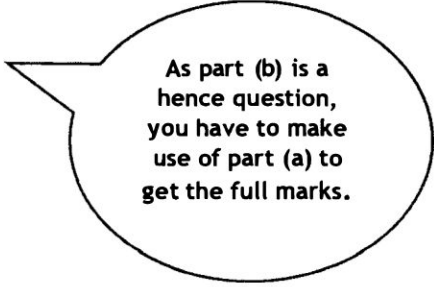
(b) By shifting the equation to the RHS, you will have $2 + 3y^2 - 11y^4 - 6y^6 = 0$

By dividing every term by y^6 , you will obtain the following equation:

$$\frac{2}{y^6} + \frac{3}{y^4} - \frac{11}{y^2} - 6 = 0$$

$$2\left(\frac{1}{y^6}\right) + 3\left(\frac{1}{y^4}\right) - 11\left(\frac{1}{y^2}\right) - 6 = 0$$

$$2\left(\frac{1}{y^2}\right)^3 + 3\left(\frac{1}{y^2}\right)^2 - 11\left(\frac{1}{y^2}\right) - 6 = 0$$



As part (b) is a hence question, you have to make use of part (a) to get the full marks.

From the original equation $f(x) = 2x^3 + 3x^2 - 11x - 6$, we can conclude that x has been replaced with $\frac{1}{y^2}$.

Hence, $\frac{1}{y^2} = 2$ or $\frac{1}{y^2} = -\frac{1}{2}$ (reject $\because \frac{1}{y^2} > 0$) or $\frac{1}{y^2} = -3$ (reject $\because \frac{1}{y^2} > 0$)

$$\therefore y = \pm \sqrt{\frac{1}{2}} = \pm \frac{\sqrt{2}}{2}$$

Partial Fractions

Partial fractions is the simplification of a single algebraic fraction into a sum of two or more simpler fractions.

$$\frac{P(x)}{Q(x)}$$

Proper Fractions

Degree of $P(x)$ is less than the degree of $Q(x)$.

Example:

$$\frac{x}{3x^2 + 4}$$

Improper Fractions

Degree of $P(x)$ is greater than, or equal to, the degree of $Q(x)$.

Example:

$$\frac{x^2 + 5}{x - 1}$$

Common Mistake

$$\frac{x + 5}{x - 1}$$

Many students forget that when the degree of $P(x)$ is equal to the degree of $Q(x)$, the fraction is an **improper** fraction.

Note!

Before starting any partial fraction questions, always check if the fraction is proper or improper.

- If the fraction is **proper**, can proceed on to using any of the 3 formulas.
- If the fraction is **improper**, convert the fraction into a proper fraction first using **LONG DIVISION**, before proceeding on to using any of the 3 formulas.

Distinct Linear Factor

Factors in the denominator are all linear

Example:

$$\frac{px + q}{(ax + b)(cx + d)} = \frac{A}{ax + b} + \frac{B}{cx + d}$$

Repeated Linear Factor

One of the factors is repeated with a square

Example:

$$\frac{px + q}{(ax + b)(cx + d)^2} = \frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{(cx + d)^2}$$

Distinct Irreducible factor

One of the factors is irreducible

Example:

$$\frac{px + q}{(ax + b)(x^2 + c^2)} = \frac{A}{ax + b} + \frac{Bx + C}{x^2 + c^2}$$

Take Note!

Be careful when selecting which formula to use for your partial fractions. Many students will make mistakes at the initial stage.

- Always check that the fractions are proper/improper.
- Always check that the denominators are factorised before making a sound decision.
If the denominator is not factorised, factorise on your own first.
- $\frac{1}{x^2}$ is considered a repeated linear factor, hence giving you $\frac{A}{x} + \frac{B}{x^2}$
- When dealing with Distinct Irreducible Factors, the degree of the numerator of the irreducible factor is always one less than the degree in the denominator.

$$\frac{Bx + C}{x^2 + c^2}$$

Degree of numerator is 1
while degree of
denominator is 2.

Worked Example 4.6

Express the following $\frac{4x+7}{x^2+6x+9}$ in partial fractions.

[Solution]

Since the degree of numerator is lower than the degree of denominator, it is a proper fraction.

Step 1: Factorise the denominator.

$$x^2 + 6x + 9 = (x + 3)^2$$

$$\frac{4x + 7}{x^2 + 6x + 9} = \frac{4x + 7}{(x + 3)^2}$$

Step 2: Apply the formula (in this case it will be Repeated Linear Factor)

$$\frac{4x + 7}{(x + 3)^2} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2}$$

Step 3: Make the denominators the same and compare the numerators.

$$\frac{4x + 7}{(x + 3)^2} = \frac{A(x + 3)}{(x + 3)^2} + \frac{B}{(x + 3)^2}$$

$$4x + 7 = A(x + 3) + B$$

Step 4: Solve for the unknowns using elimination or comparison.

By elimination,

$$\text{Substitute } x = -3 \text{ so that } A(x + 3) = A(0) = 0$$

$$4(-3) + 7 = B$$

$$B = -5$$

By comparing the coefficient of x ,

$$A = 4$$

Final answer:

$$\begin{aligned}\frac{4x + 7}{x^2 + 6x + 9} &= \frac{4}{x + 3} + \frac{-5}{(x + 3)^2} \\ &= \frac{4}{x + 3} - \frac{5}{(x + 3)^2}\end{aligned}$$

Tip! You can check your answer by subbing in a random value of x into both sides.

By substituting $x = 1$,

$$\text{LHS} = \frac{4(1)+7}{1^2+6(1)+9} = \frac{11}{16}$$

$$\text{RHS} = \frac{4}{1+3} - \frac{5}{(1+3)^2} = \frac{11}{16}$$

$$\therefore \text{RHS} = \text{LHS (Correct!)}$$

Worked Example 4.7

Express the following in terms of partial fractions

$$\frac{x^2 - 7x + 4}{(2x + 5)(x^2 + 3)}$$

[Solution]

$$\frac{x^2 - 7x + 4}{(2x + 5)(x^2 + 3)} = \frac{A}{2x + 5} + \frac{Bx + C}{x^2 + 3}$$

$$x^2 - 7x + 4 = A(x^2 + 3) + (Bx + C)(2x + 5)$$

By elimination, let $x = -\frac{5}{2}$ to eliminate $(Bx + C)(2x + 5)$.

$$\left(-\frac{5}{2}\right)^2 - 7\left(-\frac{5}{2}\right) + 4 = A\left[\left(-\frac{5}{2}\right)^2 + 3\right]$$

$$\frac{111}{4} = \frac{37}{4}A$$

$$A = 3$$

Since $(x^2 + 3)$ can never be equal to 0, we will compare coefficients.

$$x^2 - 7x + 4 = 3(x^2 + 3) + (Bx + C)(2x + 5)$$

By comparing the coefficient of x^2 ,

$$1 = 3 + 2B \Rightarrow B = -1$$

By comparing the constant,

$$4 = 3(3) + 5C \Rightarrow C = -1$$

Recall: It is always the easiest to compare the highest degree and constant!

Final answer:

$$\begin{aligned}\frac{x^2 - 7x + 4}{(2x + 5)(x^2 + 3)} &= \frac{3}{2x + 5} + \frac{-x - 1}{x^2 + 3} \\ &= \frac{3}{2x + 5} - \frac{x + 1}{x^2 + 3}\end{aligned}$$

Worked Example 4.8

Express the following in terms of partial fractions

$$\frac{2x^3 + 11x^2 - 3x - 5}{x^2 - 4}$$

[S4 HIHS P2/2014 Prelim Q5]

[Solution]

Since the degree of the numerator (3) is higher than the degree of the denominator (2), it is an improper fraction, hence we have to apply long division first.

$$\begin{aligned}\frac{2x^3 + 11x^2 - 3x - 5}{x^2 - 4} &= \frac{2x^3 + 11x^2 - 3x - 5}{(x-2)(x+2)} \\ &= 2x + 11 + \frac{5x+39}{(x-2)(x+2)}\end{aligned}$$

$$\frac{5x+39}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$5x + 39 = A(x+2) + B(x-2)$$

$$\text{Let } x = 2,$$

$$5(2) + 39 = 4A$$

$$A = \frac{49}{4}$$

$$\text{Let } x = -2,$$

$$5(-2) + 39 = -4B$$

$$B = -\frac{29}{4}$$

$$\begin{array}{r} 2x + 11 \\ x^2 - 4 \overline{) 2x^3 + 11x^2 - 3x - 5} \\ \underline{-(2x^3 - 8x)} \\ 11x^2 + 5x - 5 \\ \underline{-(11x^2 - 44)} \\ 5x + 39 \end{array}$$

Final answer:

$$\frac{2x^3 + 11x^2 - 3x - 5}{x^2 - 4} = 2x + 11 + \frac{49}{4(x-2)} - \frac{29}{4(x+2)}$$

5 Logarithms and Exponential Functions

Logarithmic Functions

A logarithmic function is the inverse of an exponential function. The general form of a logarithmic function can be expressed as $\log_a b = c$, where :

- b is the argument of the logarithm, which must be a positive real number,
- a is the base of the logarithm, which must be a positive real number different from 1,
- c is the result of the logarithmic function, representing the power to which the base a must be raised to product the number b .

Special Logarithmic Function

- 1) Common Logarithm: $\log_{10} x$
- 2) Natural Logarithm: $\log_e x = \ln x$

$\log_{10} x$ is also known as $\lg x$ on the calculator.

Exponential Function

An exponential function can be defined as $a^c = b$, where:

- a is the base, which must be a positive real number different from 1,
- c is the exponent, which can be any real number,
- b is the output value, and it will always be positive.

Relationship between Logarithmic Function and Exponential Function

$$\log_a b = c \quad \Rightarrow \quad a^c = b$$

Examples:

- 1) $\log_2 4 = 2$ implies that $2^2 = 4$
- 2) $\log_{10} 1000 = 3$ implies that $10^3 = 1000$
- 3) $\ln x = 2$ implies that $e^2 = x$

Laws of Logarithmic Functions

1) Addition

- $\log_a x + \log_a y = \log_a xy$

2) Subtraction

- $\log_a x - \log_a y = \log_a \frac{x}{y}$

Note! This rule only applies when the bases are the SAME.

3) Power

- $\log_a (x)^n = n \log_a x$

Note! This only applies when the power n is applied to x ONLY.

4) Same base

- $\log_a a = 1$

5) "1"

- $\log_a 1 = 0$

6) Change of Base Formula

- When you want to change the base of a logarithmic function from base " a " to " b ", Formula to use will be:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example:

Change the base of this equation: $\log_4 x$ to base 2.

Working:

$$\begin{aligned}\log_4 x &= \frac{\log_2 x}{\log_2 4} \\ &= \frac{\log_2 x}{\log_2 2^2} \\ &= \frac{\log_2 x}{2}\end{aligned}$$

Worked Example 5.1

Solve the equation: $\log_3(10 - 9x) - 4\log_9 x = 2$

[Solution]

Step 1: Make all the bases the same.

$$\begin{aligned}4\log_9 x &= 4\left(\frac{\log_3 x}{\log_3 9}\right) \\&= 4\left(\frac{\log_3 x}{\log_3 3^2}\right) = 4\left(\frac{\log_3 x}{2\log_3 3}\right) \\&= 4\left(\frac{\log_3 x}{2 \times 1}\right) = 2\log_3 x\end{aligned}$$

Step 2: Rewrite the equation and combine using laws of logarithmic functions.

$$\log_3(10 - 9x) - 2\log_3 x = 2$$

$$\log_3(10 - 9x) - \log_3 x^2 = 2$$

$$\log_3 \frac{(10 - 9x)}{x^2} = 2$$

Step 3: Solve for the unknown.

$$\frac{(10 - 9x)}{x^2} = 3^2$$

$$10 - 9x = 9x^2$$

$$9x^2 + 9x - 10 = 0$$

$$(3x - 2)(3x + 5) = 0$$

$$x = \frac{2}{3} \quad \text{or} \quad x = -\frac{5}{3}$$

Step 4: Check if solutions can be accepted or rejected.

x cannot be negative as x appears in : $4\log_9 x$.

Hence, only acceptable solution is $x = \frac{2}{3}$ (reject -ve)

Worked Example 5.2

Given that $p = \log_{49} q$, express in terms of p ,

(i) $\log_{49} \left(\frac{1}{q} \right)$

(ii) $\log_7 49q^2$

[Solution]

(i) Given that $p = \log_{49} q$,

$$\begin{aligned}\log_{49} \left(\frac{1}{q} \right) &= \log_{49} 1 - \log_{49} q \\ &= 0 - p \\ &= -p\end{aligned}$$

$$\begin{aligned}\text{(ii) } \log_7 49q^2 &= \log_7 49 + \log_7 q^2 \\ &= \log_7 7^2 + 2\log_7 q \\ &= 2\log_7 7 + 2\left(\frac{\log_{49} q}{\log_{49} 7}\right) \\ &= 2(1) + 2\left(\frac{p}{\log_{49} \sqrt{49}}\right) \\ &= 2 + 2\left(\frac{p}{\log_{49} 49^{\frac{1}{2}}}\right) \\ &= 2 + 2\left(\frac{p}{\frac{1}{2}\log_{49} 49}\right) \\ &= 2 + 2\left(\frac{p}{\frac{1}{2}}\right) \\ &= 2 + 4p\end{aligned}$$

Note! When the base and the argument of the logarithm are the same, the value of the logarithm will always be equal to 1.

Worked Example 5.3

Evaluate $\log_6 9 \times \log_3 16 \times \log_4 36$, without using a calculator.

[Solution]

$$\begin{aligned} & \log_6 9 \times \log_3 16 \times \log_4 36 \\ &= \frac{\lg 9}{\lg 6} \times \frac{\lg 16}{\lg 3} \times \frac{\lg 36}{\lg 4} \\ &= \frac{\lg 3^2}{\lg 6} \times \frac{\lg 4^2}{\lg 3} \times \frac{\lg 6^2}{\lg 4} \\ &= \frac{2 \lg 3}{\lg 6} \times \frac{2 \lg 4}{\lg 3} \times \frac{2 \lg 6}{\lg 4} \\ &= \frac{2 \cancel{\lg 3}}{\cancel{\lg 6}} \times \frac{2 \cancel{\lg 4}}{\cancel{\lg 3}} \times \frac{2 \cancel{\lg 6}}{\cancel{\lg 4}} \\ &= 8 \end{aligned}$$

Note! When there are 2 logarithmic functions with different bases, you will change the base to the smaller integer. When there are 3 logarithmic functions with different bases, you should change them to special logarithmic functions ($\lg x$ or $\ln x$)

Worked Example 5.4

Solve the equation $\log_x 16 + \log_2 x = 4$.

[Solution]

Given that $\log_x 16 + \log_2 x = 4$,

$$\frac{\log_2 16}{\log_2 x} + \log_2 x = 4$$

$$\frac{\log_2 2^4}{\log_2 x} + \log_2 x = 4$$

Let $y = \log_2 x$

$$\frac{4}{y} + y = 4$$

$$4 + y^2 = 4y$$

$$y^2 - 4y + 4 = 0$$

$$(y - 2)^2 = 0$$

$$y = 2$$

$$\log_2 x = 2$$

$$x = 2^2$$

$$= 4$$

Tips! You can always use substitution to form a quadratic equation when you are dealing with fractional equations.

Laws of Exponential Functions

Laws of Exponential functions are similar to Chapter 3: Indices of E-Math syllabus.

$$1) a^m \times a^n = a^{m+n}$$

$$2) a^m \div a^n = a^{m-n}$$

$$3) (a^m)^n = a^{m \times n}$$

$$4) a^m \times b^m = (ab)^m$$

$$5) a^m \div b^m = \left(\frac{a}{b}\right)^m$$

$$6a) a^{-n} = \frac{1}{a^n} \quad 6b) \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

$$7) \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

$$8) a^0 = 1$$

Exponential function questions resemble those involving indices but solving them can be more complex. For instance, you might substitute e^x with a new variable to convert the equation into a quadratic form. This method allows for the use of quadratic solving techniques, providing a practical approach to handling more intricate exponential equations.

Worked Example 5.5

Solve $e^{2x} - 5e^x + 6 = 0$, leaving your answer in exact form.

[Solution]

Step 1: Rewrite the equation to look like a quadratic equation.

$$(e^x)^2 - 5e^x + 6 = 0$$

Step 2: Replace the exponent with another letter and solve the quadratic equation as per normal.

$$\text{Let } e^x = w.$$

$$w^2 - 5w + 6 = 0$$

$$(w - 3)(w - 2) = 0$$

$$w = 3 \text{ or } w = 2$$

Step 3: Solve for the unknown x with laws of logarithm.

$$e^x = 3 \text{ or } e^x = 2$$

Note! e^x can NEVER be negative or equal to 0, hence you reject all negative and zero values

By applying $\ln$ on both sides,

$$\ln e^x = \ln 3 \text{ or } \ln e^x = \ln 2$$

$$x \ln e = \ln 3 \text{ or } x \ln e = \ln 2$$

$$x = \ln 3 \text{ or } x = \ln 2$$

$$\text{(since } \ln e = 1)$$

Worked Example 5.6

Simplify the following expression

$$\frac{81^{x+1} + 48(9^{2x})}{3^{x+3} \times 27^{x+2}}$$

Hence find the value of y when:

$$129(3^{3y}) = \frac{81^{x+1} + 48(9^{2x})}{3^{x+3} \times 27^{x+2}}$$

[Solution]

$$\frac{81^{x+1} + 48(9^{2x})}{3^{x+3} \times 27^{x+2}} = \frac{3^{4(x+1)} + 48(3^{2(2x)})}{3^{x+3} \times 3^{3(x+2)}}$$

$$= \frac{3^{4x+4} + 48(3^{4x})}{3^{x+3+3x+6}}$$

$$= \frac{3^{4x}(3^4) + 48(3^{4x})}{3^{4x+9}}$$

$$= \frac{81(3^{4x}) + 48(3^{4x})}{3^9(3^{4x})}$$

$$= \frac{129(\cancel{3^{4x}})}{19\,683(\cancel{3^{4x}})}$$

$$= \frac{43}{6561}$$

$$\text{Now, let } 129(3^{3y}) = \frac{43}{6561}$$

$$43 \times 3(3^{3y}) = \frac{43}{3^8}$$

$$43 \times (3^{3y+1}) = 43 \times 3^{-8}$$

$$3y + 1 = -8$$

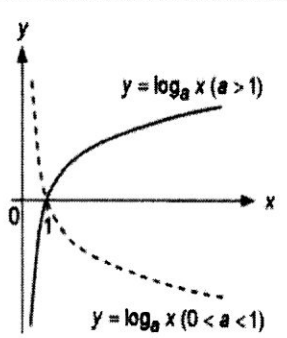
$$3y = -9$$

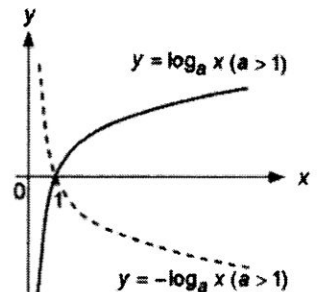
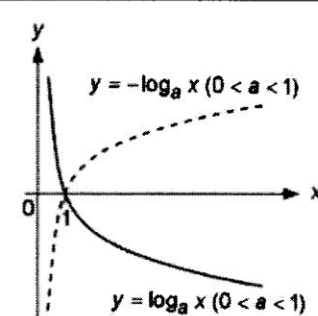
$$y = -3$$

Tip! To change the sign of your powers, bring it to the other side (i.e. from numerator to denominator or vice versa)

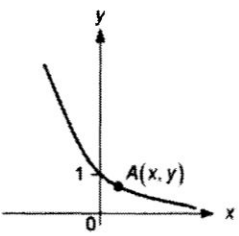
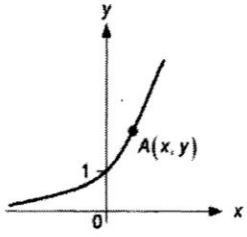
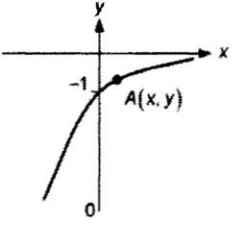
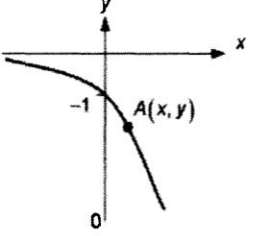
Graphs of Logarithmic and Exponential Functions

Graphs of Logarithmic Functions

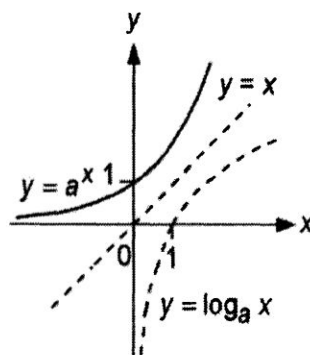
Term	Expression	Graphical Representation
$\log_a x$	$a > 1$ $0 < a < 1$	

Term	Expression	Graphical Representation
$-\log_a x$	$a > 1$ $0 < a < 1$	 

Graphs of Exponential Functions

Equation	Signum of k	Signum of a	Graph Representation
$y = ka^x$	$k > 0$	$0 < a < 1$	
	$k > 0$	$a > 1$	
	$k < 0$	$0 < a < 1$	
	$k < 0$	$a > 1$	

Do note that the graphs of $\log_a x$ and a^x are reflections of each other about the line $y = x$



Note! Always identify the asymptote first whenever you are drawing a logarithmic or exponential graph, followed by the axial-intercepts.

Worked Example 5.7

- (a) Sketch the graph of

$$y = e^{3-x} - 1$$

- (b) Find the equation of the straight line that must be drawn on the graph to obtain the solution of the equation:

$$\ln(3x + 2) = \ln 2 + (3 - x)$$

- (c) Draw this line into the same diagram and state the number of solutions of the above equation.

[Solution]

- (a) To identify the asymptote, identify the value e^{3-x} can NEVER be. Since e^{3-x} can never be negative or equal to 0, y can never be equal to or lesser than -1 .

Asymptote: $y = -1$

x - intercept

$$y = 0$$

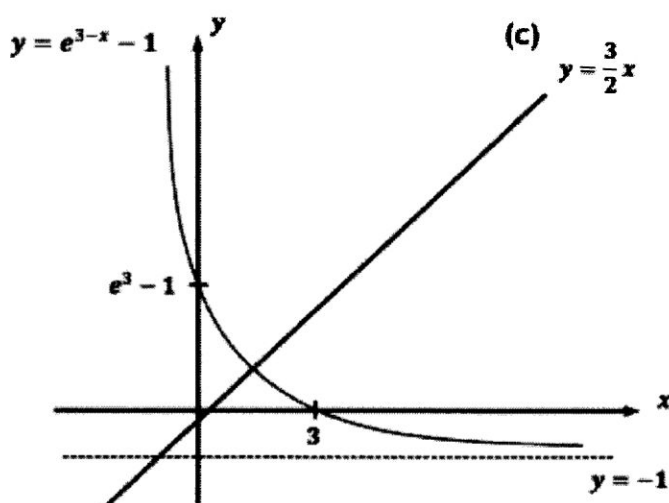
$$3 - x = 0$$

$$x = 3$$

y - intercept

$$x = 0$$

$$y = e^3 - 1$$



- (b) $\ln(3x + 2) = \ln 2 + (3 - x)$

$$\ln(3x + 2) - \ln 2 = (3 - x)$$

$$\ln\left(\frac{3x + 2}{2}\right) = (3 - x)$$

$$e^{3-x} = \frac{3x + 2}{2}$$

$$e^{3-x} = \frac{3x}{2} + 1$$

$$\frac{3}{2}x = e^{3-x} - 1$$

$\therefore$ The equation of straight line to be drawn is $y = \frac{3}{2}x$.

- (c) The number of solutions is 1 as there is only 1 intersection point between the two graphs.

Application of Logarithmic and Exponential Functions in Real-World Context

Logarithmic and Exponential functions are incredibly versatile in real-world context applications. An example includes Compound Interest Formula, $P(1 + \frac{r}{100})^n$.

Worked Example 5.8

A piece of meat is removed from the refrigerator, and it is then allowed to thaw. Its temperature, $T^\circ\text{C}$, after t minutes, is given by $P = 21 - 30e^{-0.3t}$.

- (i) Find the value of P when $t = 3.5$
- (ii) Find the value of t when $P = 10$
- (iii) Find the value of P when t becomes very large
- (iv) Hence, sketch the graph of P against t .

[Solution]

(i) When $t = 3.5$, $P = 21 - 30e^{-0.3 \times 3.5}$
 $= 10.5^\circ\text{C}$

(ii) When $P = 10$, $10 = 21 - 30e^{-0.3t}$

$$30e^{-0.3t} = 11$$

$$e^{-0.3t} = \frac{11}{30}$$

$$\ln e^{-0.3t} = \ln \frac{11}{30}$$

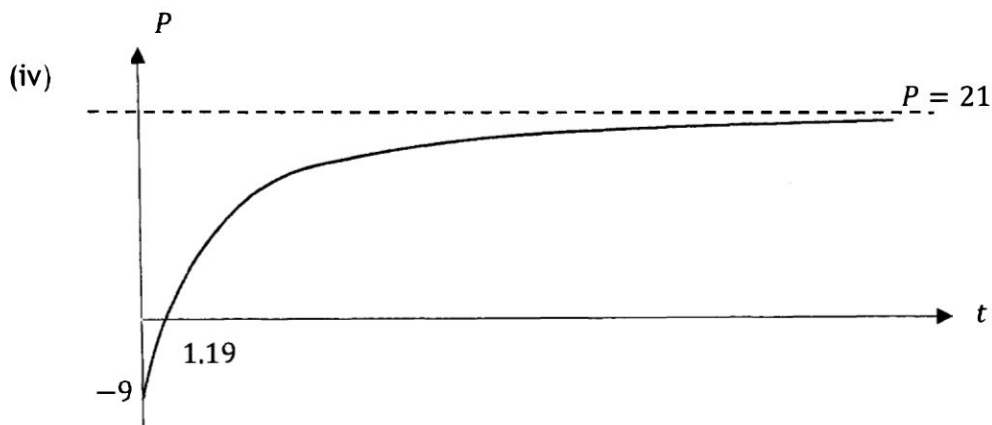
$$-0.3t = \ln \frac{11}{30}$$

$$t = \ln \frac{11}{30} \div (-0.3)$$

$$= 3.34 \text{ min (3s.f.)}$$

(iii) $P = 21 - 30(0)$

$$= 21^\circ\text{C as } e^{-0.3t} \rightarrow 0 \text{ when } t \text{ becomes very large.}$$



6 Binomial Theorem

We are exposed to binomial theorem from as early as Secondary 2, when we expanded the form $(a + b)^2$ into $a^2 + 2ab + b^2$. What allows us to derive the coefficients of each term?

By comparing the following expansions with the Pascal's triangle:

$$n = 2: (a + b)^2 = a^2 + 2ab + b^2$$

$$n = 3: (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$n = 4: (a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

We notice that the coefficients of each term follow a certain pattern. These coefficients are also known as **binomial coefficients**.

Worked Example 6.1

Using the Pascal's triangle, write out the terms of $(x + 1)^5$.

Hence, deduce the expansion of $(2x^2 + 1)^5$.

[Solution]

By Pascal's triangle, $(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

Hence,

$$\begin{aligned}
 (2x + 3)^5 &= (2x^2)^5 + 5(2x^2)^4 + 10(2x^2)^3 + 10(2x^2)^2 + 5(2x^2) + 1 \\
 &= 32x^{10} + 5(16x^8) + 10(8x^6) + 10(4x^4) + 5(2x^2) + 1 \\
 &= 32x^{10} + 80x^8 + 80x^6 + 40x^4 + 10x^2 + 1
 \end{aligned}$$

Binomial Expansion

However, using Pascal's triangle to find the coefficients for higher values of n is too tedious. The need to simplify this complex task gives rise to the equation

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where n is the n -th row of the Pascal's triangle, and r is the r -th term of the row, and the operation ' $!$ ' is factorial, ie

$$n! = n(n-1)(n-2)(n-3) \dots (3)(2)(1)$$

and by default, $0! = 1$.

It is worth noting that since Pascal's triangle is symmetrical, this also gives the property

$$\binom{n}{r} = \binom{n}{n-r}$$

Hence, the binomial expansion can now be generalised into

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}b^n$$

This is otherwise known as the **Binomial Theorem**

Worked Example 6.1

Find the x^3 term in the expansion $(2+5x)\left(1-\frac{1}{2}x\right)^8$.

[Solution]

We first want to find the expansion of $\left(1-\frac{1}{2}x\right)^8$ up to x^3 term

$$\begin{aligned}\left(1-\frac{1}{2}x\right)^8 &= 1^8 + \binom{8}{1}(1)^{8-1}\left(-\frac{1}{2}x\right)^1 + \binom{8}{2}(1)^{8-2}\left(-\frac{1}{2}x\right)^2 + \binom{8}{3}(1)^{8-3}\left(-\frac{1}{2}x\right)^3 + \dots \\ &= 1 + (8)(1)\left(-\frac{1}{2}x\right) + (28)(1)\left(\frac{1}{4}x^2\right) + (56)(1)\left(-\frac{1}{8}x^3\right) + \dots \\ &= 1 - 4x + 7x^2 - 7x^3 + \dots\end{aligned}$$

We can then multiply the x^3 and x^2 term with $(2+5x)$ to obtain the overall x^3 term.

$$\begin{aligned}\therefore (2+5x)\left(1-\frac{1}{2}x\right)^8 &= (2+5x)(1-4x+7x^2-7x^3+\dots) \\ &= \dots - 14x^3 + \dots + 35x^3 + \dots\end{aligned}$$

$\therefore$ Hence, the coefficient of x^3 in the overall expansion is 21.

Worked Example 6.2

- (a) Write down the first 3 terms, in ascending powers of x , in the expansion of

$$\left(2 - \frac{x}{3}\right)^5$$

- (b) Find the value of h , given that the first three terms in the following expansion are

$$(1 + hx + x^2)\left(2 - \frac{x}{3}\right)^5 = 32 - kx + 2kx^2 + \dots$$

[S4 CGSS P1/2011 Prelim Q4]

[Solution]

- (a) By using the Binomial Theorem,

$$\begin{aligned}\left(2 - \frac{x}{3}\right)^5 &= 2^5 + \binom{5}{1}(2)^{5-1}\left(-\frac{x}{3}\right)^1 + \binom{5}{2}(2)^{5-2}\left(-\frac{x}{3}\right)^2 + \dots \\ &= 32 - \frac{80}{3}x + \frac{80}{9}x^2 + \dots\end{aligned}$$

- (b) By observing the pattern from part (a),

$$\begin{aligned}(1 + hx + x^2)\left(32 - \frac{80}{3}x + \frac{80}{9}x^2 + \dots\right) \\ &= 32(1) + 32(hx) + 32(x^2) + \left(-\frac{80}{3}x\right)(1) + \left(-\frac{80}{3}x\right)(hx) + \left(\frac{80}{9}x^2\right)(1) + \dots \\ &= 32 + \left(32h - \frac{80}{3}\right)x + \left(32 - \frac{80}{3}h + \frac{80}{9}\right)x^2 + \dots\end{aligned}$$

By comparing coefficients,

$$32h - \frac{80}{3} = -k$$

$$80 - 96h = 3k \dots \dots (1)$$

$$32 - \frac{80}{3}h + \frac{80}{9} = 2k$$

$$-240h + 368 = 18k \dots \dots (2)$$

Substitute Equation (1)*6 into Equation (2),

$$-240h + 368 = 6(80 - 96h)$$

$$336h = 112$$

$$h = \frac{1}{3}$$

Tips!

Both 6.1 and 6.2 are very popular examination questions. Instead of expanding the full term, lookout for the terms that can give you the relevant terms you need for your answer.

Eg: If x^3 is needed, then find the term from the first bracket multiplied with the binomial expansion that can give you x^3 .

Worked Example 6.3

- (i) Find the first four terms of the expansion, in ascending powers of x , of

$$(3 + x)^5$$

- (ii) Using your answer in part (i),

- (a) find the coefficient of y^3 in the expansion of

$$(3 - y + y^2)^5,$$

- (b) estimate, leaving your answers in 3 decimal places, the value of

$$3.04^5,$$

and explain why the estimation is an accurate one.

[Solution]

- (i) By using Binomial Theorem,

$$\begin{aligned}(3 + x)^5 &= 3^5 + \binom{5}{1}(3)^{5-1}(x)^1 + \binom{5}{2}(3)^{5-2}(x)^2 + \binom{5}{3}(3)^{5-3}(x)^3 + \dots \\ &= 243 + 405x + 270x^2 + 90x^3 + \dots\end{aligned}$$

- (ii) By using part (i),

$$\begin{aligned}(3 - y + y^2)^5 &= [3 + (y^2 - y)]^5 \\ &= 243 + 405(y^2 - y) + 270(y^2 - y)^2 + 90(y^2 - y)^3 + \dots \\ &= 243 + 405(y^2 - y) + 270(y^4 - 2y^3 + y^2) + 90(y^2 - y)(y^4 - 2y^3 + y^2) + \dots \\ &= \dots + (270)(-2)y^3 + 90(-y)(y^2) + \dots \\ &= \dots - 630y^3 + \dots\end{aligned}$$

Hence, coefficient of y^3 is -630 .

- (iii) By letting $x = 0.04$, we have

$$\begin{aligned}3.04^5 &= 243 + 405(0.04) + 270(0.04)^2 + 90(0.04)^3 + \dots \\ &= 259.638 \text{ (3dp)}\end{aligned}$$

The estimation is accurate as for higher powers of n , the contribution of 0.04^n to the value of 3.04^5 becomes more and more insignificant.

General Term

The general term of a binomial can be stated as

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

The reason why it is the “ $r+1$ th” term is because the binomial coefficient of the first term, T_1 , is $\binom{n}{0}$.

This general term allows us to find the coefficient of any term of the binomial expansion, without having to write out all the terms. This is more efficient and time-saving as compared to the previous methods.

Worked Example 6.4

Find the x^5 term in the expansion of $\left(x^2 - \frac{2}{x}\right)^{10}$.

[Solution]

By finding the general $r + 1$ term,

$$\begin{aligned} T_{r+1} &= \binom{10}{r} (x^2)^{10-r} \left(-\frac{2}{x}\right)^r \\ &= \binom{100}{r} (-2)^r (x)^{20-2r} (x^{-r}) \\ &= \binom{100}{r} (-2)^r (x)^{20-3r} \end{aligned}$$

Since we are looking for the x^5 term, the power of x is 5.

$$20 - 3r = 5$$

$$3r = 15$$

$$r = 5$$

Coefficient of the x^5 term is $\binom{10}{5} (-2)^5 = -8064$

Term in x^5 of the expansion $\left(x^2 - \frac{2}{x}\right)^{10}$ is $-8064x^5$

Worked Example 6.5

In the given expansion, where $n > 0$, the first 3 terms in ascending powers of x are

$$\left(\frac{k}{x^3} + cx\right)^n = \frac{1}{x^{66}} + \frac{44}{x^{62}} + \frac{924}{x^{58}} + \dots$$

Find the values of k, n and c .

[S4 CCHY P2/2012 Prelim Q2]

[Solution]

Expanding the LHS of the equation,

$$\begin{aligned} & \left(\frac{k}{x^3} + cx\right)^n \\ &= \left(\frac{k}{x^3}\right)^n + \binom{n}{1} \left(\frac{k}{x^3}\right)^{n-1} (cx)^1 + \binom{n}{2} \left(\frac{k}{x^3}\right)^{n-2} (cx)^2 + \dots \\ &= \frac{k^n}{x^{3n}} + n \left(\frac{k^{n-1}}{x^{3n-3}}\right) (cx) + \frac{n(n-1)}{2} \left(\frac{k^{n-2}}{x^{3n-6}}\right) (c^2 x^2) + \dots \\ &= \frac{k^n}{x^{3n}} + \frac{nc k^{n-1}}{x^{3n-4}} + \frac{n(n-1)c^2 k^{n-2}}{2x^{3n-6}} + \dots \end{aligned}$$

Comparing coefficients,

$$x^{3n} = x^{66} \Rightarrow 3n = 66 \Rightarrow n = 22$$

$$k^{22} = 1 \Rightarrow k = 1$$

$$nck^{n-1} = 44 \Rightarrow (22)(c)(1) = 44$$

$$\therefore c = 2$$

Hence, the $k = 1, n = 22$ and $c = 2$.

Tips!

This is a challenging, but no doubt popular variant during exams. It is challenging as it requires the use of the following formula.

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

Whenever such questions arise, always go slow, and work out the terms carefully. If possible, memorise the first 3 sets as those are always the most popular.

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

Worked Example 6.6

In the expansion of

$$\left(x^2 - \frac{p}{2x}\right)^9,$$

where p is a positive constant, the term independent of x is 5376. Find the value of p .

[S4 CCHY P2/2012 Prelim Q2]

[Solution]

To find the value of p , we need to find the general $(r + 1)^{th}$ term

$$\begin{aligned}\text{General } (r + 1)^{th} \text{ term} &= \binom{9}{r} (x^2)^{9-r} \left(-\frac{p}{2x}\right)^r \\ &= \binom{9}{r} (x^{18-2r}) \left(-\frac{p}{2}\right)^r (x^{-r}) \\ &= \binom{9}{r} (x^{18-3r}) \left(-\frac{p}{2}\right)^r\end{aligned}$$

For the independent term of x , x^0 ,

$$x^0 = x^{18-3r}$$

$$18 - 3r = 0$$

$$r = 6$$

Given that the independent term of x is 5376

$$\therefore \binom{9}{6} \left(-\frac{p}{2}\right)^6 = 5376$$

$$84 \left(\frac{p^6}{64}\right) = 5376$$

$$p^6 = 4096$$

$$p = 4$$

7 Coordinate Geometry

This is a topic that we have covered in both lower secondary and E-Math. Here are some recap of the concepts we have learnt.

1. Properties of linear graphs

Term	Name	Definition
c	y -intercept	Represents the y -value where the line cuts the y -axis (when $x = 0$)
m	gradient	Represents the change in the y -value arising from a per unit change in x
$-\frac{c}{m}$	x -intercept	Represents the x -value where the line cuts the x -axis (when $y = 0$)

2. Length of a line segment

- Given two points $A(x_1, y_1)$ and $B(x_2, y_2)$, the distance AB is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3. Gradient of a straight line (m)

- The gradient of a can be found by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

m value	Behaviour
$m > 0$	Positive gradient, upwards sloping
$m < 0$	Negative gradient, downwards sloping
$m = 0$	Parallel to the x -axis, horizontal line
m undefined	Parallel to the y -axis, vertical line
$m_1 = m_2$	Two lines are parallel

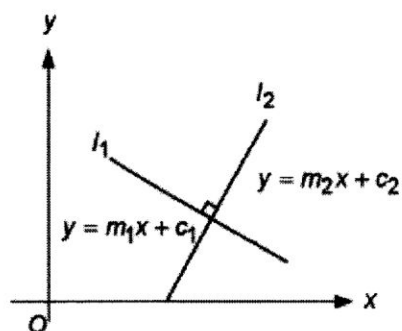
4. Equation of a straight line

- Slanted line: $y = mx + c$, where m is the gradient and c is the y -intercept
- Vertical line: $x = a$, for some value a
- Horizontal line: $y = b$, for some value b

New concepts for A-Math

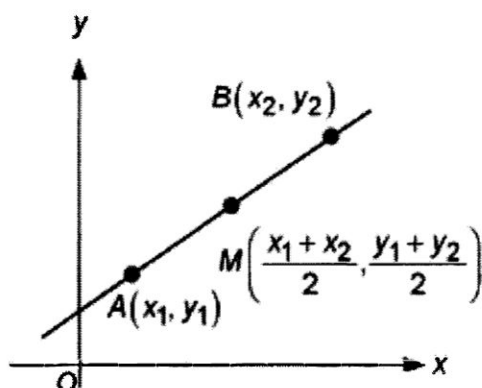
There are more concepts in A-Math for coordinate geometry. As these are self-explanatory, we won't go into too much detail into teaching them.

1. Gradient of perpendicular line



- If two lines are perpendicular, then the product of their gradients is -1 .

2. Coordinates of midpoint



- The coordinates of the midpoint $M(x, y)$, of two points $A(x_1, y_1)$ and $B(x_2, y_2)$, is given by finding the average of two points.
- This can be also found by the formula

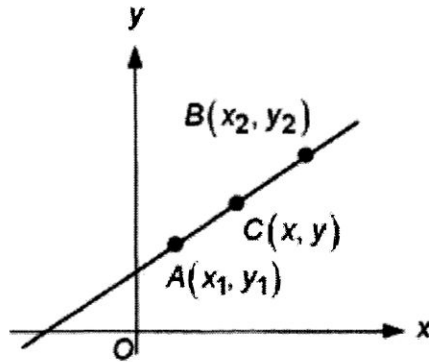
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

- **Tip!** The midpoint can be used to find the fourth vertex of a parallelogram since the two diagonals share the same midpoint.

3. Perpendicular bisector

- Combining points #1 and #2 above, we are able to get the gradient of a line's perpendicular bisector (using #1), as well as a point it passes through (#2).
- This allows us to find the equation of a perpendicular bisector.

4. Collinear points



- If three points, A, B and C are collinear, then
Gradient of AB = Gradient of BC = Gradient of AC

5. Area of rectilinear figure

- If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, ... and $M(x_m, y_m)$ are arranged in an anti-clockwise manner, the area of rectilinear figure can be found by

$$\begin{aligned} \text{Area} &= \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_m & x_1 \\ y_1 & y_2 & y_3 & \dots & y_m & y_1 \end{vmatrix} \\ &= \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_m & x_1 \\ y_1 & y_2 & y_3 & \dots & y_m & y_1 \end{vmatrix} \\ &= \frac{1}{2} [x_1y_2 + x_2y_3 + \dots + x_my_1] - [y_1x_2 + y_2x_3 + \dots + y_mx_1] \end{aligned}$$

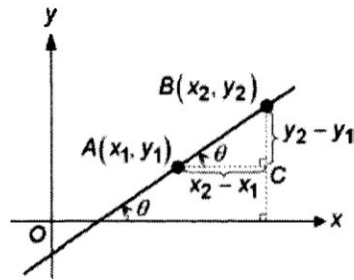
- The bars on the side of the formula are **not** modulus signs. They are a way of representing a concept called “determinants”.
- If the coordinates represented in the formula is arranged in a **clockwise** manner instead, the resultant area will be **negative**.

6. Angle that straight line makes with the x -axis

- Recall that the formula for finding gradient of a straight line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Note that in trigonometry, $y_2 - y_1$ is also the distance of the “opposite”, and $x_2 - x_1$ is also the distance of the “adjacent”, as shown in the diagram below.



- Hence, we have

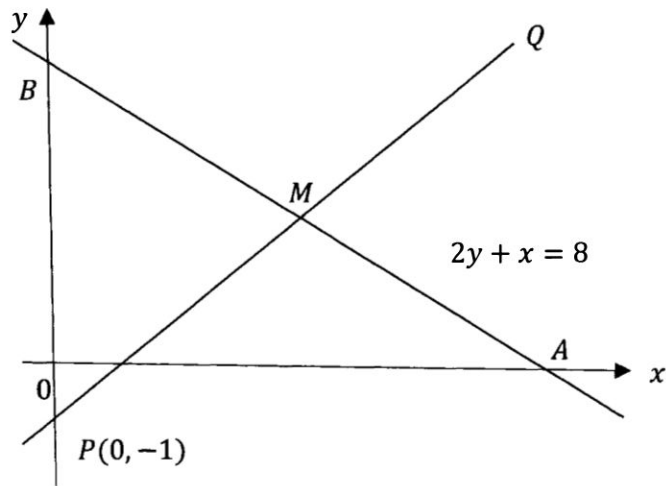
$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$$

where θ measures the angle from the positive x -direction to the line in an anti-clockwise direction.

Tip!

Typically for O-Levels, the concepts will all be tested simultaneously, in one question, in different parts. Sometimes each question is independent of one another, so don't be afraid to try each of them even if you are unable to complete the previous.

Worked Example 7.1



In the diagram shown above, the line $2y + x = 8$ cuts the x -axis at A and the y -axis at B . The point P has coordinates $(0, -1)$, and the point M lies on AB such that PM is perpendicular to AB . The line $2y + x = 8$ is the perpendicular bisector of PQ

- (a) Find the coordinates of A and of B .
- (b) Find the equation of PM .
- (c) Find the coordinates of M .
- (d) Show that the coordinates of Q are $(4, 7)$.
- (e) Find the area of the quadrilateral $APBQ$.

[Solution]

- (a) At A , $y = 0$

$$\therefore x = 8$$

$$\text{At } B, x = 0$$

$$\therefore y = 4$$

Hence, the coordinates are $A(8, 0)$ and $B(0, 4)$

- (b) $2y + x = 8$

$$y = -\frac{1}{2}x + 4$$

$$\text{Gradient of } AB = -\frac{1}{2}$$

Since AB is perpendicular to PM ,

$$\text{Gradient of } PM = \frac{-1}{(-\frac{1}{2})} = 2$$

$$\therefore y - (-1) = 2(x - (0))$$

Equation of line PM is $y = 2x - 1$

(c) $y = -\frac{1}{2}x + 4 \dots\dots (1)$

$y = 2x - 1 \dots\dots (2)$

Substitute Equation (2) into Equation (1)

$$-\frac{1}{2}x + 4 = 2x - 1$$

$$-x + 8 = -4x - 2$$

$$-5x = -10$$

$$x = 2$$

Substitute $x = 2$ into Equation (2)

$$y = 2(2) - 1 = 3$$

$$\therefore M(2,3)$$

(d) Let (x, y) be the coordinate of point Q . Since M is the mid-point,

$$\left(\frac{0+x}{2}, \frac{-1+y}{2}\right) = (2,3)$$

$$x = 4, y = 7$$

Hence, $Q(4,7)$

(e) To find the area, we use the shoelace method

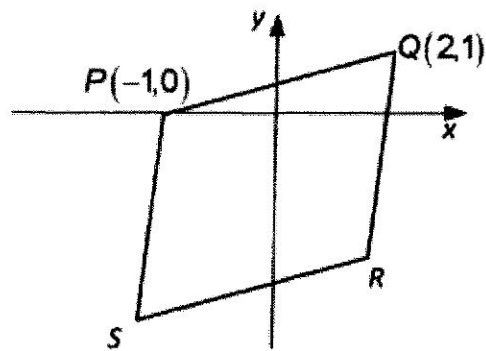
$$\text{Area} = \frac{1}{2} \begin{vmatrix} 4 & 0 & 0 & 8 & 4 \\ 7 & 4 & -1 & 0 & 7 \end{vmatrix}$$

$$= \frac{1}{2} |(16 + 56) - (-8)|$$

$$= \frac{1}{2} |80|$$

$$= 40 \text{ units}^2$$

Worked Example 7.2



$PQRS$ is a rhombus where $P(-1,0)$ and $Q(2,1)$

Given that SQ is parallel to the line $y = x + 3$ and centre of the rhombus lies on the y -axis.

Find,

- the equation of PR ,
- the coordinates of R ,
- the equation of SR ,
- the area of the rhombus $PQRS$.

[Solution]

- (a) Gradient of $SQ = 1$

$$\therefore \text{Gradient of } PR = -\frac{1}{1} = -1$$

$$y - (0) = -1(x - (-1))$$

$$y = -x - 1$$

- (b) Let R be (x_1, y_1)

$$\text{Length of } PQ = \sqrt{(-1 - 2)^2 + (0 - 1)^2} = \sqrt{10}$$

$$\text{Length of } PR = \sqrt{(x_1 - 2)^2 + (y_1 - 1)^2}$$

$$\sqrt{10} = \sqrt{(x_1 - 2)^2 + (y_1 - 1)^2}$$

$$(x_1)^2 - 4(x_1) + 4 + (y_1)^2 - 2(y_1) + 1 = 10$$

$$(x_1)^2 - 4(x_1) + (y_1)^2 - 2(y_1) = 5 \dots \dots (1)$$

Substitute the equation of line PR into Equation (1)

$$(x_1)^2 - 4(x_1) + (-x_1 - 1)^2 - 2(-x_1 - 1) = 5$$

$$(x_1)^2 - 4(x_1) + (x_1)^2 + 2(x_1) + 1 + 2(x_1) + 2 = 5$$

$$2(x_1)^2 - 2 = 0$$

$$(x_1)^2 - 1 = 0$$

$$x_1 = 1 \quad \text{or} \quad x_1 = -1 \text{ (N. A.)}$$

Substitute $x_1 = 1$ into Equation (2)

$$y_1 = -(1) - 1$$

$$= -2$$

$$\therefore R(1, -2)$$

(c) Since $PQRS$ is a rhombus, the sides are parallel to one another.

$$\text{Gradient of } PQ = \frac{1-0}{2-(-1)} = \frac{1}{3}$$

$$\therefore \text{Gradient of } SR = \frac{1}{3}$$

$$\therefore y - (-2) = \frac{1}{3}(x - 1)$$

$$y + 2 = \frac{1}{3}x - \frac{1}{3}$$

$$3y + 6 = x - 1$$

$$3y = x - 7$$

(d) By shoelace method

Area of $PQRS$

$$= 2(\text{Area of } PQR)$$

$$= 2 \left(\frac{1}{2} \begin{vmatrix} -1 & 1 & 2 & -1 \\ 0 & -2 & 1 & 0 \end{vmatrix} \right)$$

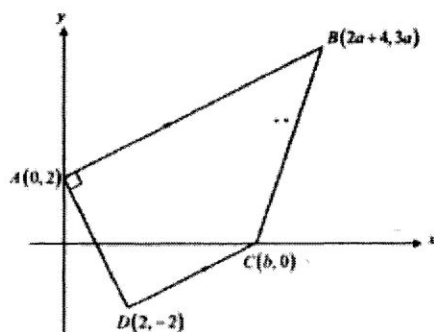
$$= |(2 + 1 + 0) - (0 + (-4) + (-1))|$$

$$= 3 + 5$$

$$= 8 \text{ units}^2$$

Worked Example 7.3

Solutions to this question by accurate drawing will not be accepted.



The diagram, not drawn to scale, shows a trapezium $ABCD$ in which AB is parallel to DC and $\angle BAD = 90^\circ$. The vertices of the trapezium are the points $A(0, 2)$, $B(2a + 4, 3a)$, $C(b, 0)$ and $D(2, -2)$

- Given that the length of AB is $4\sqrt{5}$ units, find the value of a , where $a > 0$
- Find the equation of AB
- Find the value of b
- Find the perpendicular bisector of AB
- Hence, or otherwise, show that C lies on perpendicular bisector of AB
- Find the area of the trapezium $ABCD$

[S4 NGHS P2/2015 PRELIM Qn 9]

[Solution]

- Since the length is $4\sqrt{5}$ units,

$$\sqrt{(2a + 4 - 0)^2 + (3a - 2)^2} = 4\sqrt{5}$$

$$4a^2 + 16a + 16 + 9a^2 - 12a + 4 = 16(5)$$

$$13a^2 + 4a - 60 = 0$$

$$(13a + 30)(a - 2) = 0$$

$$a = -\frac{30}{13} (\text{rej} \because a > 0) \quad \text{or} \quad a = 2$$

- To find the equation of AB ,

$$\text{Gradient}_{AB} = \frac{6-2}{8-0}$$

$$= \frac{1}{2}$$

$$\therefore y = \frac{1}{2}x + 2$$

(c) Since AB is parallel to DC , AB have the same gradient as DC .

$$\frac{0-(-2)}{b-2} = \frac{1}{2}$$

$$b - 2 = 4$$

$$b = 6$$

(d) To find the perpendicular bisector of AB , we need to find the midpoint of AB .

$$\text{Midpoint} = \left(\frac{8+0}{2}, \frac{6+2}{2} \right) = (4, 4)$$

$$\text{Gradient of perpendicular bisector} = \frac{-1}{\left(\frac{1}{2}\right)} = -2$$

$$\therefore y - 4 = -2(x - 4)$$

$$y = -2x + 12$$

(e) When $x = 6$,

$$y = -2(6) + 12 = 0$$

As the point satisfies the equation, point C lies on the perpendicular bisector.

(f) To find the area of trapezium $ABCD$,

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 0 & 2 & 6 & 8 & 0 \\ 2 & -2 & 0 & 6 & 2 \end{vmatrix}$$

$$= \frac{1}{2} |(16 + 36) - (4 - 12)|$$

$$= \frac{1}{2} (60)$$

$$= 30 \text{ units}^2$$

Linear Law

The concept of linear law is about converting non-linear equations into a linear form, allowing us to use the concept of “gradient” and “y-intercept” to solve for unknown values. In general, we will use the equation $Y = mX + c$, where Y and X are functions of y and x .

Non-Linear Equation	Transformation
$y = ax + bx^2$	$\frac{y}{x} = bx + a$ $Y = \frac{y}{x}, \quad X = x^2$ Gradient: $m = b$ Y-intercept: $c = a$
$y = \frac{a}{x} - b$	$y = a\left(\frac{1}{x}\right) - b$ $Y = y, \quad X = \frac{1}{x}$ Gradient: $m = a$ Y-intercept: $c = -b$ (alternative: $yx = -bx + a$)
$y = a\sqrt{x} + \frac{b}{\sqrt{x}}$	$y\sqrt{x} = ax + b$ $Y = y\sqrt{x}, \quad X = x$ Gradient: $m = a$ Y-intercept: $c = b$ (alternative: $\frac{y}{\sqrt{x}} = \frac{b}{x} + a$)
$xy = \frac{a}{x} - bx$	$y = \frac{a}{x^2} - b$ $Y = y, \quad X = \frac{1}{x^2}$ Gradient: $m = a$ Y-intercept: $c = -b$ (alternative: $yx^2 = -bx^2 + a$)
$y = ab^x$	$\lg y = x \lg b + \lg a$ $Y = \lg y, \quad X = x$ Gradient: $m = \lg b$ Y-intercept: $c = \lg a$

Worked Example 7.4

Variables x and y are related by the equation $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$, where a and b are constants to be determined. The table below shows measured values of x and y

x	1	2	3	4	5	6
y	5.7	5.6	5.9	6.2	6.6	6.9

- (a) On a piece of graph paper, plot $y\sqrt{x}$ against x , using a scale of 2cm to represent 1 unit on the $y\sqrt{x}$ axis. Draw a straight-line graph to represent the equation $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$.
- (b) Use your graph to estimate the value of a and b .
- (c) On the same diagram, draw the line representing the equation $y = \frac{3x}{\sqrt{x}}$ and hence find the value for which $x = \frac{b}{3-a}$. [S4 SCGS P1/2012 PRELIM Qn 5]

[Solution]

(a) $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$

Multiplying by $\sqrt{x}$ throughout: $y\sqrt{x} = ax + b \dots\dots (1)$

x	1	2	3	4	5	6
y	5.7	5.6	5.9	6.2	6.6	6.9
$y\sqrt{x}$	5.7	7.9	10.2	12.4	14.8	16.9

$Y = mX + C$ where a is the gradient, and b is the y -intercept

Graph is drawn on the next page

- (b) To find a , we need to find the gradient of the line

$$a = \frac{14.8-5.7}{5-1}$$

$$= 2.275 \dots$$

$$= 2.28(3.s.f)$$

To find b , we can read the y -intercept off the graph

$$b = 3.3$$

(c) $y = \frac{3x}{\sqrt{x}}$

$$y\sqrt{x} = 3x \dots\dots (2)$$

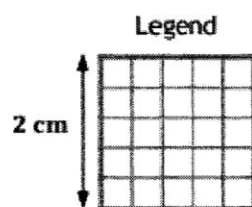
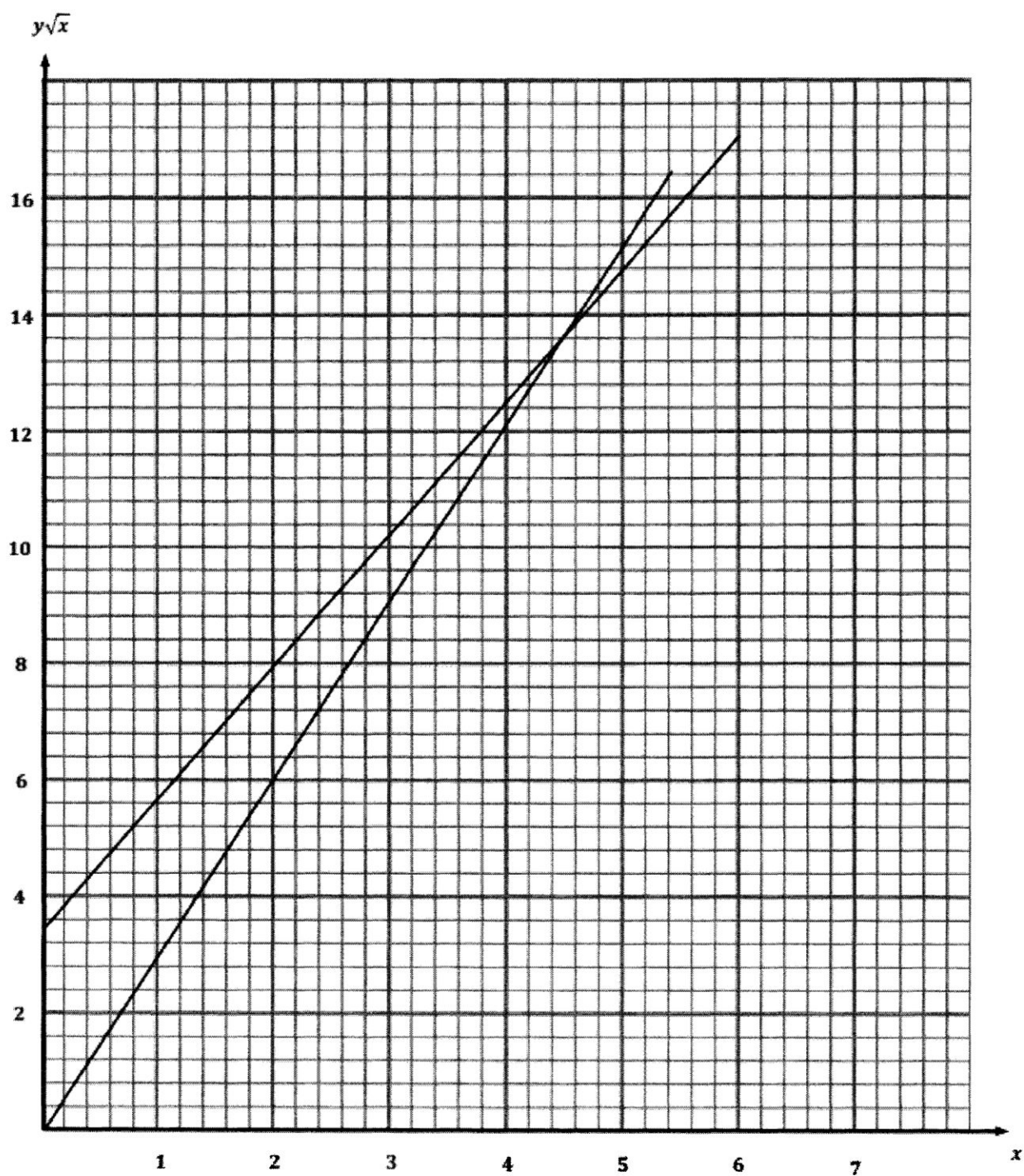
Graph is drawn on the next page

$$x = \frac{b}{3-a} \Rightarrow 3x - ax = b \Rightarrow 3x = ax + b$$

We can obtain $3x = ax + b$ by equating Equation (1) and Equation (2). Hence, to find the value of the following, we are looking for the intersection point between the 2 lines.

$\therefore$ From the graph, $x = 4.5$

Graphical Solution for Linear Law Question



Worked Example 7.5

Answer the whole of this question on a sheet of graph paper.

Variables x and y are related by the equation $y = ab^x$, where a and b are constants to be determined. The table shows experimental value of x and y .

x	1	2	3	4
y	28	200	1590	7950

- Using a scale of 4 cm to represent one unit on both axes, plot $\lg y$ against x to obtain a straight line.
- Use your graph to estimate the value of a and b .
- From the graph, find the value of x and $y = 10^{2.75}$.

[Solution]

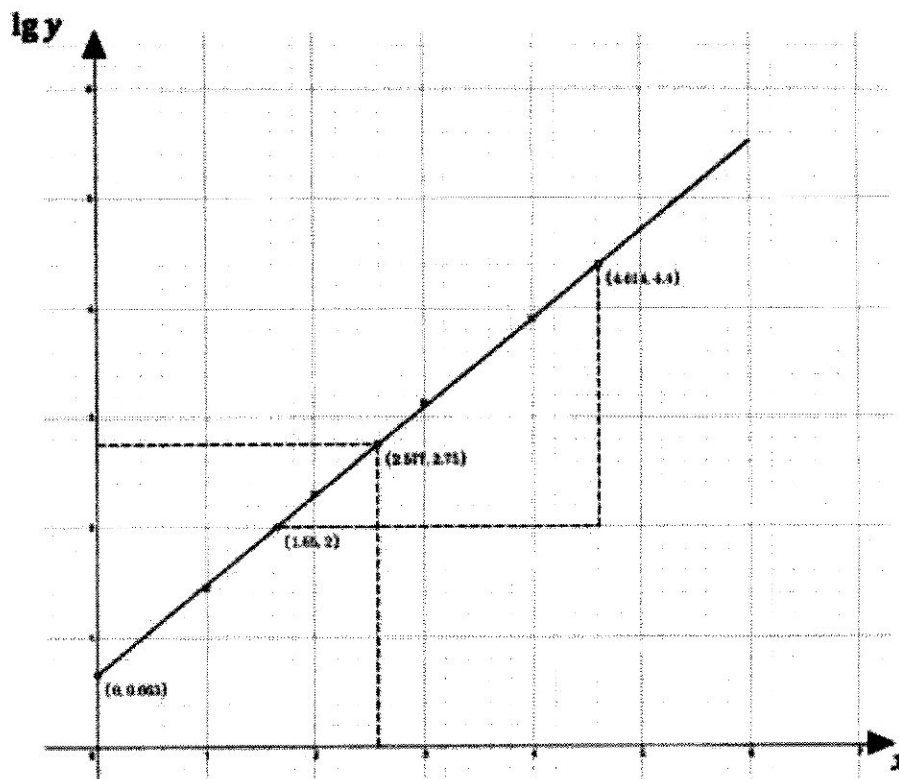
(a) $y = ab^x$

$$\lg y = \lg(ab^x)$$

$$\lg y = x \lg b + \lg a$$

$$Y = X(m) + C$$

x	1	2	3	4
y	28	200	1590	7950
$\lg y$	1.45	2.30	3.20	3.90



(b) To find the value of a , we need to find the y -intercept of the graph

$$\lg a = 0.663$$

$$a = 10^{0.663}$$

$$= 4.60256 \dots$$

$$= 4.60(3\text{s.f})$$

To find the value of b , we need to find the gradient of the graph

$$\lg b = \frac{4.4-2}{4.614-1.65}$$

$$= \frac{200}{247}$$

$$b = 10^{\frac{200}{247}}$$

$$= 6.45233 \dots$$

$$= 6.45 (3\text{s.f})$$

(c) When $y = 10^{2.75}$, $\lg y = 2.75$

$$\therefore x = 2.58$$

Equation of Circles

We can express the equation of circles in two forms.

	Standard	General
Equation	$(x - h)^2 + (y - k)^2 = r^2$	$x^2 + y^2 + 2gx + 2fy + c = 0$
Centre	(h, k)	$(-g, -f)$
Radius	r	$\sqrt{g^2 + f^2 - c}$

We can obtain the standard form of the equation of circle by completing the square from the general form:

$$\begin{aligned}x^2 + y^2 + 2gx + 2fy + c &= x^2 + 2gx + y^2 + 2fy + c \\&= x^2 + 2gx + g^2 + y^2 + 2fy + f^2 + c - f^2 - g^2 \\&= (x + g)^2 + (y + f)^2 + c - f^2 - g^2\end{aligned}$$

Hence, the general equation of circle can be algebraically manipulated into

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c$$

which gives us the centre and radius of the circle as shown in the table above.

Position of point relative to circle

To test/determine if a point is inside, on or outside of a circle, we can use the compare the length of the radius with the length between the point and the centre of the circle:

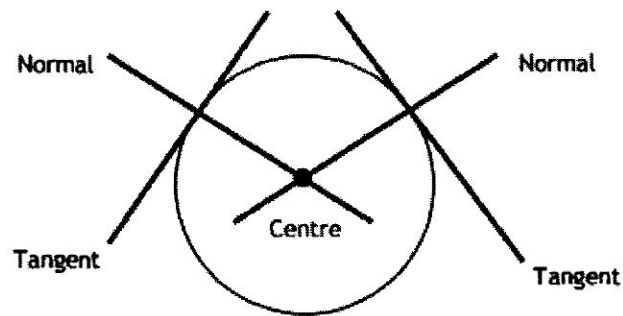
Test	Result
Inside the circle	Radius > Length
On the circle	Radius = Length
Outside the circle	Radius < Length

Note: Another test to determine if a point is on the circle is to substitute the point into the equation of the circle. If the equation holds, the point is on the circle.

Note

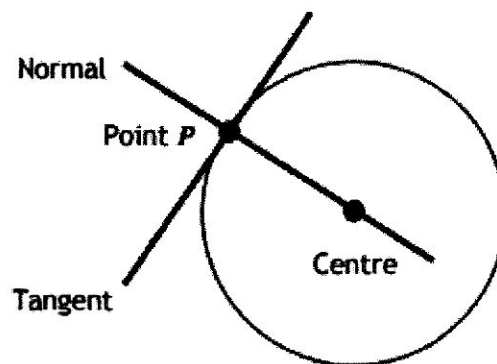
The concept of **normal and perpendicular bisectors** is usually tested in this chapter. Usually the equation of tangent, and the coordinate it touches the circle will be given, allowing us to find the equation of normal easily.

- The normal passes through the centre of the circle.
- Any two normals of a circle intersect at the centre of the circle.



Tip

How to find equation of tangent?



1. Find the gradient of the line connecting the point in question and the centre of the circle. This line corresponds to the normal of the circle at the point.
2. Find the gradient of the normal, and hence the gradient of the tangent.
(Recall the relationship between the two gradients?)
3. Substitute the gradient of tangent and coordinate of point P into $y = mx + c$ to find the equation of tangent.

Worked Example 7.6

The line $y + 2x = 5$ cuts the circle with equation $x^2 + y^2 - 4x - 6y = 12$ at points A and B .

- (a) Find the coordinates of the two points
- (b) Given that $p > 0$ and $D(3, p)$ is a point on the circle, find the value of p . Leave your answer in the form $a + b\sqrt{c}$.
- (c) Show that AB is not the diameter of the circle

[S4 DSS P1/2011 PRELIM Qn 4]

[Solution]

(a) $y + 2x = 5$

$$y = 5 - 2x \dots \dots (1)$$

$$x^2 + y^2 - 4x - 6y = 12 \dots \dots (2)$$

Substitute Equation (1) into Equation (2)

$$x^2 + (5 - 2x)^2 - 4x - 6(5 - 2x) = 12$$

$$x^2 + 25 - 20x + 4x^2 - 4x - 30 + 12x = 12$$

$$5x^2 - 12x - 17 = 0$$

$$(5x - 17)(x + 1) = 0$$

$$x = \frac{17}{5} \quad \text{or} \quad x = -1$$

Substitute $x = \frac{17}{5}$ into (1)

Substitute $x = -1$ into (1)

$$y = 5 - 2\left(\frac{17}{5}\right)$$

$$y = 5 - 2(-1)$$

$$= -\frac{9}{5}$$

$$= 7$$

$$\therefore A\left(\frac{17}{5}, -\frac{9}{5}\right) \quad B(-1, 7) \quad [\text{or vice versa as it is not specific}]$$

(b) Substitute the point $(3, p)$ into Equation (2)

$$(3)^2 + (p)^2 - 4(3) - 6(p) = 12$$

$$9 + p^2 - 12 - 6p - 12 = 0$$

$$p^2 - 6p - 15 = 0$$

$$p = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-15)}}{2(P)}$$

$$= \frac{6 \pm 4\sqrt{4}}{2}$$

$$= 3 \pm \sqrt{2}$$

$$\text{Since } p > 0, \quad p = 3 + \sqrt{2}$$

- (c) To show that AB is not the diameter of the circle, we need to show that the length of AB is either shorter or longer than the diameter of the circle

$$\text{Length of } AB = \sqrt{\left(\frac{17}{5} - (-1)\right)^2 + \left(-\frac{9}{5} - 7\right)^2} = \sqrt{\frac{484}{5}} = 9.83869 \dots$$

$$x^2 + y^2 - 4x - 6y = 12$$

$$x^2 + y^2 + 2(-2)x + 2(-3)y + (-12) = 0$$

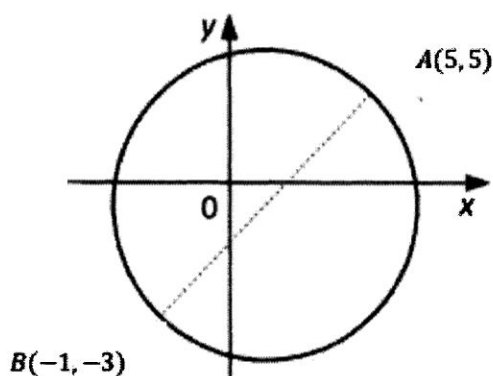
$$g = -2 \quad f = -3 \quad c = -12$$

$$\begin{aligned} \text{Radius of the circle} &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{(-2)^2 + (-3)^2 - (-12)} \\ &= \sqrt{25} \\ &= 5 \text{ units} \end{aligned}$$

$$\text{Diameter of the circle} = 5 \times 2 = 10 \text{ units}$$

Since the diameter of the circle is longer than the length of AB , AB is not the diameter.

Worked Example 7.7



In the diagram above, the points $A(5,5)$ and $B(-1,-3)$ are opposite ends of a diameter of a circle

- (a) Find the equation of the circle in its general form.
- (b) If $C(6,k)$ lies on the circle, find the value of k , where $k < 0$.
- (c) Hence, prove that $\triangle ABC$ is an isosceles right-angled triangle.

[Solution]

- (a) To find the equation of the circle, we need to find the radius and the centre of the circle. The centre of the circle, it corresponds to the midpoint of the line AB

$$\text{Centre of circle} = \text{Midpoint of } AB = \left(\frac{-1+5}{2}, \frac{-3+5}{2} \right) = (2,1)$$

$$\text{Radius of circle} = \frac{\sqrt{(5-(-1))^2 + (5-(-3))^2}}{2} = 5 \text{ units}$$

$$\therefore (x-2)^2 + (y-1)^2 = 5^2$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 25$$

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

- (b) Since $C(6,k)$ lies on the circle,

$$(6)^2 + k^2 - 4(6) - 2k - 20 = 0$$

$$k^2 - 2k - 8 = 0$$

$$(k+2)(k-4) = 0$$

$$k = -2 \quad \text{or} \quad k = 4 \text{ (rej } \because k < 0)$$

(c) To prove it is an isosceles triangle, we need to find 2 lengths of equal length.

Length of $AB = 10$ units

Length of $AC = \sqrt{(5-6)^2 + (5-(-2))^2} = \sqrt{50}$ units

Length of $AB = \sqrt{(-1-6)^2 + (-3-(-2))^2} = \sqrt{50}$ units

$\therefore \triangle ABC$ is an isosceles triangle

To prove a right-angled triangle, we need to show that the product of the gradients of 2 lines of the triangle is -1

Gradient of $AC = \frac{5-6}{5-(-2)} = -\frac{1}{7}$

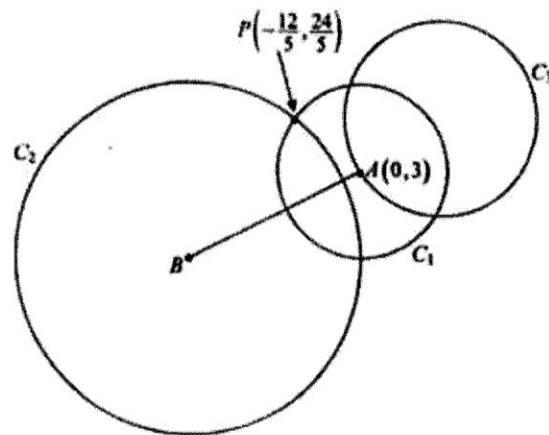
Gradient of $AB = \frac{-1-6}{-3-(-2)} = 7$

Gradient of $AC \times$ Gradient of $AB = -\frac{1}{7} \times 7 = -1$

$\therefore \triangle ABC$ is a right-angled triangle

$\therefore \triangle ABC$ is an isosceles right-angles triangle

Worked Example 7.8



Circles C_1 , C_2 and C_3 are shown in the diagram above.

C_1 has centre $A(0,3)$ and a radius of 3 units.

- (i) Find the equation of circle C_1 .

C_2 has centre B has the equation $(x+a)^2 = 36 - y^2$, where $a > 0$. It is given that C_2 intersects C_1 at the point $P(-\frac{12}{5}, \frac{24}{5})$. Find,

- (ii) the value of a ,
 (iii) the length of AB .

The equation of C_3 may be written in the form $x^2 + y^2 - 8x + ky + 27 = 0$.

Given that C_3 passes through the point $A(0,3)$, find,

- (iv) the value of k ,
 (v) the coordinates of the centre and radius of C_3

[S4 DHS P1/2011 PRELIM Qn 10]

Solution

- (i) Since the centre is $A(0,3)$ and radius 3 units,

$$(x-0)^2 + (y-3)^2 = 3^2$$

$$x^2 + (y-3)^2 = 9$$

- (ii) To find the value of a , substitute P into C_2

$$\left(-\frac{12}{5} + a\right)^2 = 36 - \left(\frac{24}{5}\right)^2$$

$$\frac{144}{25} - \frac{24}{5}a + a^2 - \frac{324}{25} = 0$$

$$a^2 - \frac{24}{5}a - \frac{36}{5} = 0$$

$$5a^2 - 24a - 36 = 0$$

$$(5a + 6)(a - 6) = 0$$

$$a = -\frac{6}{5} \text{ (rej } \because a > 0) \quad \text{or} \quad a = 6$$

(iii) Based on the equation, the centre of C_2 is $(-6,0)$

$$\text{Length of } AB = \sqrt{(-6 - 0)^2 + (0 - 3)^2} = 3\sqrt{5} \text{ units}$$

(iv) Since $A(0,3)$ cuts C_3 , substitute $A(0,3)$ into the equation of C_3

$$(0)^2 + (3)^2 - 8(0) + k(3) + 27 = 0$$

$$3k = -36$$

$$k = -12$$

(v) Hence, the equation of C_3 is $x^2 + y^2 - 8x - 12y + 27 = 0$

$$\text{Centre} = \left(-\frac{(-8)}{2}, \frac{(-12)}{2} \right) = (4,6)$$

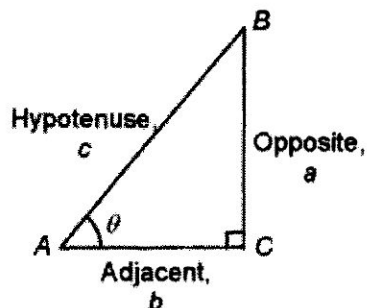
$$\text{Radius} = \sqrt{(4)^2 + (6)^2 - 27} = 5 \text{ units}$$

8 Trigonometry

This topic would be one of the biggest topics in A-Math. However, don't be fazed. Let's try to combine different concepts together so we are not caught by surprised in exams!

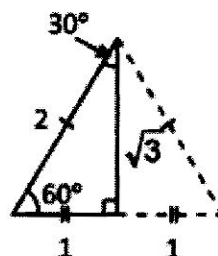
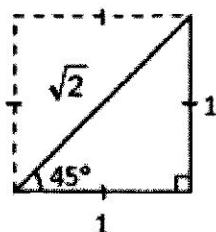
Trigo Basics

1. Trigo Ratios of Acute Angles (TOA CAH SOH)



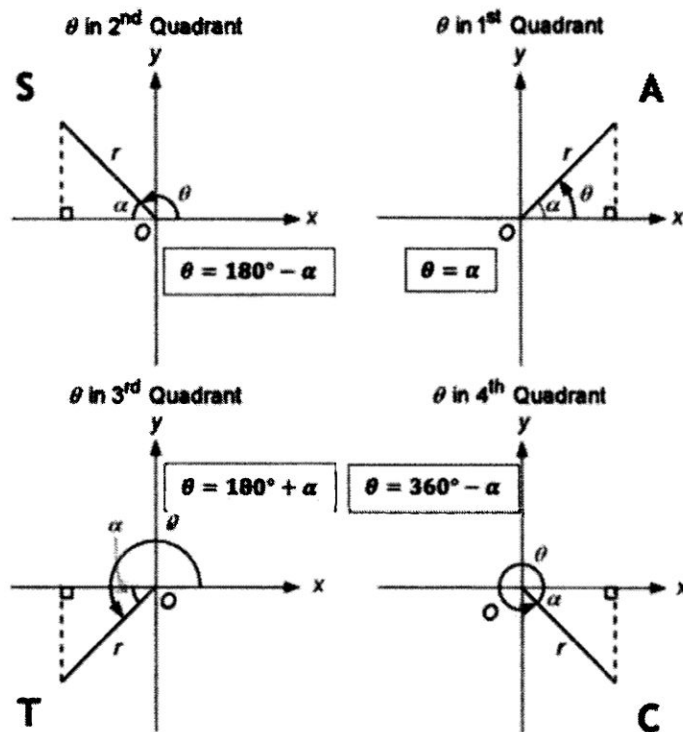
Ratio	Formula
sine	$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$
cosine	$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$
tangent	$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$

2. Special Angles



θ	0° 0 rad	30° $\frac{1}{6}\pi$ rad	45° $\frac{1}{4}\pi$ rad	60° $\frac{1}{3}\pi$ rad	90° $\frac{1}{2}\pi$ rad
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

3. Four Quadrants (ASTC)



Quad	1 st	2 nd	3 rd	4 th
opposite	+	+	-	-
adjacent	+	-	-	+
hypotenuse	+	+	+	+
θ	$0^\circ \leq \theta \leq 90^\circ$	$90^\circ \leq \theta \leq 180^\circ$	$180^\circ \leq \theta \leq 270^\circ$	$270^\circ \leq \theta \leq 360^\circ$
$\sin \theta$	$\sin \theta$	$\sin \theta$	$-\sin \theta$	$-\sin \theta$
$\cos \theta$	$\cos \theta$	$-\cos \theta$	$-\cos \theta$	$\cos \theta$
$\tan \theta$	$\tan \theta$	$-\tan \theta$	$\tan \theta$	$-\tan \theta$

4. Reciprocal Ratios

Ratio	Formula
secant	$\sec \theta = \frac{1}{\cos \theta}$
cosecant	$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$
cotangent	$\cot \theta = \frac{1}{\tan \theta}$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

$$\cot \theta = \frac{1}{\tan \theta}$$

5. Trigonometric Identities

Identity	Formula
Quotient	$\tan A = \frac{\sin A}{\cos A}$
	$\cot A = \frac{\cos A}{\sin A}$
Reciprocal	$\sec A = \frac{1}{\cos A}$
	$\operatorname{cosec} A = \frac{1}{\sin A}$
	$\cot A = \frac{1}{\tan A}$
Pythagorean	$\sin^2 A + \cos^2 A = 1^*$
	$\sec^2 A = 1 + \tan^2 A^*$
	$\operatorname{cosec}^2 A = 1 + \cot^2 A^*$
Addition Formula	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B^*$
	$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B^*$
	$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
Double Angle Formula (gives us Half Angle Formula)	$\sin 2A = 2 \sin A \cos A^*$
	$\cos 2A = \cos^2 A - \sin^2 A^*$ $= 2 \cos^2 A - 1^*$ $= 1 - 2 \sin^2 A^*$
	$\tan A = \frac{2 \tan A}{1 - \tan^2 A}^*$

Worked Example 8.1.1.

Given that x and y are acute angles, and that $\cos x = \frac{4}{5}$ and $\frac{\cos x}{\tan y} = \frac{4\sqrt{3}}{5}$, find in exact form,

(a) $\sin 2x$

(b) $\cos^2\left(\frac{y}{2}\right)$

[S4 VS P2/2011 PRELIM Qn 7(a)]

[Solution]

(a) Since x is an acute angle, x is in the first quadrant

$$\sin 2x = 2 \sin x \cos x$$

$$= 2 \left(\frac{3}{5}\right) \left(\frac{4}{5}\right)$$

$$= \frac{24}{25}$$

(b) Since y is an acute angle, y is in the first quadrant

$$\frac{\cos x}{\tan y} = \frac{4\sqrt{3}}{5}$$

$$\tan y = \frac{5}{4\sqrt{3}} \times \cos x$$

$$= \frac{1}{\sqrt{3}}$$

Since $\cos 2A = 2 \cos^2 A - 1$,

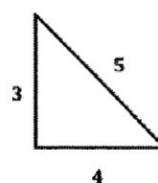
$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\cos^2\left(\frac{y}{2}\right) = \frac{1 + \cos y}{2}$$

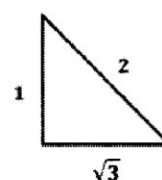
$$= \frac{1 + \frac{\sqrt{3}}{2}}{2}$$

$$= \frac{2 + \sqrt{3}}{4}$$

	$\sin x$	$\cos x$	$\tan x$
x	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$



	$\sin y$	$\cos y$	$\tan y$
y	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$



Worked Example 8.1.2

It is given that $\sin A = \frac{9}{41}$ and $\cos B = -\frac{20}{29}$, where both angles A and B lie in the same quadrant. Find, without calculating the values of A and B , the exact value of

(a) $\tan(A - B)$

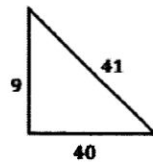
(b) $\sec\left(\frac{\pi}{2} - A\right)$

[S4 NBSS P2/2012 PRELIM Qn 4]

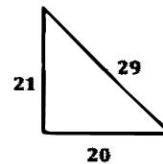
Solution

Since $\sin A > 0$ and $\cos B < 0$, we can conclude A and B lie in the second quadrant.

	$\sin A$	$\cos A$	$\tan A$
A	$\frac{9}{41}$	$-\frac{40}{41}$	$-\frac{9}{40}$



	$\sin B$	$\cos B$	$\tan B$
B	$\frac{21}{29}$	$-\frac{20}{29}$	$-\frac{21}{20}$



(a) To find $\tan(A - B)$, we can use the addition formula

$$\begin{aligned}\therefore \tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\&= \frac{\left(-\frac{9}{40}\right) - \left(-\frac{21}{20}\right)}{1 + \left(-\frac{9}{40}\right)\left(-\frac{21}{20}\right)} \\&= \frac{660}{989}\end{aligned}$$

$$\begin{aligned}\text{(b) } \therefore \sec\left(\frac{\pi}{2} - A\right) &= \frac{1}{\cos\left(\frac{\pi}{2} - A\right)} \\&= \frac{1}{\sin A} \\&= \frac{1}{\left(\frac{9}{41}\right)} \\&= \frac{41}{9}\end{aligned}$$

Worked Example 8.1.3

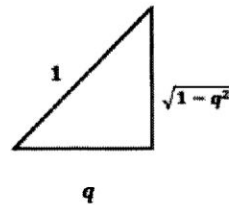
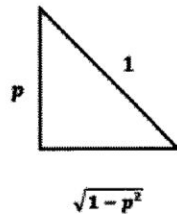
It is given that $\sin A = p$ and $\cos B = q$, where A and B are obtuse and acute angles respectively. Find, in terms of p and q ,

- (a) $\tan 2A$
 (b) $\cos(A - B)$
 (c) $\sin\left(\frac{1}{2}B\right)$

[Solution]

Since A is an obtuse angle, A lies in the second quadrant.

Since B is an acute angle, B lies the first quadrant



	$\sin A$	$\cos A$	$\tan A$
A	$\frac{p}{1}$	$-\frac{\sqrt{1-p^2}}{1}$	$-\frac{p}{\sqrt{1-p^2}}$

	$\sin B$	$\cos B$	$\tan B$
B	$\frac{\sqrt{1-q^2}}{1}$	$\frac{q}{1}$	$\frac{\sqrt{1-q^2}}{q}$

- (a) Using the ratios above,

$$\begin{aligned}
 \tan 2A &= \frac{2 \tan A}{1 - \tan^2 A} \\
 &= \frac{2 \left(-\frac{p}{\sqrt{1-p^2}} \right)}{1 - \left(-\frac{p}{\sqrt{1-p^2}} \right)^2} \\
 &= \frac{-\frac{2p}{\sqrt{1-p^2}}}{\left(1 - \frac{p^2}{1-p^2} \right)} \\
 &= \left(-\frac{2p}{\sqrt{1-p^2}} \right) \div \frac{1-p^2-p^2}{1-p^2} \\
 &= \left(-\frac{2p}{\sqrt{1-p^2}} \right) \left(\frac{1-p^2}{1-2p^2} \right) \\
 &= -\frac{2p\sqrt{1-p^2}}{1-2p^2}
 \end{aligned}$$

- (b) Using the ratios above,

$$\begin{aligned}
 \cos(A - B) &= \cos A \cos B + \sin A \sin B \\
 &= \left(-\frac{\sqrt{1-p^2}}{1} \right) \left(\frac{q}{1} \right) + \left(\frac{p}{1} \right) \left(\frac{\sqrt{1-q^2}}{1} \right) \\
 &= -q\sqrt{1-p^2} + p\sqrt{1-q^2}
 \end{aligned}$$

- (c) $\cos 2B = 1 - 2 \sin^2 B$

Using the ratios above,

$$\begin{aligned}
 \sin\left(\frac{1}{2}B\right) &= \sqrt{\frac{1 - \cos B}{2}} \\
 &= \sqrt{\frac{1-q}{2}}
 \end{aligned}$$

Common Mistakes

$$\tan 2A \neq 2 \tan A$$

$$\sin(A + B) \neq \sin A + \sin B$$

Always remember to use the formulas you learnt, do not create your own 😊

Worked Example 8.2

Without using a calculator, find the exact value of

$$\frac{\sin 105^\circ}{\sec 105^\circ}$$

[S4 SCGS P1/2012 PRELIM Qn 8(a)]

Solution

Using special angles,

$$\begin{aligned}\frac{\sin 105^\circ}{\sec 105^\circ} &= \frac{\sin 105^\circ}{\left(\frac{1}{\cos 105^\circ}\right)} \\&= (\sin 105^\circ)(\cos 105^\circ) \\&= [\sin(60^\circ + 45^\circ)][\cos(60^\circ + 45^\circ)] \\&= [\sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ][\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ] \\&= \left[\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\right]\left[\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\right] \\&= -\frac{1}{4}\end{aligned}$$

Worked Example 8.3.1

Prove that

$$\frac{\sin x}{\operatorname{cosec} x - \cot x} = 1 + \cos x$$

[Solution]

LHS

$$\begin{aligned} &= \frac{\sin x}{\operatorname{cosec} x - \cot x} \\ &= \frac{\sin x}{\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)} \\ &= (\sin x) \left(\frac{\sin x}{1 - \cos x}\right) \\ &= \frac{\sin^2 x}{1 - \cos x} \\ &= \frac{1 - \cos^2 x}{1 - \cos x} \\ &= \frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x} \\ &= 1 + \cos x = \text{RHS} \end{aligned}$$

Worked Example 8.3.2

Prove the following identity

$$\frac{1 + \cos 2\theta + \sin 2\theta}{1 - \cos 2\theta + \sin 2\theta} = \cot \theta$$

[S4 SCGS P2/2017 PRELIM Qn 3]

[Solution]

LHS

$$\begin{aligned} &= \frac{1 + \cos 2\theta + \sin 2\theta}{1 - \cos 2\theta + \sin 2\theta} \\ &= \frac{1 + 2 \sin \theta \cos \theta + (2 \cos^2 \theta - 1)}{1 - (1 - 2 \sin^2 \theta) + 2 \sin \theta \cos \theta} \\ &= \frac{2 \sin \theta \cos \theta + 2 \cos^2 \theta}{2 \sin \theta \cos \theta + 2 \sin^2 \theta} \\ &= \frac{2 \cos \theta (\sin \theta + \cos \theta)}{2 \sin \theta (\sin \theta + \cos \theta)} \\ &= \frac{\cos \theta}{\sin \theta} \\ &= \cot \theta = \text{RHS} \end{aligned}$$

Worked Example 8.3.3

Find the value of m for which

$$\frac{1 + \cos 2x + \sin 2x}{\cos x + \sin x} \equiv m \cos x$$

[Solution]

$$\begin{aligned}\frac{1 + \cos 2x + \sin 2x}{\cos x + \sin x} &= \frac{1 + (2 \cos^2 x - 1) + 2 \sin x \cos x}{\cos x + \sin x} \\&= \frac{2 \cos^2 x + 2 \sin x \cos x}{\cos x + \sin x} \\&= \frac{2 \cos x (\cos x + \sin x)}{\cos x + \sin x} \\&= 2 \cos x \\m &= 2\end{aligned}$$

Worked Example 8.3.4

Show that, when $-90^\circ < x < 90^\circ$

$$\sqrt{\frac{1 - \sin x}{1 + \sin x}} = \sec x - \tan x$$

[Solution]

$$\sqrt{\frac{1 - \sin x}{1 + \sin x}} = \sec x - \tan x$$

$$\frac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2$$

LHS

$$\begin{aligned}&= \frac{1 - \sin x}{1 + \sin x} \\&= \frac{(1 - \sin x)(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} \\&= \frac{(1 - \sin x)^2}{1 - \sin^2 x} \\&= \frac{1 - 2 \sin x + \sin^2 x}{\cos^2 x} \\&= \frac{1}{\cos^2 x} - 2 \left(\frac{\sin x}{\cos^2 x} \right) + \tan^2 x \\&= \sec^2 x - 2 \sec x \tan x + \tan^2 x \\&= (\sec x - \tan x)^2 = \text{RHS}\end{aligned}$$

Stretch Question

Why must x be acute for the above equation to hold? (*hint: consider different cases*)

Worked Example 8.4.1

Show that

$$\frac{1 - \cos 2A}{3 + \cos 2A} = \frac{\sin^2 A}{1 + \cos^2 A}$$

Hence, or otherwise, find the values of A for $0^\circ \leq A \leq 180^\circ$ given that

$$\frac{1 - \cos 2A}{3 + \cos 2A} = \frac{1}{7}$$

[S4 HIHS P1/2009 PRELIM Qn 4]

[Solution]

$$\begin{aligned} \text{LHS} &= \frac{1 - \cos 2A}{3 + \cos 2A} \\ &= \frac{1 - (1 - 2 \sin^2 A)}{3 + (2 \cos^2 A - 1)} \\ &= \frac{2 \sin^2 A}{2 + 2 \cos^2 A} \\ &= \frac{\sin^2 A}{1 + \cos^2 A} \\ &= \text{RHS (shown)} \end{aligned}$$

$$\frac{1 - \cos 2A}{3 + \cos 2A} = \frac{1}{7} \Rightarrow \frac{\sin^2 A}{1 + \cos^2 A} = \frac{1}{7}$$

$$7 \sin^2 A = 1 + \cos^2 A$$

$$7(1 - \cos^2 A) = 1 + \cos^2 A$$

$$7 - 7 \cos^2 A = 1 + \cos^2 A$$

$$-8 \cos^2 A = -6$$

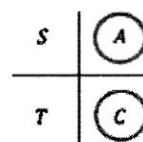
$$\cos^2 A = \frac{3}{4}$$

$$\cos A = \frac{\sqrt{3}}{2} \quad \text{or} \quad \cos A = -\frac{\sqrt{3}}{2}$$

$$\cos A = \frac{\sqrt{3}}{2}$$

Basic Angle $\alpha = 30^\circ$

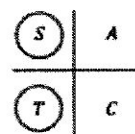
$$A = 30^\circ \quad \text{or} \quad A = 360^\circ - 30^\circ = 330^\circ$$



$$\cos A = -\frac{\sqrt{3}}{2}$$

Basic Angle $\alpha = 30^\circ$

$$A = 180^\circ - 30^\circ \quad \text{or} \quad A = 180^\circ + 30^\circ = 210^\circ$$



Worked Example 8.4.2

Given that $0^\circ \leq y \leq 360^\circ$, solve the following equation

$$4\operatorname{cosec}^2 y = 7 - \cot^2 y + 2 \cot y$$

[S4 CCHS(M) P2/2012 PRELIM Qn 5]

[Solution]

$$4\operatorname{cosec}^2 y = 7 - \cot^2 y + 2 \cot y$$

$$4(1 + \cot^2 y) = 7 - \cot^2 y + 2 \cot y$$

$$4 + 4 \cot^2 y = 7 - \cot^2 y + 2 \cot y$$

$$5 \cot^2 y - 2 \cot y - 3 = 0$$

$$(5 \cot y + 3)(\cot y - 1) = 0$$

$$\cot y = -\frac{3}{5} \quad \text{or} \quad \cot y = 1$$

$$\tan y = -\frac{5}{3} \quad \text{or} \quad \tan y = 1$$

$$\tan y = -\frac{5}{3}$$

$$\text{Basic angle } a = \tan^{-1}\left(\frac{5}{3}\right)$$

$$y = 180^\circ - \tan^{-1}\left(\frac{5}{3}\right) \quad \text{or} \quad y = 360^\circ - \tan^{-1}\left(\frac{5}{3}\right)$$

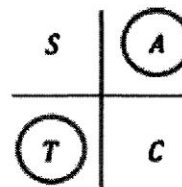
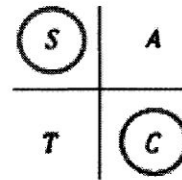
$$y = 120.96375 \dots^\circ \quad y = 300.96375 \dots^\circ$$

$$y = 121.0^\circ (1d.p) \quad y = 301.0^\circ (1d.p)$$

$$\tan y = 1$$

$$\text{Basic angle } a = 45^\circ$$

$$y = 45^\circ \quad \text{or} \quad y = 180^\circ + 45^\circ = 225^\circ$$



Worked Example 8.4.3

- (i) Given that $5(\cos y + \sin y)(2 \cos y - \sin y) = 7$, show that

$$12 \tan^2 y - 5 \tan y - 3 = 0$$

- (ii) Hence, solve the following equation for $0^\circ < x < 180^\circ$,

$$5(\cos y + \sin y)(2 \cos y - \sin y) = 7$$

[S4 CGSS P2/2012 PRELIM Qn 5(a) (Modified)]

Solution

(a) $5(\cos y + \sin y)(2 \cos y - \sin y) = 7$

$$5[2 \cos^2 y - \cos y \sin y + 2 \cos y \sin y - \sin^2 y] = 7$$

$$10 \cos^2 y + 5 \cos y \sin y - 5 \sin^2 y = 7$$

$$\frac{10 \cos^2 y + 5 \cos y \sin y - 5 \sin^2 y}{\cos^2 y} = \frac{7}{\cos^2 y}$$

$$\frac{10 \cos^2 y}{\cos^2 y} + \frac{5 \cos y \sin y}{\cos^2 y} - \frac{5 \sin^2 y}{\cos^2 y} = 7 \sec^2 y$$

$$10 + 5 \tan y - 5 \tan^2 y = 7 \sec^2 y$$

$$10 + 5 \tan y - 5 \tan^2 y = 7 + 7 \tan^2 y$$

$$12 \tan^2 y - 5 \tan y - 3 = 0 \text{ (shown)}$$

(b) $5(\cos y + \sin y)(2 \cos y - \sin y) = 7$

$$12 \tan^2 y - 5 \tan y - 3 = 0$$

$$(4 \tan y - 3)(3 \tan y + 1) = 0$$

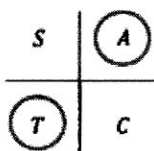
$$\tan y = \frac{3}{4} \quad \text{or} \quad \tan y = -\frac{1}{3}$$

$$\text{Basic angle } a = \tan^{-1}\left(\frac{3}{4}\right)$$

$$y = \tan^{-1}\left(\frac{3}{4}\right)$$

$$= 36.869 \dots^\circ$$

$$= 36.9^\circ \text{ (1d.p.)}$$



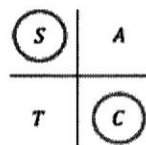
$$\tan y = -\frac{1}{3}$$

$$\text{Basic angle } a = \tan^{-1}\left(\frac{1}{3}\right)$$

$$y = 180^\circ - \tan^{-1}\left(\frac{1}{3}\right)$$

$$= 161.565 \dots^\circ$$

$$= 161.6^\circ \text{ (1d.p.)}$$



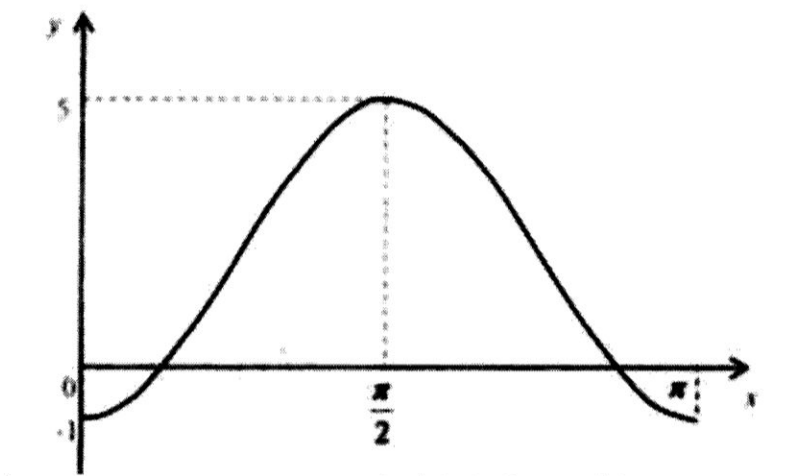
Graphs of Trigonometric Functions

Trigonometric Ratio	Graphical Representation	Features
$y = \sin \theta$		Amplitude = 1 Period = 360° or 2π
$y = \cos \theta$		Amplitude = 1 Period = 360° or 2π
$y = \tan \theta$		Period = 180° or π

Take Note

- For tangent graphs, the dotted lines represent vertical asymptotes. At these vertical asymptotes, the graph approaches very close to it but never touches / crosses them.
- a**
 - $|a|$ is the amplitude of the graph
 - Distance of the highest / lowest points from the equilibrium position
 - $a < 0$ implies the graph is flipped (reflected about the x -axis)
- b**
 - Determines period of the graph (how long the graph completes 1 cycle)
 - $b = \frac{360^\circ}{\text{period}}$
- c**
 - How much the equilibrium line has shifted up or shifted down from the origin. (translation)

Worked Example 8.5.1



The diagram above shows the graph, where a, b and c are constants, of

$$y = a \cos bx + c$$

- Use your graph to determine the value of a, b and c .
- By using the values of a, b and c found in part (a), determine the equation of the straight line that needs to be drawn on the same diagram to solve

$$\sec bx = \frac{a\pi}{x - \pi c}$$

[S4 ACS(I) P1/2015 PRELIM Qn 5]

[Solution]

- Based on the graph,

$$a = -3$$

$$\text{Period} = \pi, b = 2$$

$$c = 2$$

- To determine the equation

$$\cos 2x = \frac{x - 2\pi}{-3\pi}$$

$$-3\pi \cos 2x = x - 2\pi$$

$$2\pi - 3\pi \cos 2x = x$$

$$\pi(2 - 3 \cos 2x) = x$$

$$2 - 3 \cos 2x = \frac{x}{\pi}$$

The line to be drawn is: $y = \frac{x}{\pi}$

Worked Example 8.5.2

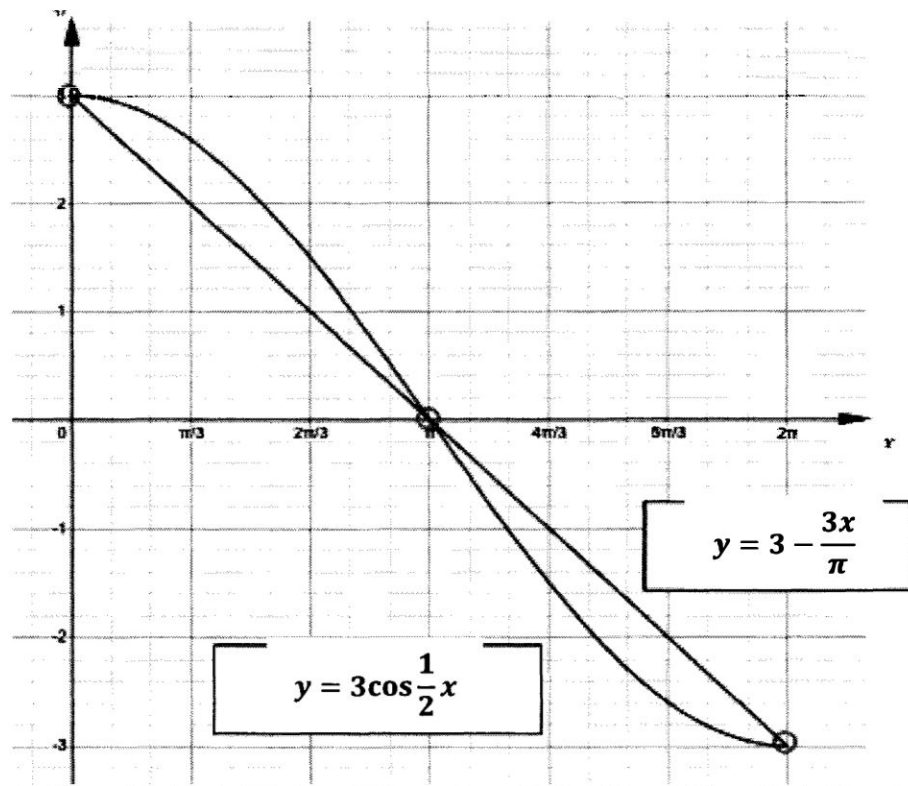
Sketch the following curve between $0 \leq x \leq 2\pi$

$$y = 3 \cos \frac{1}{2}x$$

Hence, or otherwise, use your curve and find the number of solutions for the following equation

$$\cos \frac{1}{2}x + \frac{1}{\pi} = 1$$

[Solution]



$$\cos \frac{1}{2}x + \frac{x}{\pi} = 1$$

$$\cos \frac{1}{2}x = 1 - \frac{x}{\pi}$$

$$3 \cos \frac{1}{2}x = 3 \left(1 - \frac{x}{\pi}\right)$$

Since we have the graph of $y = 3 \cos \frac{1}{2}x$, we need to draw the line $y = 3 - \frac{3x}{\pi}$

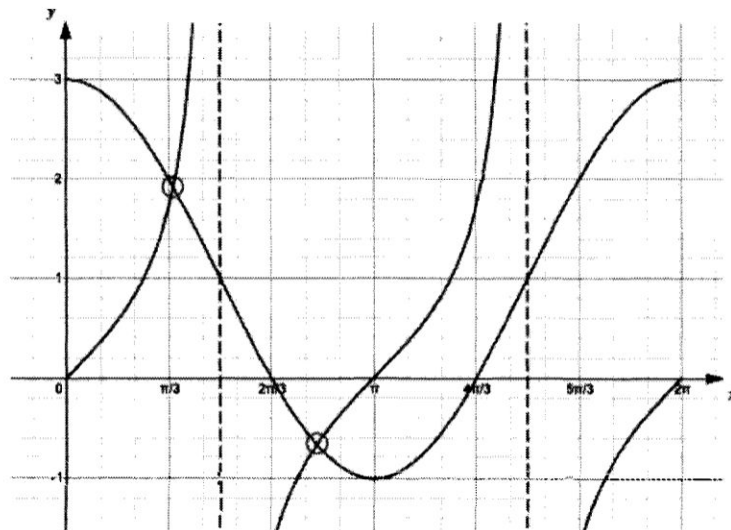
Number of solutions = 3

Worked Example 8.5.3

On the same diagram, sketch the graph of $y = \tan x$ and $y = 2 \cos x + 1$ for the interval $0 \leq x \leq 2\pi$. Hence state the number of solutions, in this interval, of the equation

$$\frac{1}{2}(\tan x - 1) = \cos x$$

[Solution]



$$\frac{1}{2}(\tan x - 1) = \cos x$$

$$\tan x - 1 = 2 \cos x$$

$$\tan x = 2 \cos x + 1$$

Since we have the graph of $y = \tan x$, we need to draw the graph of $y = 2 \cos x + 1$

$\therefore$ Number of solutions = 2

Tips/Thinking Process

Always start each sketching question by finding the amplitude and the period. Once these have been found, the curve will be significantly easier to draw

If possible, sketch out some working curves (pseudo curves) and perform some shifting to ensure that the vertical translations are done accurately

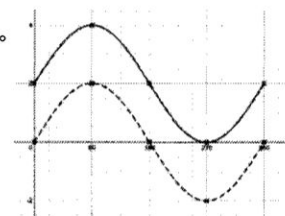
Example: Sketch the graph of $y = 2 \sin x + 2$ for $0 \leq x \leq 360^\circ$

Solution: Amplitude: 2, Period 360°

The dotted line is $y = 2 \sin x$ is our working

while the solid curve is the actual solution.

The crosses are important points for us to take note of when translating!



R-Formula

Cosine Form:

$$a \cos A \pm b \sin A = R \cos(\theta \mp \alpha)$$

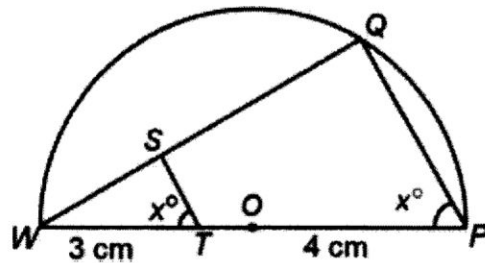
Sine Form:

$$a \sin A + b \cos A = R \sin(\theta \pm \alpha)$$
$$\alpha = \tan^{-1} \left(\frac{b}{a} \right), \quad R = \sqrt{a^2 + b^2}$$

NOTE:

- 1) Proving R Formulas is always about using TOA CAH SOH
- 2) So, you need to create more RIGHT ANGLE TRIANGLES
- 3) NEVER EVER SLICE THE GIVEN ANGLE, you can slice right angles

Worked Example 8.6.1



The diagram shows a triangle PQW inside a semicircle, centre O and radius 4 cm. Given that $\angle WPQ = \angle WTS = x^\circ$ and $WT = 3$ cm

- Show that $PQ + QS + ST = 11 \cos x^\circ + 5 \sin x^\circ$
- Express $11 \cos x^\circ + 5 \sin x^\circ$ in the form $R \cos(x^\circ - a^\circ)$
- Find the maximum value of $PQ + QS + ST$ and the corresponding value of x
- Find the value of x for which $PQ + QS + ST = 12$ cm

[S4 VS P2/2007 PRELIM Qn 5]

[Solution]

- $\angle WQP = 90^\circ$ (in a semicircle)

$$\cos \angle STW = \frac{ST}{WT} \Rightarrow ST = 3 \cos x$$

$$\sin \angle STW = \frac{SW}{WT} \Rightarrow SW = 3 \sin x$$

$$\cos \angle QPW = \frac{QP}{WP} \Rightarrow QP = 8 \cos x$$

$$\sin \angle QPW = \frac{QW}{WP} \Rightarrow QW = 8 \sin x$$

$$\begin{aligned} \therefore PQ + QS + ST &= 3 \cos x + 8 \cos x + (8 \sin x - 3 \sin x) \\ &= 11 \cos x + 5 \sin x \end{aligned}$$

- $11 \cos x + 5 \sin x = R \cos(x - a)$

$$\tan a = \frac{5}{11}$$

$$a = \tan^{-1}\left(\frac{5}{11}\right)$$

$$= 24.4^\circ (1 \text{ d.p.})$$

$$R = \sqrt{(11)^2 + (5)^2}$$

$$= 12.1 (3 \text{ s.f.})$$

$$\therefore 11 \cos x + 5 \sin x = 12.1 \cos(x - 24.4^\circ)$$

- Max value of $PQ + QS + ST$ is 12.1
Corresponding value of x is 24.4°

- $12.1 \cos(x - 24.4^\circ) = 12$

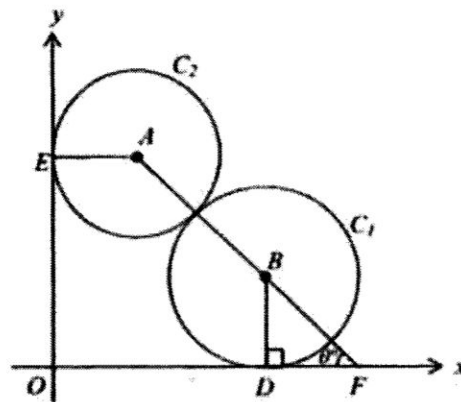
$$\left[x - \tan^{-1}\left(\frac{5}{11}\right) \right] = \frac{12}{\sqrt{(11)^2 + (5)^2}}$$

$$x = \tan^{-1}\left(\frac{5}{11}\right) + \frac{12}{\sqrt{(11)^2 + (5)^2}}$$

$$= 31.16532 \dots$$

$$= 31.2^\circ (1 \text{ d.p.})$$

Worked Example 8.6.2



The figure shows two circles, C_1 and C_2 which touch each other and lie on the xy -plane and shown below. C_1 has radius 4 units and touches the x -axis at D , C_2 has radius 3 units and touches the y -axis at E . The line AB , joining the centres of C_2 and C_1 meets at the x -axis at F and $\angle BFO = \theta^\circ$

(a) Obtain expressions for OD and OE

(b) Show that

$$ED^2 = 74 + 56 \sin \theta + 42 \cos \theta$$

(c) Express ED^2 in the form $74 + R \cos(\theta - a)$ where $R > 0$ and $0^\circ < a < 90^\circ$

(d) By considering the extreme positions in which both circles can touch the x -axis and both circles touch the y -axis, correcting to one decimal place, show that

$$8.2^\circ \leq \theta \leq 81.8^\circ$$

[S4 FMS(S) P2/2015 PRELIM Qn 11]

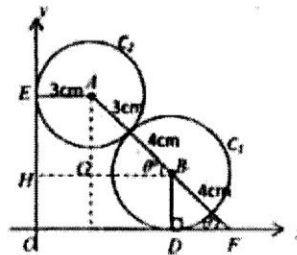
[Solution]

(a) $OD = EA + GB$

$$= 3 + 7 \cos \theta$$

$$OE = DB + HE$$

$$= 4 + 7 \sin \theta$$



(b) By Pythagoras' Theorem,

$$ED^2 = OD^2 + OE^2$$

$$= (3 + 7 \cos \theta)^2 + (4 + 7 \sin \theta)^2$$

$$= 9 + 42 \cos \theta + 49 \cos^2 \theta + 16 + 56 \sin \theta + 49 \sin^2 \theta$$

$$= 25 + 56 \sin \theta + 42 \cos \theta + 49(\sin^2 \theta + \cos^2 \theta)$$

$$= 25 + 56 \sin \theta + 42 \cos \theta + 49$$

$$= 74 + 56 \sin \theta + 42 \cos \theta$$

(c) $ED^2 = 74 + 56 \sin \theta + 42 \cos \theta$

$$74 + R \cos(\theta - \alpha) = 74 + 56 \sin \theta + 42 \cos \theta$$

$$R = \sqrt{56^2 + 42^2}$$

$$= 70$$

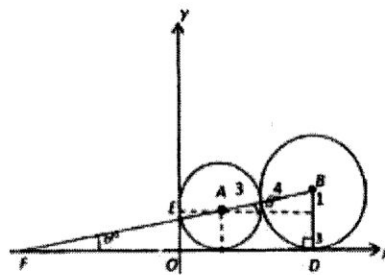
$$\alpha = \tan^{-1}\left(\frac{56}{42}\right)$$

$$= 53.130 \dots$$

$$= 53.1^\circ \text{ (1d.p.)}$$

$$\therefore ED^2 = 74 + 70 \cos(\theta - 53.1^\circ)$$

(d) Case 1: Both circles touch the y-axis

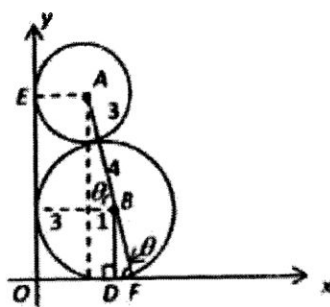


$$\sin \theta = \frac{1}{7}$$

$$\theta = 8.213210 \dots$$

$$= 8.2^\circ \text{ (1d.p.)}$$

Case 2: Both circles touch the y-axis



$$\cos \theta = \frac{1}{7}$$

$$\theta = 81.786789 \dots$$

$$= 81.8^\circ \text{ (1d.p.)}$$

9 Differentiation and its Applications

Differentiation is one of the main concepts in a broader umbrella known as Calculus. The following table will summarise some of the techniques and skills used.

Differentiation Formulae

Type	x	$ax + b$ (Chain Rule)
Algebraic	$\frac{d}{dx}(a) = 0$	
	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}((ax + b)^n) = a \cdot [ax + b]^{n-1}$
Exponential	$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(e^{ax+b}) = a \cdot e^{ax+b}$
Logarithmic	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\ln(ax + b)) = a \cdot \left(\frac{1}{ax + b}\right)$
Trigonometric	$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sin(ax + b)) = a \cdot (\cos(ax + b))$
	$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cos(ax + b)) = a \cdot -(\sin(ax + b))$
	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\tan(ax + b)) = a \cdot (\sec^2(ax + b))$

Chain rule states that we differentiate the functions one-by-one in a chain form, from the outer-most function to the inner-most function.

Using $\frac{d}{dx}(\ln(3x + 2))$ as an example,

1. Differentiate the outer-most function, in this case, $\ln x$.
2. Differentiate the next inner function, in this case $3x + 2$.

$$\text{Hence, } \frac{d}{dx}(\ln(3x + 2)) = \frac{1}{3x+2} \cdot 3 = \frac{3}{3x+2}$$

Interestingly enough, when we differentiate $\ln x$, we technically also apply chain rule.

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \cdot \frac{d}{dx}(x) = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

Tip!

Factorise the constant from the differential before applying our differentiation techniques!

$$\frac{d}{dx}(5x^2) = 5 \frac{d}{dx}(x^2) = 5(2x) = 10x$$

Worked Example 9.1

Differentiate the following equations with respect to x

(a) $y = \frac{2}{3x^2-2}$

(b) $y = \frac{x^2-1}{\sqrt{x}}$

(c) $y = 3 \sin^2(5x + 7)$

(d) $y = \ln(x^2 + 5x)^2$

(e) $y = e^{\sin 3x}$

[Solution]

(a) $y = \frac{2}{3x^2-2} = 2(3x^2 - 2)^{-1}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= 2[(-1)(3x^2 - 2)^{-2} \cdot (6x)] \\ &= -\frac{12x}{(3x^2-2)^2}\end{aligned}$$

(b) $y = \frac{x^2-1}{\sqrt{x}} = x^{\frac{3}{2}} - x^{\frac{1}{2}}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} \\ &= \frac{3x^2+1}{2x\sqrt{x}}\end{aligned}$$

(c) $y = 3 \sin^2(5x + 7)$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (3)[2 \sin(5x + 7) \cdot \cos(5x + 7) \cdot 5] \\ &= 30 \sin(5x + 7) \cos(5x + 7)\end{aligned}$$

(d) $y = \ln(x^2 + 5x)^2 = 2 \ln(x^2 + 5x)$

$$\therefore \frac{dy}{dx} = 2 \left(\frac{2x+5}{x^2+5x} \right) = \frac{4x+10}{x^2+5x}$$

(e) $y = e^{\sin 3x}$

$$\therefore \frac{dy}{dx} = 3 \cos 3x (e^{\sin 3x})$$

Sum and Difference of Differentials

Assuming that u and v are functions of x , we have

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Using questions (d) and (e) from Worked Example 9.1 as an example,

$$\begin{aligned}\frac{d}{dx}(\ln(x^2 + 5x)^2 + e^{\sin 3x}) &= \frac{d}{dx}(\ln(x^2 + 5x)^2) + \frac{d}{dx}(e^{\sin 3x}) \\ &= \frac{4x + 10}{x^2 + 5x} + 3 \cos 3x (e^{\sin 3x})\end{aligned}$$

Product and Quotient Rule

Assuming u and v are functions of x , we have

$$\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + \frac{dv}{dx} \cdot u$$

“Differentiate the first, Keep the second + Differentiate the second, Keep the first”

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2}$$

“Differentiate the top, Keep the bottom - Differentiate the bottom, Keep the top, over Bottom Squared”

Worked Example 9.1 (redo)

Differentiate the following equations with respect to x using product rule and quotient rule

(a) $y = \frac{2}{3x^2-2}$

(b) $y = \frac{x^2-1}{\sqrt{x}}$

[Solution]

(a) Product Rule (although not necessary, it still works)

$$y = 2(3x^2 - 2)^{-1}$$

$$\begin{aligned}\frac{dy}{dx} &= 0(3x^2 - 2)^{-1} + (-1)(3x^2 - 2)^{-2}(6x)(2) \\ &= -12x(3x^2 - 2)^{-2}\end{aligned}$$

Quotient Rule

$$\frac{dy}{dx} = \frac{0(3x^2-2)-(6x)(2)}{(3x^2-2)^2} = -\frac{12x}{(3x^2-2)^2}$$

(b) Product Rule

$$y = (x^2 - 1)(\sqrt{x})^{-1}$$

$$\begin{aligned}\frac{dy}{dx} &= 2x(\sqrt{x})^{-1} + (-1)(\sqrt{x})^{-2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)(x^2 - 1) \\ &= \frac{2x\sqrt{x} - \frac{x^2-1}{2\sqrt{x}}}{(\sqrt{x})^2} \\ &= \frac{3x^2 + 1}{2x\sqrt{x}}\end{aligned}$$

Quotient Rule

$$\begin{aligned}y &= \frac{(x^2-1)}{x^{\frac{1}{2}}} \\ \frac{dy}{dx} &= \frac{(2x)(\sqrt{x}) - \frac{1}{2}(x)^{-\frac{1}{2}}(x^2-1)}{x} = \frac{4x^2-x^2+1}{2x\sqrt{x}} = \frac{3x^2+1}{2x\sqrt{x}}\end{aligned}$$

Note!

All answers, even those from the previous question, are the same! Hence, we need to be smart when choosing which method to use when differentiating. You can get the same answer, but some methods are a lot more efficient than others 😊

Worked Example 9.2

Differentiate the following expressions with respect to x .

(a) $y = x(3x - 5)^{\frac{5}{3}}$

(b) $y = \frac{5}{2x \sec x}$

(c) $y = \sin(4x - 1) \tan x$

(d) $y = e^{2x} \sqrt{3x + 5}$

[Solution]

$$\begin{aligned} \text{(a)} \quad \frac{dy}{dx} &= (3x - 5)^{\frac{5}{3}} + \frac{5}{3} (3x - 5)^{\frac{2}{3}} (3)(x) \\ &= (3x - 5)^{\frac{2}{3}} (3x - 5 + 5x) \\ &= (8x - 5)(3x - 5)^{\frac{2}{3}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad y &= \frac{5 \cos x}{2x} \\ \frac{dy}{dx} &= \frac{5(-\sin x)(x) - 1(\cos x)}{x^2} = \frac{-5x \sin x - \cos x}{2x^2} \end{aligned}$$

$$\text{(c)} \quad \frac{dy}{dx} = 4 \cos(4x - 1) \tan x + \sec^2 x \sin(4x - 1)$$

$$\begin{aligned} \text{(d)} \quad y &= e^{2x} \sqrt{3x + 5} \\ y &= e^{2x} (3x + 5)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= (2)e^{2x} \sqrt{3x + 5} + e^{2x} \left[\frac{1}{2} (3x + 5)^{-\frac{1}{2}} \cdot (3) \right] \\ &= 2e^{2x} \sqrt{3x + 5} + \frac{3e^{2x}}{2\sqrt{3x + 5}} \\ &= e^{2x} \left[\frac{(2\sqrt{3x + 5})(2\sqrt{3x + 5}) + 3(1)}{2\sqrt{3x + 5}} \right] \\ &= e^{2x} \left[\frac{4(3x + 5) + 3}{2\sqrt{3x + 5}} \right] \\ &= e^{2x} \left[\frac{12x + 20 + 3}{2\sqrt{3x + 5}} \right] \\ &= e^{2x} \left[\frac{12x + 23}{2\sqrt{3x + 5}} \right] \end{aligned}$$

Increasing and Decreasing Functions

The sign of $\frac{dy}{dx}$ can tell us the behaviour of the graph of y as x varies.

Property	Condition
Increasing function	$\frac{dy}{dx} > 0$
Decreasing function	$\frac{dy}{dx} < 0$

If we want to determine if $\frac{dy}{dx}$ is increasing or decreasing, we observe the sign of $\frac{d^2y}{dx^2}$.

Worked Example 9.3

A curve has the equation $y = f(x)$, where $x > 0$ and

$$f(x) = \frac{2x - 3}{4 + 3x}$$

- Obtain an expression for $f'(x)$
- Determine, with explanation, whether $f(x)$ is an increasing or decreasing function
- Determine whether the gradient of the curve is an increasing or decreasing function

[S4 FMS(S) P2/2014 PRELIM Qn 7]

[Solution]

$$\begin{aligned}
 \text{(a) } f'(x) &= \frac{(4+3x)(2) - (2x-3)(3)}{(4+3x)^2} \\
 &= \frac{8+6x-6x+9}{(4+3x)^2} \\
 &= \frac{17}{(4+3x)^2}
 \end{aligned}$$

- (b) To determine if the function is increasing or decreasing, we need to analyse $f'(x)$

$$f'(x) = \frac{17}{(4+3x)^2}$$

$$\text{Since } (4 + 3x)^2 > 0, \frac{17}{(4+3x)^2} > 0$$

Since $f'(x) > 0$, $f(x)$ is an increasing function

- (c) To determine if the gradient of the curve is increasing or decreasing, we need to analyse $f''(x)$

$$f'(x) = 17(4 + 3x)^{-2}$$

$$f''(x) = -34(4 + 3x)^{-3}(3)$$

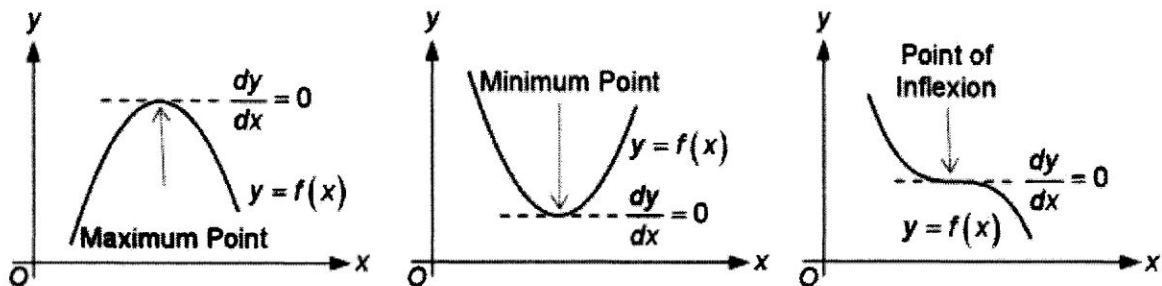
$$= -\frac{102}{(4+3x)^3}$$

$$\text{Since } (4 + 3x)^3 > 0, -\frac{102}{(4+3x)^3} < 0$$

Since $f''(x) < 0$, the gradient of the curve is a decreasing function

Stationary Points

Stationary points are points where the curve has a gradient of zero. They come in two forms, maximum/minimum points, and stationary points of inflexion.



To find the coordinate of the stationary point, we first find $\frac{dy}{dx}$, and solve for the value of x where $\frac{dy}{dx} = 0$. We then use this value of x to find the y -coordinate of the stationary point.

First Derivative Test

By substituting values of x into $\frac{dy}{dx}$, we can observe the nature of the stationary point.

x	x^-	x	x^+
$\frac{dy}{dx}$ sign		0	
shape		—	
conclusion			

shape			
-------	--	--	--

Maximum point

shape			
-------	--	--	--

Minimum point

shape			
-------	--	--	--

Stationary point of inflexion
(increasing)

shape			
-------	--	--	--

Stationary point of inflexion
(decreasing)

This test is a fool-proof method and will not fail, but might be a bit tedious in substituting the value of x . Students who are confident in differentiating may want to consider the second derivative test.

Second Derivative Test

By differentiating $\frac{dy}{dx}$ with respect to x again, we can observe the rate of change of $\frac{dy}{dx}$.

$\frac{d^2y}{dx^2}$ at $x = c$	Conclusion
> 0	Minimum point at $x = c$
$= 0$	Inconclusive nature
< 0	Maximum point at $x = c$

If the nature of the stationary point is inconclusive, we need to use the first derivative test to determine its nature.

Worked Example 9.4

Given the curve with equation $y = -2x^3 + 3x^2 + 4$, determine the coordinates and nature of each stationary point. [S4 ANDSS P2/2012 PRELIM Qn 6]

[Solution]

(a) $y = -2x^3 + 3x^2 + 4 \dots\dots (1)$

$$\frac{dy}{dx} = -6x^2 + 6x \dots\dots (2)$$

For a stationary point, $\frac{dy}{dx} = 0$

$$-6x(x - 1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

Substitute $x = 0$ into (1)

$$\begin{aligned} y &= -2(0)^3 + 3(0)^2 + 4 \\ &= 4 \end{aligned}$$

Substitute $x = 1$ into (1)

$$\begin{aligned} y &= -2(1)^3 + 3(1)^2 + 4 \\ &= 5 \end{aligned}$$

Hence the coordinates of the stationary points are (0, 4) and (1, 5)

$$\frac{d^2y}{dx^2} = -12x + 6$$

When $x = 0$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -12(0) + 6 \\ &= 6 > 0 \end{aligned}$$

$\therefore$ (0, 4) is a minimum point

When $x = 1$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -12(1) + 6 \\ &= -6 < 0 \end{aligned}$$

$\therefore$ (1, 5) is a maximum point

Worked Example 9.5

It is given that the curve $y = (2x)(3x - 2)^3$ has 2 stationary points

(a) Find $\frac{dy}{dx}$, factorising your expression completely.

(b) Find the x -coordinates of the stationary points.

(c) Determine the nature of the stationary points.

[S4 XMSS P1/2011 PRELIM Qn 3]

[Solution]

(a) $y = (2x)(3x - 2)^3$

$$\begin{aligned}\frac{dy}{dx} &= (2x)(3)(3x - 2)^2(3) + (2)(3x - 2)^3 \\ &= 2(3x - 2)^2[(x)(3)(3) + (3x - 2)] \\ &= 2(3x - 2)^2(9x + 3x - 2) \\ &= 2(3x - 2)^2(12x - 2) \\ &= 4(3x - 2)^2(6x - 1)\end{aligned}$$

(b) At the stationary point, $\frac{dy}{dx} = 0$

$$\therefore 4(3x - 2)^2(6x - 1) = 0$$

$$(3x - 2)^2 = 0 \quad \text{or} \quad (6x - 1) = 0$$

$$x = \frac{2}{3} \quad \text{or} \quad x = \frac{1}{6}$$

(c) First-Derivative Test

$x = \frac{2}{3}$ $= 0.667$	$x = 0.657$	$x = 0.667$	$x = 0.677$
	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
			

$x = \frac{1}{6}$ $= 0.167$	$x = 0.157$	$x = 0.167$	$x = 0.177$
	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
			

$\therefore x = \frac{2}{3}$ is a stationary point of inflection

$\therefore x = \frac{1}{6}$ is a minimum point

Second-Derivative Test

$$y = (2x)(3x - 2)^3$$

$$\text{From (i), } \frac{dy}{dx} = 4(3x - 2)^2(6x - 1)$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 4 \left\{ \frac{dy}{dx} [(3x - 2)^2(6x - 1)] \right\} \\ &= 4[(2)(3x - 2)(3)(6x - 1) + (3x - 2)^2(6)] \\ &= 4\{6(3x - 2)[(6x - 1) + (3x - 2)]\} \\ &= 4[6(3x - 2)(9x - 3)] \\ &= 4[18(3x - 2)(3x - 1)] \\ &= 72(3x - 2)(3x - 1) \end{aligned}$$

$$\text{When } x = \frac{2}{3},$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 72 \left(3 \left(\frac{2}{3} \right) - 2 \right) \left(3 \left(\frac{2}{3} \right) - 1 \right) \\ &= 0 \end{aligned}$$

This implies that the Second Derivative Test for $x = \frac{2}{3}$ is inconclusive. Hence, we need to perform the First Derivative Test to find the nature of this stationary point

	$x = 0.657$	$x = 0.667$	$x = 0.677$
$x = \frac{2}{3}$ $= 0.667$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
			

The stationary point at $x = \frac{2}{3}$ is a point of inflection.

$$\text{When } x = \frac{1}{6},$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 72 \left(3 \left(\frac{1}{6} \right) - 2 \right) \left(3 \left(\frac{1}{6} \right) - 1 \right) \\ &= 54 > 0 \end{aligned}$$

Since $\frac{d^2y}{dx^2} > 0$, the stationary point at $x = \frac{1}{6}$ is a minimum point.

Maxima and Minima

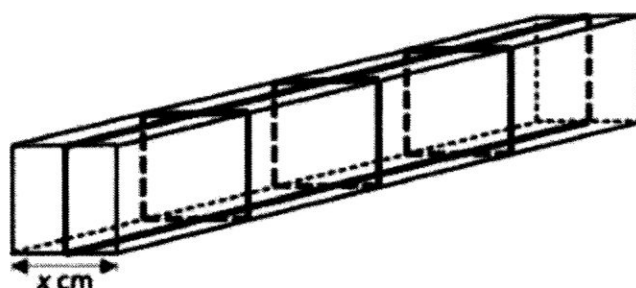
The concepts in maxima/minima questions are similar to questions involving stationary points, with a context added into the questions. Remember that for a stationary point,

$$\frac{dy}{dx} = 0$$

Tip!

- If we are unable to show the first part of the question, move on to other parts! We can still get full credit for the other parts, using the given equation in the first part.
- Remember that we always have to prove that the value calculated will give a maximum / minimum value, to get the full credit of the question.

Worked Example 9.6



The diagram shows a box in the shape of a cuboid with a square cross-section of side x cm. The volume of the box is 3500 cm^3 . Four pieces of tape are fastened round the box as shown in the diagram. The pieces are parallel to the edges of the box.

- (a) Given that the total length of the 4 pieces of tape is L cm, show that

$$L = 14x + \frac{7000}{x^2}$$

- (b) Given that x can vary, find the stationary value of L and determine the nature of this stationary value.

[Solution]

- (a) Length of 3 square tapes = $3(4x)$

$$= 12x$$

$$\text{Length of rectangular tape} = 2\left(\frac{3500}{x^2}\right) + 2x$$

$$\therefore \text{Total length of tape used} = 2\left(\frac{3500}{x^2}\right) + 2x + 12x$$

$$= 14x + \frac{7000}{x^2} \text{ (shown)}$$

(b) $L = 14x + \frac{7000}{x^2}$

Since x can vary,

$$\begin{aligned}\frac{dL}{dx} &= 14 + \frac{x^2(0) - 7000(2x)}{(x^2)^2} \\ &= 14 - \frac{14000}{x^3}\end{aligned}$$

For a stationary value of L , $\frac{dL}{dx} = 0$

$$14 - \frac{14000}{x^3} = 0$$

$$14x^3 - 14000 = 0$$

$$x^3 = 1000$$

$$x = 10$$

To determine the nature of the stationary value,

$$\begin{aligned}\frac{d^2L}{dx^2} &= \frac{x^3(0) - 14000(3x^2)}{(x^3)^2} \\ &= -\frac{42000}{x^4}\end{aligned}$$

$$\frac{d^2L}{dx^2}, \text{ when } x = 10$$

$$= -\frac{42000}{(10)^4}$$

$$= -4\frac{1}{5} < 0$$

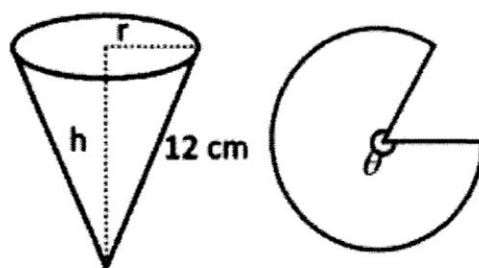
Since the value of $\frac{d^2L}{dx^2} < 0$, the nature of this stationary value is maximum.

$$\therefore L = 14(10) + \frac{7000}{(10)^2}$$

$$= 210$$

Hence, the maximum length of the tape being used is 210 cm.

Worked Example 9.7



Jack wishes to make a cone to hold water. He makes a right circular cone, without overlap, of depth h cm, radius r cm and slant height 12 cm

- (a) Show that the volume of the cone, V cm³ is given by the equation

$$V = 48\pi h - \frac{1}{3}\pi h^3$$

- (b) Find the value of h for which V has a stationary value. Find this value of V and determine whether it is a maximum or minimum
- (c) Given that Peter uses material that is originally in the form of a sector of angle θ radians. Show that θ is approximately 5.13 when V is stationary

[S4 VS P1/2005 PRELIM Qn 4]

[Solution]

- (a) By Pythagoras' Theorem,

$$r^2 + h^2 = 12^2$$

$$r^2 = 144 - h^2$$

$$\therefore \text{Volume of cone} = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi h(144 - h^2)$$

$$= 48\pi h - \frac{1}{3}\pi h^3 \text{ (shown)}$$

- (b) Since we know that V has a stationary value, $V' = 0$

$$V = 48\pi h - \frac{1}{3}\pi h^3$$

$$V' = 48\pi - \pi h^2$$

$$\therefore 48\pi - \pi h^2 = 0$$

$$\pi h^2 = 48\pi$$

$$h^2 = 48$$

$$h = \sqrt{48} \text{ (rej } -\sqrt{48})$$

$$= 6.928203$$

$$= 6.93\text{cm (3 s.f)}$$

$$\begin{aligned}\therefore V &= 48\pi(\sqrt{48}) - \frac{1}{3}\pi(\sqrt{48})^3 \\ &= 696.4989559 \dots \\ &= 696 \text{ cm}^3 \text{ (3 s.f.)}\end{aligned}$$

To determine its nature, we perform the second derivative test

$$\begin{aligned}V' &= 48\pi - \pi h^2 \\ V'' &= -2\pi h \\ &= -2\pi(\sqrt{48}) \\ &= -43.531184 \dots < 0\end{aligned}$$

Since $V'' < 0$, this is a maximum value

(c) Surface area = πrl

$$\begin{aligned}&= \pi(\sqrt{144 - h^2})(12) \\ &= \pi(\sqrt{144 - (48)^2})(12) \\ &= 12\sqrt{96}\pi\end{aligned}$$

$$\therefore \text{Area of sector} = \frac{1}{2}r^2\theta$$

$$12\sqrt{96}\pi = \frac{1}{2}l^2\theta$$

$$12\sqrt{96}\pi = \frac{1}{2}(12)^2\theta$$

$$\theta = 5.13019932 \dots$$

$$= 5.13 \text{ rad (3 s.f.) (shown)}$$

Tangents and Normals

A tangent to a curve at $x = a$ is a straight line that touches the curve at $x = a$, but if extended does not cross $x = a$.

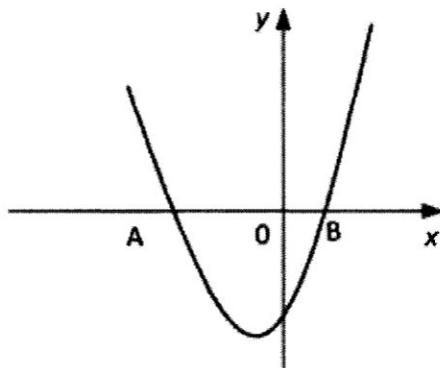
A normal to a curve at $x = a$ is a straight line that passes through $x = a$ that is perpendicular to the tangent.

Recall that if the gradient of the tangent is m , the gradient of the normal is $-\frac{1}{m}$. Hence, this gives us the equation of tangent and normal as:

Equation of tangent: $y - y_1 = m(x - x_1)$

Equation of normal: $y - y_1 = -\frac{1}{m}(x - x_1)$

Worked Example 9.8



The diagram shows the curve $y = x^2 + x - 2$. The curve crosses the x -axis at the points $A(a, 0)$ and $B(b, 0)$ where $a < b$

- (a) Find the value of a and b
- (b) Show that the normal to the curve at A has the equation $x - 3y + 2 = 0$
- (c) The tangent to the curve at B meets the normal at A at the point C . Find the coordinates of C

[S4 GMSS P2/2018 PRELIM Qn 9 (modified)]

Solution

(a) $y = x^2 + x - 2$

Since A and B cuts the x -axis, $y = 0$

$$\therefore x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2 \quad \text{or} \quad x = 1$$

$$\therefore a = -2, b = 1$$

(b) $y = x^2 + x - 2$

$$\frac{dy}{dx} = 2x + 1$$

At $A(-2,0)$,

$$\begin{aligned}\text{Gradient at } A &= \left. \frac{dy}{dx} \right|_{x=-2} \\ &= 2(-2) + 1 \\ &= -3\end{aligned}$$

Since the normal to the curve at A is perpendicular to the tangent at A

$$\begin{aligned}\text{Normal at } A &= \frac{-1}{(-3)} \\ &= \frac{1}{3}\end{aligned}$$

$\therefore$ Using $A(-2,0)$,

$$y - 0 = \frac{1}{3}(x - (-2))$$

$$3y = x + 2 \dots \dots (1)$$

$$x - 3y + 2 = 0 \text{ (shown)}$$

(c) At $B(1,0)$,

$$\begin{aligned}\text{Gradient at } B &= \left. \frac{dy}{dx} \right|_{x=1} \\ &= 2(1) + 1 \\ &= 3\end{aligned}$$

$\therefore$ Using $B(1,0)$,

$$y - 0 = 3(x - 1)$$

$$y = 3x - 3 \dots \dots (2)$$

Substitute Equation (2) into Equation (1),

$$3(3x - 3) = x + 2$$

$$9x - 9 = x + 2$$

$$8x = 11$$

$$x = \frac{11}{8}$$

Substitute $x = \frac{11}{8}$ into Equation (2)

$$y = 3\left(\frac{11}{8}\right) - 3 = \frac{9}{8}$$

$$\therefore C\left(1\frac{3}{8}, 1\frac{1}{8}\right)$$

Worked Example 9.9

- (a) Find the equation of the normal to the curve $y = e^{\sin x}$ at the point $x = \pi$.
- (b) The equation of a curve is $y = 2x^2 + 7x + 5$. Show that the line $y = 19x - 13$ is a tangent to the curve, and find the coordinates of the point of contact.

[Solution]

(a) $\frac{dy}{dx} = \cos x \cdot e^{\sin x}$

$$\left. \frac{dy}{dx} \right|_{x=\pi} = \cos \pi \cdot e^{\sin \pi}$$

$$= -e^0$$

$$= -1$$

Hence, the gradient of the normal to the curve is 1.

When $x = \pi$,

$$y = e^{\sin \pi} = e^0 = 1$$

$$1 = 1(\pi) + c$$

$$c = 1 - \pi$$

Hence, the equation of the normal to the curve is $y = x + 1 - \pi$.

(b) $\frac{dy}{dx} = 4x + 7$

Since the gradient of the tangent is 19, we have $4x + 7 = 19$

$$\therefore x = 3$$

Substitute $x = 3$ into the equation of the curve,

$$y = 2(3)^2 + 7(3) = 44$$

$$44 = 19(3) + c$$

$$c = 44 - 57 = -13$$

Hence, the equation of the tangent is $y = 19x - 13$ (shown)

From the above calculations, the coordinates of the point of contact is (3, 44).

Rate of Change

Usually, questions will imbue the element of time into rate of change questions. These questions are quite specific and easy to spot.

For example:

- x is increasing at a rate of $0.2 \text{ m/s} \Rightarrow \frac{dx}{dt} = 0.2$
- y is decreasing at a rate of $0.5 \text{ m/s} \Rightarrow \frac{dy}{dt} = -0.5$

To formulate the equation, start the process by writing the term on the left and expanding it out to the right. On the right hand side, the denominator of the first fraction and the numerator of the second fraction should be left blank.

$$\frac{dy}{dx} = \frac{dy}{\quad} \times \frac{\quad}{dx}$$

Both blanked terms must be the same. Most of the time it will be the time-element (dt) or one of the expressions in the question, giving us:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Although we can see it as the dt “cancels off”, mathematically it is actually inaccurate as $\frac{d}{dx}$ is a function, not a fraction.

Worked Example 9.10

A container is given such that when the depth of liquid in it is h cm, the volume of the liquid, V cm³, is given by the equation

$$V = \frac{1}{3(5-2h)^2}$$

Water is leaking out of the container at a constant rate of 5 cm³/s. Find the rate at which the depth is decreasing when $h = 2$. [S4 TKGS P1/2013 PRELIM Qn 11]

[Solution]

We are looking for the value of $\frac{dh}{dt}$ when $h = 2$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$V = \frac{1}{3}(5-2h)^{-2}$$

$$\frac{dV}{dh} = \frac{1}{3}(-2(5-2h)^{-3})(-2)$$

$$= \frac{4}{3(5-2h)^3}$$

$$\therefore \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dh}, \text{ when } h = 2$$

$$= \frac{1}{\left(\frac{4}{3((5-2(2))^3)}\right)} \times 5$$

$$= \frac{15}{4} \text{ cm/s}$$



Worked Example 9.11

A sphere has radius r and its surface area is given by A . At $r = 8$, the volume of a sphere increases at a rate of $20 \text{ m}^3\text{s}^{-1}$

- (a) Find the rate of increase of the radius r at this instant
- (b) Hence, find the rate of increase of the surface area A at this instant

[S4 SCGS P1/2011 PRELIM Qn 10]

[Solution]

- (a) To find the rate of increase,

$$\therefore \frac{dr}{dt} = \frac{dr}{dv} \times \frac{dv}{dt}$$

$$v = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dr} = \frac{4}{3}\pi r^3$$

$$= 4\pi r^2$$

$$\therefore \frac{dr}{dt} = \frac{1}{4\pi r^2} (20)$$

$$= \frac{5}{\pi r^2}$$

When $r = 8$,

$$\frac{dr}{dt} = \frac{5}{\pi(8)^2}$$

$$= \frac{5}{64\pi}$$

Hence, the rate of increase of the radius is $\frac{5}{64\pi} \text{ ms}^{-1}$

- (b) We want to find the rate of increase of the surface area A when $r = 8$

$$\Rightarrow \frac{dA}{dt}$$

$$\therefore \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$A = 4\pi r^2$$

$$\frac{dA}{dr} = 4(2)\pi r$$

$$= \frac{40\pi r}{64\pi}$$

$$= \frac{5}{8}r$$

When $r = 8$,

$$\frac{dA}{dt} = \frac{5}{8}(8) = 5$$

Hence, the rate of increase of the surface area A is $5 \text{ m}^2\text{s}^{-1}$

10 Integration and its Applications

Integration is seen as the reverse of differentiation. For example, we have:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\int nx^{n-1} dx = x^n$$

The idea now is to form a set of rules for us to be able to integrate easily. In general:

- The outcome of integration is the starting point for differentiation
- If we multiplied by a constant during differentiation, we will now **divide**
- We still apply chain rule

Indefinite Integrals

Type	x	$ax + b$ (Chain Rule)
Algebraic	$\int 0 dx = c$ (where c is an arbitrary constant)	
	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$	$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c$
Exponential	$\int e^x dx = e^x + c$	$\int e^{ax+b} dx = \frac{e^x}{a} + c$
Logarithmic	$\int \frac{1}{x} dx = \ln x + c$	$\int \frac{1}{ax + b} dx = \frac{\ln(ax + b)}{a} + c$
Trigonometric	$\int \cos x dx = \sin x + c$	$\int \frac{1}{\cos(ax + b)} dx = \frac{\sin(ax + b)}{a} + c$
	$\int \sin x dx = -\cos x + c$	$\int \sin(ax + b) dx = -\frac{\cos(ax + b)}{a} + c$
	$\int -\sin x dx = \cos x + c$	$\int -\sin(ax + b) dx = \frac{\cos(ax + b)}{a} + c$
	$\int \sec^2 x dx = \tan x + c$	$\int \sec^2(ax + b) dx = \frac{\tan(ax + b)}{a} + c$

Sum and Difference of Integrals

Similar to differentiation, assuming that u and v are functions of x , we have

$$\int (u + v) dx = \int u dx \pm \int v dx$$

Tip!

$$\int a f(x) \, dx = a \int f(x) \, dx$$

Similar to differentiation, removing the constants from the function helps simplify the integration process.

Worked Example 10.1

(a) $\int \frac{5-2x^2}{x^2} \, dx$

(b) $\int \frac{1}{4} \sqrt{x} \, dx$

(c) $\int \frac{2}{\sqrt{3-4x}} \, dx$

(d) $\int \tan^2 x \, dx$

[Solution]

$$\begin{aligned}
 \text{(a)} \quad \int \frac{5-2x^2}{x^2} \, dx &= \int \left(\frac{5}{x^2} - \frac{2x^2}{x^2} \right) \, dx \\
 &= \int \frac{5}{x^2} \, dx - \int 2 \, dx \\
 &= 5 \int \frac{1}{x^2} \, dx - 2x + c \\
 &= 5 \int x^{-2} \, dx - 2x + c \\
 &= 5 \left[\frac{x^{-1}}{-1} \right] - 2x + c \\
 &= -\frac{5}{x} - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{1}{4} \sqrt{x} \, dx &= \frac{1}{4} \int x^{\frac{1}{2}} \, dx \\
 &= \frac{\frac{1}{4} \left(x^{\frac{3}{2}} \right)}{\frac{3}{2}} \\
 &= \frac{x^{\frac{3}{2}}}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{2}{\sqrt{3-4x}} \, dx &= 2 \int \frac{1}{\sqrt{3-4x}} \, dx \\
 &= 2 \int (3-4x)^{-\frac{1}{2}} \, dx \\
 &= 2 \left[\frac{(3-4x)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)(-4)} \right] + c \\
 &= -\sqrt{3-4x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \tan^2 x \, dx &= \int (\sec^2 x - 1) \, dx \\
 &= \tan x - x + c
 \end{aligned}$$

Definite Integrals

When evaluating definite integrals, we first find the indefinite integral, then substitute the values of the upper limit and lower limit. In general, we have

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

where $f(x)$ is the function before integrating, and $F(x)$ is the result after integration.

Notice that there is no arbitrary constant $+c$ here. We can take it that the arbitrary constant cancels each other out.

Definite Integrals	Formulae
Equal limits	$\int_a^a f(x) dx = 0$
Interchanging limits	$\int_a^b f(x) dx = -\int_b^a f(x) dx$
Common limits	$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$

Worked Example 10.2

With reference to your answers in 10.1, evaluate the following definite integrals, leaving your answer in exact form.

(a) $\int_{-2}^3 \frac{5-2x^2}{x^2} dx$

(b) $\int_1^4 \left(\frac{1}{4}\sqrt{x}\right) dx$

(c) $\int_{-1}^0 \frac{2}{\sqrt{3-4x}} dx$

(d) $\int_0^{\frac{\pi}{4}} \tan^2 x dx$

[Solution]

(a) $\int_{-2}^3 \frac{5-2x^2}{x^2} dx = \left[-\frac{5}{x} - 2x\right]_{-2}^3 = \left(-\frac{5}{3} - 2(3)\right) - \left(-\frac{5}{2} - 2(2)\right) = -\frac{7}{6}$

(b) $\int_1^4 \left(\frac{1}{4}\sqrt{x}\right) dx = \frac{1}{4} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \frac{1}{4} \left[\frac{(4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{(1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right] = \frac{7}{6}$

(c) $\int_{-1}^0 \frac{2}{\sqrt{3-4x}} dx = \left[-\sqrt{3-4x}\right]_{-1}^0 = -\sqrt{3} + \sqrt{7}$

(d) $\int_0^{\frac{\pi}{4}} \tan^2 x dx = [\tan x - x]_0^{\frac{\pi}{4}} = \left[\tan\left(\frac{\pi}{4}\right) - \left(\frac{\pi}{4}\right)\right] - [\tan(0) - (0)] = 1 - \frac{\pi}{4}$

Differentiation & Integration Questions

This next section is dedicated to a specific group of questions, where students are expected to differentiate in the first part, and use that result to integrate in the second part.

These kinds of questions are always known to be one of the common pitfalls of students. Students always struggle with the latter half of the question, as they cannot find the relationship between what they have differentiated and what they need to integrate.

Example

Show that

$$\frac{d}{dx}[(x+1)\sqrt{2x-1}] = \frac{3x}{\sqrt{2x-1}}$$

Hence, evaluate the following integral

$$\int_5^{13} \frac{2x}{3\sqrt{2x-1}} dx$$

[S4 BHSS P2/2011 PRELIM Qn 2]

How to solve the “Hence” part?

Since this is a hence question, students need to remember that their solutions must use the previous part. The line below (to start of the question) is usually the most difficult to get

$$\therefore \int_5^{13} \frac{2x}{3\sqrt{2x-1}} dx = \frac{2}{3} \int_5^{13} \frac{1}{3} \left(\frac{3x}{\sqrt{2x-1}} \right) dx = \frac{2}{9} \int_5^{13} \left(\frac{3x}{\sqrt{2x-1}} \right) dx$$

We first factorise out the constant of $\frac{2}{3}$, and want to manipulate the fraction such that we get the result after differentiation.

However, by just multiplying the fraction by 3, we are changing the value of the fraction! So we multiply by another $\frac{1}{3}$ to compensate for the factor of 3.

Once we get the result after differentiation, we can then use the previous part to integrate.

$$\frac{2}{9} \int_5^{13} \left(\frac{3x}{\sqrt{2x-1}} \right) dx = \frac{2}{9} [(x+1)\sqrt{2x-1}]_5^{13} = \frac{2}{9} (14\sqrt{25} - 6\sqrt{9}) = \frac{104}{9}$$

Worked Example 10.3

Prove that

$$(2 - \sqrt{x})(4 + 2\sqrt{x} + x) = 8 - x\sqrt{x}$$

Hence, evaluate the following integral

$$\int_0^4 \frac{8 - x\sqrt{x}}{2 - \sqrt{x}} dx$$

[S4 FHSS P2/2015 PRELIM Qn 3]

[Solution]

$$\text{LHS} = (2 - \sqrt{x})(4 + 2\sqrt{x} + x)$$

$$= 8 + 4\sqrt{x} + 2x - 4\sqrt{x} - 2x - x\sqrt{x}$$

$$= 8 - x\sqrt{x}$$

$$= \text{RHS (shown)}$$

$$\therefore \int_0^4 \frac{8 - x\sqrt{x}}{2 - \sqrt{x}} dx = \int_0^4 \frac{(2 - \sqrt{x})(4 + 2\sqrt{x} + x)}{(2 - \sqrt{x})} dx$$

$$= \int_0^4 (4 + 2\sqrt{x} + x) dx$$

$$= \left[4x + \frac{2x^{\frac{3}{2}}}{(\frac{3}{2})} + \frac{x^2}{2} \right]_0^4$$

$$= \left[4x + \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 \right]_0^4$$

$$= [4(4) + \frac{4}{3}(4)^{\frac{3}{2}} + \frac{1}{2}(4)^2] - [4(0) + \frac{4}{3}(0)^{\frac{3}{2}} + \frac{1}{2}(0)^2]$$

$$= \frac{104}{3}$$

Worked Example 10.4

Differentiate the following with respect to x

$$\frac{x^2 + 1}{2x - 3}$$

Hence, evaluate the following

$$\int_0^2 \frac{x^2 - 3x}{3(2x - 3)^2} dx$$

[S4 RI P2/2012 PRELIM Qn 7]

[Solution]

$$\begin{aligned}\frac{d}{dx} \left[\frac{x^2 + 1}{2x - 3} \right] &= \frac{(2x - 3)(2x) - (x^2 + 1)(2)}{(2x - 3)^2} \\&= \frac{4x^2 - 6x - 2x^2 - 2}{(2x - 3)^2} \\&= \frac{4x^2 - 6x - 2x^2 - 2}{(2x - 3)^2} \\&= \frac{2x^2 - 6x - 2}{(2x - 3)^2}\end{aligned}$$

$$\begin{aligned}\therefore \int_0^2 \frac{x^2 - 3x}{3(2x - 3)^2} dx &= \frac{1}{3} \int_0^2 \frac{1}{2} \left(\frac{2x^2 - 6x}{(2x - 3)^2} \right) dx \\&= \frac{1}{6} \int_0^2 \left(\frac{2x^2 - 6x - 2}{(2x - 3)^2} + \frac{2}{(2x - 3)^2} \right) dx \\&= \frac{1}{6} \left[\frac{x^2 + 1}{2x - 3} \right]_0^2 + \frac{1}{3} \int_0^2 (2x - 3)^{-2} dx \\&= \frac{1}{6} \left(\frac{5}{1} - \frac{1}{-3} \right) + \frac{1}{3} \left[\frac{(2x - 3)^{-1}}{(-1)(2)} \right]_0^2 \\&= \frac{1}{6} \left(\frac{16}{3} \right) - \frac{1}{6} (1^{-1} - (-3)^{-1}) \\&= \frac{2}{3}\end{aligned}$$

Worked Example 10.5

- (a) Differentiate the following with respect to x

$$y = \sin^3 x$$

- (b) Show that

$$\cos^3 x = \cos x - \cos x \sin^2 x$$

- (c) Hence or otherwise, evaluate the following

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{\cos^3 x}{2} \right) dx$$

[S4 CGS P2/2011 PRELIM Qn 9]

[Solution]

- (a) $y = \sin^3 x$

$$\frac{dy}{dx} = 3 \sin^2 x \cos x$$

- (b) $\cos^3 x = \cos x - \cos x \sin^2 x$

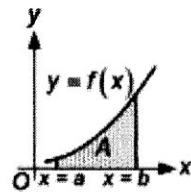
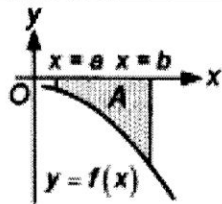
$$\begin{aligned} \text{LHS} &= \cos^3 x \\ &= \cos x (\cos^2 x) \\ &= \cos x (1 - \sin^2 x) \\ &= \cos x - \cos x \sin^2 x \\ &= \text{RHS (shown)} \end{aligned}$$

- (c) Hence,

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{\cos^3 x}{2} \right) dx &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^3 x \, dx \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos x - \cos x \sin^2 x) \, dx \\ &= \frac{1}{2} \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \, dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \sin^2 x \, dx \right] \\ &= \frac{1}{2} \left[\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \, dx - \frac{1}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3 \sin^2 x \cos x \, dx \right] \\ &= \frac{1}{2} \left\{ \left[\sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{1}{3} \left[\sin^3 x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \right\} \\ &= \frac{1}{2} \left\{ \left[\sin \left(\frac{\pi}{2} \right) - \sin \left(\frac{\pi}{6} \right) \right] - \frac{1}{3} \left[\sin^3 \left(\frac{\pi}{2} \right) - \sin^3 \left(\frac{\pi}{6} \right) \right] \right\} \\ &= \frac{1}{2} \left[\left(\frac{1}{2} \right) - \frac{1}{3} \left(\frac{7}{8} \right) \right] \\ &= \frac{5}{48} \end{aligned}$$

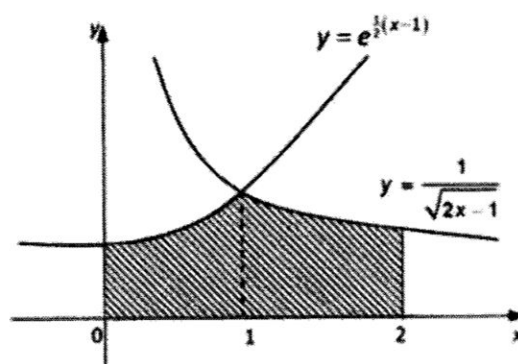
Area under by curve $y = f(x)$

One practical use of integration is to find the area under the curve, using the concept of definite integrals. However, if the area is “above” the curve, the definite integral gives a negative value. Hence, we have to multiply by -1 to get the area.

Region	Formulae	Diagram representation
Above the x -axis	$A = \int_a^b f(x) dx$	
Below the x -axis	$A = -\int_a^b f(x) dx$	

Worked Example 10.6

Find the area of the shaded region.



[Solution]

$$\begin{aligned}
 \text{Shaded region} &= \int_0^1 e^{\frac{1}{2}(x-1)} dx + \int_1^2 (2x-1)^{-\frac{1}{2}} dx \\
 &= \left[\frac{e^{\frac{1}{2}(x-1)}}{\left(\frac{1}{2}\right)} \right]_0^1 + \left[\frac{(2x-1)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)(2)} \right]_1^2 \\
 &= \left[2e^{\frac{1}{2}(x-1)} \right]_0^1 + \left[\sqrt{2x-1} \right]_1^2 \\
 &= \left[\left(2e^{\frac{1}{2}((1)-1)} \right) - \left(2e^{\frac{1}{2}((0)-1)} \right) \right] + \left[\left(\sqrt{2(2)-1} \right) - \left(\sqrt{2(1)-1} \right) \right] \\
 &= \left(2 - 2e^{-\frac{1}{2}} \right) + (\sqrt{3} - 1) \\
 &= \sqrt{3} + 1 - \frac{2}{\sqrt{e}}
 \end{aligned}$$

Worked Example 10.7

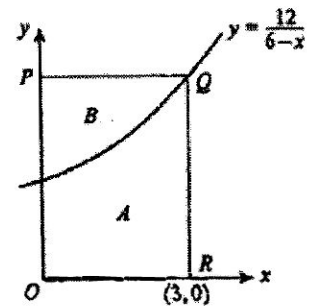
The diagram shows the rectangle $OPQR$ and part of the curve

$$y = \frac{12}{6-x}$$

Given that R is $(3,0)$

- (a) Find the exact area of region A .
- (b) Hence, find the area of region B .

[S4 ANDSS P2/2012 PRELIM Qn 12(a)]



Solution

$$\begin{aligned} \text{(a) Area} &= \int_0^3 \frac{12}{6-x} dx \\ &= [-12 \ln(6-x)]_0^3 \\ &= [-12 \ln(3) + 12 \ln(6)] \\ &= -12 \ln 3 + 12 \ln 3 + 12 \ln 2 \\ &= 12 \ln 2 \text{ units}^2 \end{aligned}$$

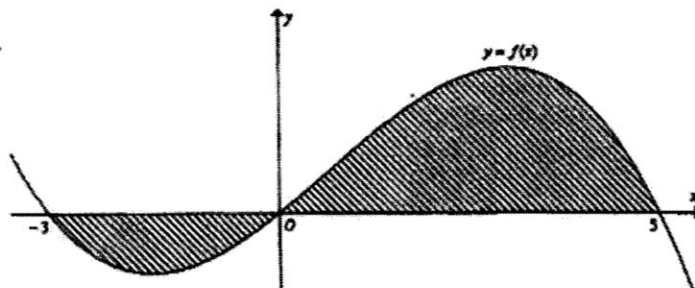
$$\text{(b) When } x = 3, y = \frac{12}{6-3} = 4$$

Hence, $Q(3,4)$

$$\begin{aligned} \text{Area of B} &= 3(4) - 12 \ln 2 \\ &= 12(1 - \ln 2) \end{aligned}$$

Worked Example 10.8

The diagram shows the graph of $y = f(x)$



(a) Given that

$$\int_{-3}^0 f(x) dx = p$$

$$\int_0^5 f(x) dx = q$$

Evaluate the following, in terms of p and/or q

(i) $\int_{-3}^5 3f(x) dx$

(ii) $\int_0^{-3} 1 - f(x) dx$

(b) Hence, find the area of the shaded region bounded by the curve $y = f(x)$ and the x -axis, in terms of p and/or q

[S4 ANDSS P2/2012 PRELIM Qn 12(b)]

Solution

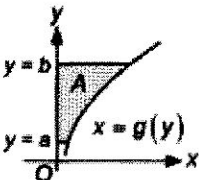
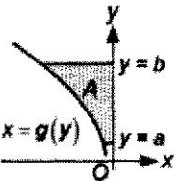
$$\begin{aligned} \text{(a) (i) } \int_{-3}^5 3f(x) dx &= 3 \left[\int_{-3}^0 f(x) dx + \int_0^5 f(x) dx \right] \\ &= 3(p + q) \end{aligned}$$

$$\begin{aligned} \text{(ii) } \int_0^{-3} 1 - f(x) dx &= \int_{-3}^0 f(x) - 1 dx \\ &= \int_{-3}^0 f(x) - \int_{-3}^0 1 dx \\ &= q - [x]_{-3}^0 \\ &= q - [0 - (-3)] \\ &= q - 3 \end{aligned}$$

$$\begin{aligned} \text{(b) Area} &= \int_{-3}^0 f(x) dx + \int_0^5 f(x) dx \\ &= -p + q \end{aligned}$$

Area under by curve $x = f(y)$

Using integration, we are also able to find the area bounded by the curve and the y -axis. However, we have to ensure that x is made the subject, allowing us to integrate a function in y , and the bounds are on the y -axis.

Region	Formulae	Diagram representation
Right of y -axis	$A = \int_a^b g(y) dy$	
Left of y -axis	$A = - \int_a^b g(y) dy$	

Worked Example 10.7 (redo)

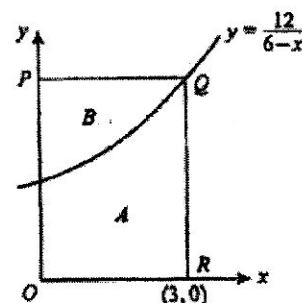
The diagram shows the rectangle $OPQR$ and part of the curve

$$y = \frac{12}{6-x}$$

Given that R is $(3,0)$, show that the area of region B is

$$12(1 - \ln 2) \text{ units}^2.$$

[S4 ANDSS P2/2012 PRELIM Qn 12(a)]



Solution

$$6y - xy = 12$$

$$x = \frac{6y - 12}{y} = 6 - \frac{12}{y}$$

$$\text{When } x = 3, y = \frac{12}{6-3} = 4 \Rightarrow Q(3,4)$$

$$\text{When } x = 0, y = \frac{12}{6-0} = 2$$

$$\text{Area of } B = \int_2^4 6 - \frac{12}{y} dy$$

$$= [6y - 12 \ln y]_2^4$$

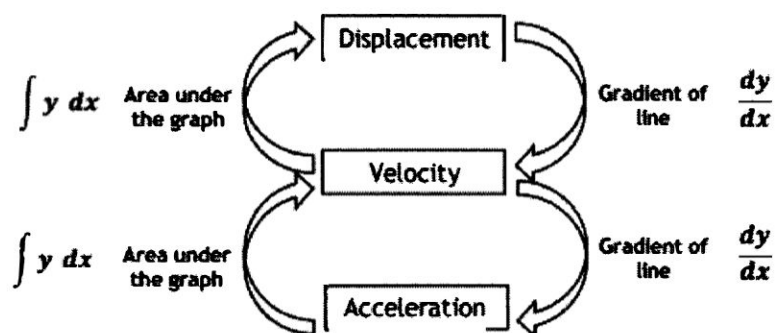
$$= 24 - 12 \ln 4 - 12 + 12 \ln 2$$

$$= 12 - 24 \ln 2 + 12 \ln 2$$

$$= 12 - 12 \ln 2 = 12(1 - \ln 2)$$

Kinematics

Although some students feel like they are disadvantaged if they do not take Physics, we actually require very little Physics knowledge. The relationship between each measurement can be found in the diagram below.



Distance and speed are similar to displacement and velocity, with the former being scalar quantities and the latter being vector quantities. Vector quantities are scalar quantities with an additional dimension - direction. Hence, vector quantities can be negative.

Worked Example 10.9

A particle travels in a straight line so that its velocity $v \, \text{ms}^{-1}$, from a fixed point O is given by the equation $v = 3(t^2 - 4t + 3)$, where t is the time in seconds after passing O . Find

- (a) the value of t when the particle is instantaneously at rest
- (b) the acceleration of the particle when $t = 0.5$
- (c) the displacement of the particle at $t = 1$
- (d) the total distance travelled by the particle in the first 4 seconds

[S4 CHS P1/2012 PRELIM Qn 4]

[Solution]

(a) $v = 0 \Rightarrow 3(t^2 - 4t + 3) = 0$

$$t^2 - 4t + 3 = 0$$

$$(t - 1)(t - 3) = 0$$

$$\therefore t = 1 \quad \text{or} \quad t = 3$$

(b) $\frac{dv}{dt} = 6t - 12$

$$\left. \frac{dv}{dt} \right|_{t=0.5} = 6(0.5) - 12 = -9 \, \text{ms}^{-2}$$

$$(c) \int v \, dt = \int (6t^2 - 18t + 12) dt$$

$$= 2t^3 - 9t^2 + 12t + c$$

$$\therefore s = 2t^3 - 9t^2 + 12t + c$$

When $t = 0, s = 0$

$$(0) = 2(0)^3 - 9(0)^2 + 12(0) + c$$

$$\therefore c = 0$$

$$\therefore s = 2t^3 - 9t^2 + 12t$$

When $t = 1,$

$$\therefore s = 2(1)^3 - 9(1) + 12(1) = 5 \text{ m}$$

When $t = 2,$

$$\therefore s = 2(2)^3 - 9(2) + 12(2) = 4 \text{ m}$$

When $t = 3,$

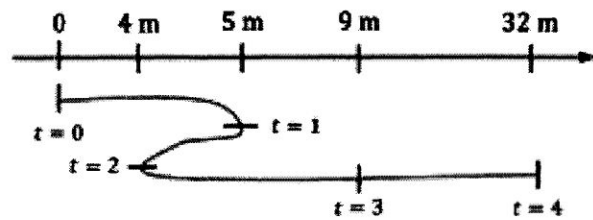
$$\therefore s = 2(3)^3 - 9(3) + 12(3) = 9 \text{ m}$$

When $t = 4,$

$$\therefore s = 2(4)^3 - 9(4) + 12(4) = 32 \text{ m}$$

Based on the diagram drawn, total distance travelled by the particle = $32 + 1 + 1$

$$= 34 \text{ m}$$



$$(d) v = 6t^2 - 18t + 12$$

We want to find the time interval when velocity is increasing: $\frac{dv}{dt} > 0$

$$\therefore \frac{dv}{dt} = 12t - 18$$

$$12t - 18 > 0$$

$$t > \frac{18}{12}$$

$$> 1.5 \text{ sec}$$

Tip!

Part (d) is a little unorthodox if students do not understand what it means by the velocity is increasing. Since the velocity is increasing, it implies that the particle is accelerating. Hence, the differential function of the velocity (acceleration) is greater than 0.

An obvious hint is to remember that when something is increasing, the differential function of that something is greater than 0.

Worked Example 10.10

A car moves along a straight road with velocity $v \text{ ms}^{-1}$. On reaching its maximum velocity, the brakes are applied and the car decelerates at a constant rate of 6 ms^{-2} , before coming to rest.

Given that $v = 8t - \frac{1}{2}t^2 + 4$, find

- (a) the maximum velocity reached by the car,
- (b) the instantaneous acceleration 3 seconds into the journey,
- (c) the total time taken for the whole journey,
- (d) the total distance travelled by the car. [S4 TKSS P2/2018 PRELIM Qn 6]

[Solution]

- (a) At the maximum velocity, $\frac{dv}{dt} = 0$

$$\frac{dv}{dt} = 8 - t = 0$$

$$\therefore t = 8$$

$$v|_{t=8} = 8(8) - \frac{1}{2}(8)^2 + 4 = 36 \text{ ms}^{-1}$$

- (b) $a = \frac{dv}{dt} = 8 - t$

$$a|_{t=3} = 5 \text{ ms}^{-2}$$

- (c) Since the car decelerates at a constant rate, $\frac{dv}{dt} = -6$.

$$\text{Time taken to decelerate until } 0 = 36 \div 6 = 6 \text{ sec}$$

$$\text{Total time taken} = 8 + 6 = 14 \text{ sec}$$

OR

$$\therefore \frac{36-0}{8-t_1} = -6$$

$$t_1 = 14$$

$$\therefore \text{Total time for the whole journey} = 14 \text{ sec}$$

- (d) Area under the curve $= \int_0^8 \left(8t - \frac{1}{2}t^2 + 4 \right) dt + \frac{1}{2}(6)(36)$

$$= \left[4t^2 - \frac{1}{6}t^3 + 4t \right]_0^8 + 108$$

$$= \left[4(8)^2 - \frac{1}{6}(8)^3 + 4(8) \right] - \left[4(0)^2 - \frac{1}{6}(0)^3 + 4(0) \right] + 108$$

$$= \frac{932}{3} \text{ m}$$

11 Proof in Plane Geometry

There are many properties learnt in E-Math that we can use in this chapter. The two new concepts we will learn in this chapter are the **Mid-Point Theorem** and **Alternate Segment Theorem**.

Properties involving a pair of parallel lines and a transversal

Property	Characteristic	Diagram Representation
Corresponding angles	Corresponding angles are equal $\alpha = \beta$	
Alternative angles	Alternative angles are equal $\alpha = \beta$	
Interior angles	Sum of interior angles add up to 180° $\alpha + \beta = 180^\circ$	

Definition of transversal: a line that intersects two or more other (parallel) lines

Properties involving triangles

Property	Characteristics	Diagram Representation
Sum of 3 angles in the triangle is 180°	$\alpha + \beta + \gamma = 180^\circ$	
Exterior angle is equal to the sum of interior opposite angles	$\alpha + \beta = \gamma$	
Isosceles Triangle	$AB = AC$ $\angle ACB = \angle ABC$	
Equilateral Triangle	$AB = BC = CA$ $\angle ACB = \angle ABC = \angle BAC = 60^\circ$	

Properties of quadrilaterals

Type	Characteristics	Diagram Representation
Square	- All sides are equal - All interior angles are right angles - Diagonals are equal and cut at right angles - Diagonals bisect the interior angles and makes angles of 45° with the sides	
Rectangle	- Opposite sides are equal - All interior angles are right angles - Diagonals are equal in length	
Parallelogram	- Opposite sides are equal - Opposite angles are equal - Diagonals are equal in length	
Rhombus	- All sides are equal in length - Opposite angles are equal - Diagonals cut at right angles and bisect its interior angles - Diagonals are equal in length	
Trapezium	1 pair of parallel sides	

Tests for congruent triangles

Name	Test	Diagram Representation
'side-side-side' SSS	$AB = XY$ $BC = YZ$ $AC = XZ$	
'angle-side-angle' ASA	$AB = XY$ $\angle CAB = \angle ZXY$ $\angle ABC = \angle XYZ$	
'side-angle-side' SAS	$AB = XY$ $BC = YZ$ $\angle ABC = \angle XYZ$	
'right angle-hypotenuse-side' RHS	$AB = XY$ $BC = YZ$ $\angle ACB = \angle XYZ = 90^\circ$	

Take Note

For the SAS test, the angle chosen must be the included angle in-between the 2 chosen sides for the test

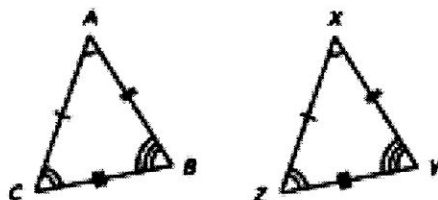
Tests for similar triangles

Name	Test	Diagram Representation
'angle-angle-angle' AAA	$\angle CAB = \angle ZXY$ $\angle ABC = \angle XYZ$ $\angle BCA = \angle YZX$	
Corresponding sides same ratio	$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$	
Ratio of 2 sides same + 1 included angle between the 2 chosen sides	$\angle ABC = \angle XYZ$ $\frac{AB}{XY} = \frac{BC}{YZ}$	

Take Note

The naming conventions of Congruency & Similarity Questions are important! Many students make mistake when defining the sides that they want to use in their arguments

Example: To show that $\triangle ABC \equiv \triangle XYZ$



To show that 2 sides are equal in length, the direction of how students define the length must stay consistent (Either all Clockwise or Anti-Clockwise)

$$AC = XZ \quad \text{OR} \quad CA = ZX$$

To show that 2 angles are equal in size, the direction of how students define the angles must stay consistent (Either all Clockwise or Anti-Clockwise)

$$\angle ABC = \angle XYZ \quad \text{OR} \quad \angle CBA = \angle ZYX$$

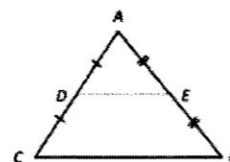
Midpoint Theorem

Midpoint Theorem

If $AD = DC$ and $AE = EB$

then

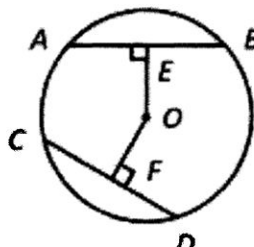
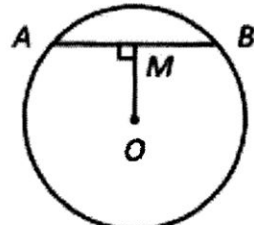
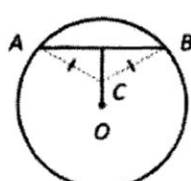
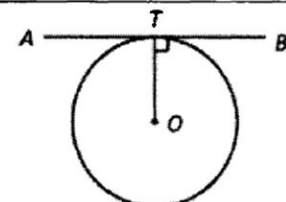
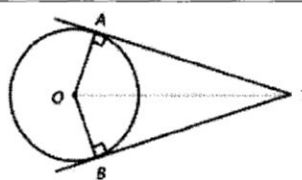
$$DE \parallel CB \text{ \& } DE = \frac{1}{2}CB$$



Midpoint theorem states that if a line segment (in this case DE) is formed by connecting two midpoints of the two sides of a triangle, then that segment must be parallel to the third side, and its length is exactly half of the third side.

This theorem can be easily proven using similar triangles $\triangle ADE$ is similar to $\triangle ACB$, giving us the ratio $\frac{AD}{AC} = \frac{AE}{AB} = \frac{DE}{CB} = \frac{1}{2}$. It is also uncommon, and not easy to spot in exams.

Circle Properties (lengths)

Theorem	Property	Diagram Representation
Chord Theorem	If two chords are equidistant from the centre of the circle ($OE = OF$) then the length of the chords are the same ($AB = CD$)	
	If the length of two chords are the same ($AB = CD$) then they are equidistant from the centre of the circle ($OE = OF$)	
Perpendicular Bisector Theorem	A line segment drawn from the centre to the midpoint of the chord is perpendicular to the chord If $AM = MB$, then $AB \perp OM$	
	The perpendicular bisector of a chord passes through the centre	
	Every point on the perpendicular bisector of a line segment is equidistant from the end points of the segment $AC = BC$	
Tangent Theorem	A line perpendicular to the tangent at the point of contact passes through the centre of the circle $\angle OTA = \angle OTB = 90^\circ$	
	Tangents drawn from an external point to a circle are equal $AT = BT, \angle AOT = \angle BOT$	

Circle Properties (angles)

Property	Diagram Representation
Angles in the same segment $\angle BAO = \angle DCO$	

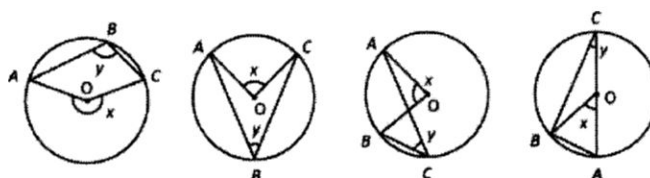
Tip!

Always look out for the “butterfly” shape to use this property. The only condition is that all 4 vertices of this “butterfly” must touch the circumference of the circle

Angle at centre is 2 times the angle at the circumference $\angle AOC = \angle ABC$ $x = 2y$	
--	--

Tip!

There are many variations of this theorem. Always look out for this arrow shape to use this property. This arrow head must also consist of the centre of the circle



Common Mistakes

Many students get tricked by the third figure as it looks like a “butterfly” - angle in same segment. Recall that angle in the same segment only applies if all 4 vertices touch the circumference.

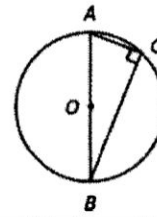
Many students get also tricked by this figure



Many students think that $\alpha = 2\beta$ when in actual fact there is no special relationship between α and β UNLESS if the 2 lines above are tangents that extend to a point, then

$$\alpha + \beta = 180^\circ$$

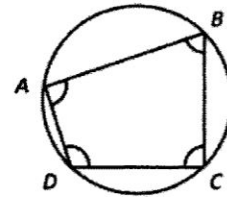
Angle in a semicircle
 $\angle ACB = 90^\circ$



Tip!

This property can only be used if AOB is the diameter of the circle. We should not assume unless it is explicitly given in the question!

Opposite angles of a cyclic quadrilateral
 are 180° (supplementary)
 $\angle ADC + \angle ABC = 180^\circ$

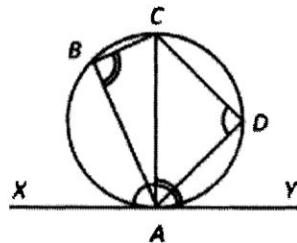


Tip!

This property can only be used when the quadrilateral is a cyclic quadrilateral - all 4 vertices must lie on the circumference of the circle. We should not assume unless it is explicitly given in the question!

Alternate Segment Theorem

This is a new theorem that is **highly** tested in exams. To use it, find a tangent to the circle, and find a triangle (inscribed in the circle) with one of its vertices touching the tangent.



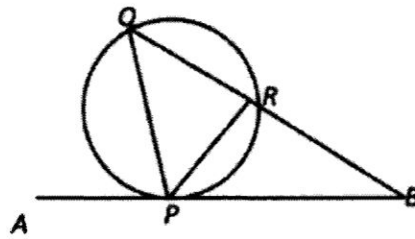
In the figure above, XY is tangent to the circle at point A , and $\triangle CBA$ and $\triangle CDA$ are triangles that share a point with the tangent (point A)

Alternate segment theorem states that the angle formed between the triangle and the tangent is equal to the angle in the opposite segment. In the case above, we have

$$\angle CAY = \angle CBA \text{ and } \angle BAX = \angle BCA$$

$$\angle CAX = \angle CDA \text{ and } \angle DAY = \angle DCA$$

Worked Example 11.1



The diagram above shows a triangle PQR inscribed in a circle. It is given that AB is a tangent to the circle at P , and PR bisects $\angle QPB$. Prove that

- (a) $RQ = RP$
- (b) $BP^2 = QB \times BR$

[Solution]

- (a) Let $\angle BPR = \theta^\circ$

$$\angle BPR = \angle PQR \text{ (alternate segment theorem)}$$

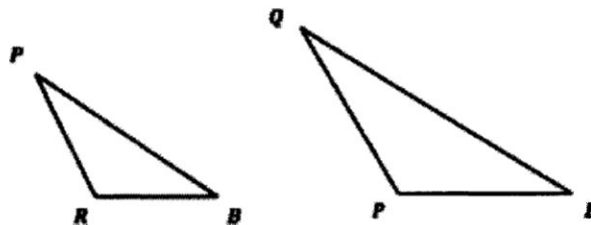
$$\text{Since } PR \text{ bisects } \angle QPB, \angle BPR = \angle RPQ = \theta^\circ$$

$$\text{Hence, in } \triangle PQR, \angle RPQ = \angle PQR = \theta^\circ$$

This implies that $\triangle PQR$ is an isosceles triangle

$$\therefore RQ = RP$$

- (b)



$$\angle BRP = 2\theta^\circ \text{ (exterior } \angle = \text{sum of interior opposite } \angle \text{)}$$

$$\angle BPQ = 2\theta^\circ (\angle BPR = \angle RPQ)$$

$$\angle BPR = \angle PQR = \theta \text{ (in alternate segment)}$$

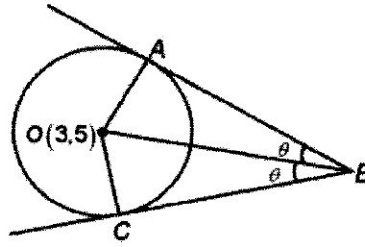
$\therefore \triangle BPR$ is similar to $\triangle BQP$ (AA similarity test)

$$\text{By similarity ratios, } \frac{BP}{BR} = \frac{PQ}{RP} = \frac{QB}{PB}$$

$$\frac{BP}{BR} = \frac{QB}{PB}$$

$$BP^2 = QB \times BR \text{ (shown)}$$

Worked Example 11.2



The diagram below shows a circle with centre $O(3,5)$. It is given that AB and CB are tangents to the circle, $\angle ABO = \angle OBC = \theta^\circ$, and the length of OB is 2 units.

Show that the equation of the circle can be expressed as

$$\frac{x^2 + y^2}{2} = 3x + 5y - 16 - \cos 2\theta$$

[Solution]

Since BA is a tangent to the circle, $\angle OAB = 90^\circ$, $\triangle OAB$ is a right-angled triangle

Let the radius of the circle be r ,

$$\sin \theta = \frac{r}{2}$$

$$r = 2 \sin \theta$$

Hence, using the standard equation of a circle: $(x - a)^2 + (y - b)^2 = r^2$

$$(x - 3)^2 + (y - 5)^2 = (2 \sin \theta)^2$$

$$x^2 - 6x + 9 + y^2 - 10y + 25 = 4 \sin^2 \theta$$

$$x^2 + y^2 = 6x + 10y - 34 + 4 \sin^2 \theta$$

$$\frac{x^2 + y^2}{2} = 3x + 5y - 17 + 2 \sin^2 \theta$$

$$\frac{x^2 + y^2}{2} = 3x + 5y - 16 - 1 + 2 \sin^2 \theta$$

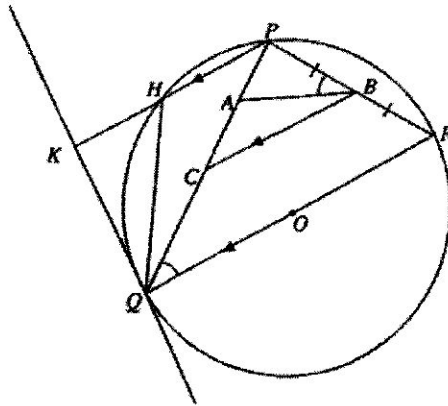
$$\frac{x^2 + y^2}{2} = 3x + 5y - 16 - (1 - 2 \sin^2 \theta)$$

$$\frac{x^2 + y^2}{2} = 3x + 5y - 16 - \cos 2\theta \text{ (shown)}$$

Notes

This is a fairly simply manipulation question, which only makes use of one circle property concept. However, it requires students to use one trigonometric property, and know the standard form of the equation of a circle, which makes it more challenging.

Worked Example 11.3



In the diagram above, it is given that QR is a diameter of the circle, B is the midpoint of PR and $\angle PQR = \angle PBA$. PK , BC and QR are parallel to each other. KQ is a tangent to the circle at point Q . Prove that

(a) $\triangle PBA$ is similar to $\triangle PQR$

(b) $PA = \frac{PR^2}{2PQ}$

(c) $PC:PQ = 1:2$

(d) $\angle KHQ = \angle PRQ$

[S4 CCHS(Y) P2/2012 PRELIM Qn 2]

[Solution]

(a) $\angle APB = \angle RPQ$ (common angle) (A)

$$\angle PBA = \angle PQR \text{ (given) (A)}$$

By the AA similarity test, $\triangle PBA$ is similar to $\triangle PQR$ (shown)

(b) Since B is the midpoint of PR , by the midpoint theorem,

$$PB = \frac{1}{2}PR$$

$$\frac{PA}{PR} = \frac{\frac{1}{2}PR}{PQ}$$

$$PA = \frac{PR^2}{2PQ} \text{ (shown)}$$

(c) Since BC is parallel to RQ and $PB:PR = 1:2$

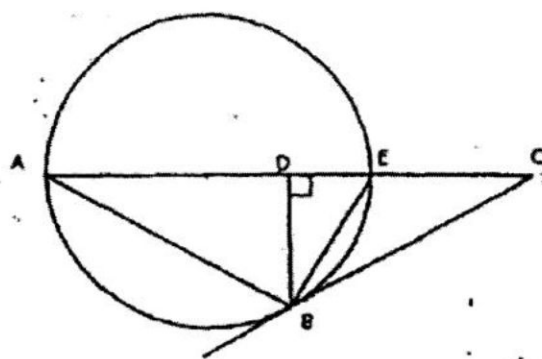
$$PC:PQ = 1:2 \text{ (shown)}$$

(d) $\angle PHQ = 180^\circ - \angle KHQ$ (angles on a straight line)

$$= 180^\circ - \angle PRQ \text{ (opposite angles in a cyclic quadrilateral)}$$

$$\therefore \angle KHQ = \angle PRQ \text{ (shown)}$$

Worked Example 11.4



In the figure shown above, the tangent to the circle at B meets the diameter AE produced at C . Given that BD is perpendicular to AC , prove that

- (a) EB bisects $\angle CBD$
- (b) $\triangle ABC$ is similar to $\triangle BEC$
- (c) $AB \times BC = AC \times BE$

[S4 SJI P1/2014 PRELIM Qn 11(a)]

[Solution]

- (a) $\angle CBE = \angle BAE$ (alternate segment theorem)

$\angle ABE = 90^\circ$ (angle in a semicircle, AE is the diameter)

$$\angle DBE = \angle ABE - \angle ABD$$

$$= \angle ADB - \angle ABD$$

$$= \angle BAE$$

$$= \angle CBE$$

Hence, EB bisects $\angle CBD$ (shown)

- (b) $\angle CBE = \angle CAB$ (alternate segment theorem) (A)

$\angle ACB$ is a common angle (A)

By the AA similarity test, $\triangle ABC$ is similar to $\triangle BEC$ (shown)

- (c) Since $\triangle ABC$ is similar to $\triangle BEC$

$$\frac{AB}{BE} = \frac{AC}{BC}$$

$$AB \times BC = AC \times BE \text{ (shown)}$$

