NAME:	Solution	( )		CLASS:	

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YISHUN TOWN SECONDARY SCHOOL

PRELIMINARY EXAMINATION 2024
SECONDARY 4 EXPRESS / 5 NORMAL ACADEMIC
ADDITIONAL MATHEMATICS PAPER 1 (4049/01)

DATE : 21 August 2024 **DAY : Wednesday**

DURATION : 2 h 15 min **MARKS : 90**

READ THESE INSTRUCTIONS FIRST

Do not turn over the cover page until you are told to do so.
Write your name, class and class index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total marks for this paper is 90.

	MARKS	
	OBTAINED	FULL
1		4
2		4
3		5
4		6
5		6
6		6
7		8
8		7
9		8
10		9
11		10
12		8
13		9
TOTAL		90

This question paper consists of **21** printed pages and **1** blank page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for $\Delta\ ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

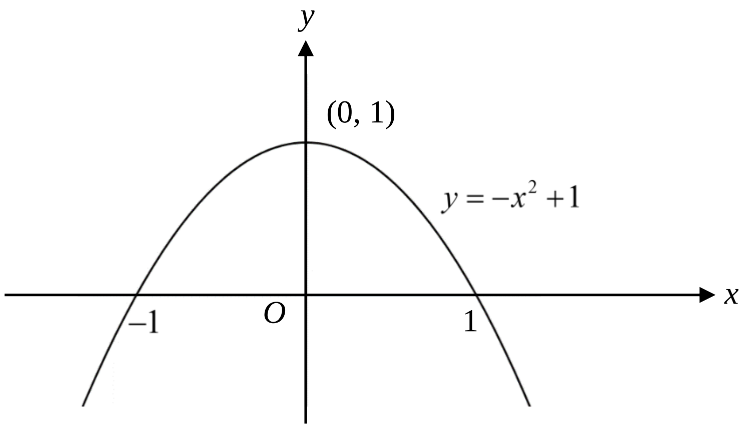
$$\Delta = \frac{1}{2} ab \sin C$$

Answer all questions.

1 **(a)** Write down the turning point of the curve $y = -x^2 + 1$, stating clearly whether it is a maximum or minimum point. [2]

Maximum point (0, 1)

(b) In a clearly labelled diagram, sketch the graph of $y = -x^2 + 1$, indicating clearly the coordinates of the turning point and axial intercepts. [2]



2 A cylinder has radius $(2-\sqrt{2})$ cm and volume $\pi(7\sqrt{8}-18)$ cm^3. Find, without using a calculator, the height of the cylinder, in cm, in the form $(a+b\sqrt{2})$, where a and b are integers. [4]

Volume, $\pi(2-\sqrt{2})^2 h = \pi(7\sqrt{8}-18)$

$$
\begin{aligned}
h &= \frac{\pi\left(7\sqrt{8}-18\right)}{\pi\left(2-\sqrt{2}\right)^2} \\
&= \frac{7\sqrt{8}-18}{4-4\sqrt{2}+2} \\
&= \frac{14\sqrt{2}-18}{6-4\sqrt{2}} \times \frac{6+4\sqrt{2}}{6+4\sqrt{2}} \\
&= \frac{84\sqrt{2}+112-108-72\sqrt{2}}{36-32} \\
&= \frac{12\sqrt{2}+4}{4} \\
&= 1+3\sqrt{2}
\end{aligned}
$$

The height of the cylinder is $(1+3\sqrt{2})$ cm.

3 **(a)** Explain why the principal value of $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ cannot be $-\frac{\pi}{4}$. [1]

For any $y = \cos\theta$, $0 \le \cos^{-1} y \le \pi$,

Hence principal value cannot be $-\frac{\pi}{4}$.

*Accept since $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$, $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) \ne -\frac{\pi}{4}$.

(b) It is given that $\cos\theta = -p$, where $p > 0$, and θ is an obtuse angle. Without using a calculator and leaving your answers in terms of *p*, find the value of

(i) $\cos\left(\frac{\pi}{2} - \theta\right)$, [2]

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$= \sqrt{1 - p^2}$$

(ii) $\cot^2\theta$. [2]

$$\cot^2\theta = \frac{1}{\tan^2\theta}$$
$$= \frac{p^2}{1 - p^2}$$

4 **(a)** Find the remainder when $4x^3+3x^2+11x-3$ is divided by $x+2$. [2]

Let $\mathrm{P}(x)=4x^3+3x^2+11x-3$

$$\begin{aligned}\mathrm{P}(-2)&=4(-8)+3(4)+11(-2)-3\\&=-45\end{aligned}$$

The remainder is -45.

(b) When a polynomial $\mathrm{g}(x)$ is divided by $x-1$ and $2x-1$, the remainders are 1 and 2 respectively. Find the remainder when $\mathrm{g}(x)$ is divided by $(x-1)(2x-1)$. [4]

Degree of remainder is less than degree of divider.
Let the remainder be $ax+b$.

Then $\mathrm{g}(x)=(x-1)(2x-1)\,\mathrm{Q}(x)+ax+b$

$\mathrm{g}(1)=1$

$a+b=1\cdots\cdots(1)$

$\mathrm{g}\left(\frac{1}{2}\right)=2$

$\frac{1}{2}a+b=2\cdots\cdots(2)$

$(1)-(2):\quad \frac{1}{2}a=-1$

$a=-2$

Sub. into (1): $b=3$

Therefore the remainder is $-2x+3$.

5 **(a)** Prove the identity

$$\cos^2 A\cos^2 B - \sin^2 A\sin^2 B - \sin(A+B)\sin(A-B) = \cos 2A. \qquad [4]$$

Proof 1

$$\begin{aligned}\text{LHS} &= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B - \sin(A+B)\sin(A-B)\\ &= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B\\ &\quad -(\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)\\ &= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B - \sin^2 A\cos^2 B + \cos^2 A\sin^2 B\\ &= \cos^2 A\left(\cos^2 B + \sin^2 B\right) - \sin^2 A\left(\sin^2 B + \cos^2 B\right)\\ &= \cos^2 A - \sin^2 A\\ &= \cos 2A\\ &= \text{RHS} \quad \text{(proven)}\end{aligned}$$

Proof 2

$$\begin{aligned}\text{LHS} &= \left(\cos A\cos B\right)^2 - \left(\sin A\sin B\right)^2 - \sin(A+B)\sin(A-B)\\ &= \left(\cos A\cos B + \sin A\sin B\right)\left(\cos A\cos B - \sin A\sin B\right)\\ &\quad - \sin(A+B)\sin(A-B)\\ &= \cos(A-B)\cos(A+B) - \sin(A+B)\sin(A-B)\\ &= \cos(A+B+A-B)\\ &= \cos 2A\\ &= \text{RHS} \quad \text{(proven)}\end{aligned}$$

(b) Hence find the value of A for $-\dfrac{\pi}{2} \le A \le 0$ such that

$$\cos^2 A\cos^2 B - \sin^2 A\sin^2 B - \sin(A+B)\sin(A-B) = -0.241\ . \qquad [2]$$

$\cos^2 A\cos^2 B - \sin^2 A\sin^2 B - \sin(A+B)\sin(A-B) = -0.241$

$\cos 2A = -0.241$

Reference angle $= 1.3274$

$2A = -(\pi - 1.3274)$

$A = -0.907$ (3 s.f.)

6 **(a)** The curve $\frac{x}{y}+y=1$ and the line $x-2y+6=0$ intersect at the points A and B. Find the x-coordinate of A and of B. [3]

$\frac{x}{y}+y=1 \quad \cdots\cdots (1)$

$x-2y+6=0$

$x=2y-6 \quad \cdots\cdots (2)$

Sub. (2) into (1),

$\frac{2y-6}{y}+y=1$

$2y-6+y^2=y$

$y^2+y-6=0$

$(y-2)(y+3)=0$

$y=2 \quad \text{or} \quad y=-3$

$x=-2 \qquad\quad x=-12$

(b) Hence solve for the value(s) of x in the simultaneous equations

$$\frac{\lg x}{\lg y} + \lg y = 1,$$

$$\lg \frac{x}{y^2} = -6.$$

[3]

$$\frac{\lg x}{\lg y} + \lg y = 1 \quad \cdots\cdots(1)$$

$$\lg \frac{x}{y^2} = -6$$

$$\lg x - 2\lg y + 6 = 0 \quad \cdots\cdots(2)$$

From **(a)**, $\lg x = -2$ or $\lg x = -12$

$x = 10^{-2}$ $\qquad$ $x = 10^{-12}$

7 The equation of a curve is $y = \dfrac{\ln x^2}{x^2}$, where $x > 0$.

(a) Show that the gradient of the curve can be written as $\dfrac{p + q\ln x}{x^r}$, where p, q and r are integers. [2]

$$y = \frac{2\ln x}{x^2}$$

$$\frac{dy}{dx} = \frac{\frac{2}{x}(x^2) - 4x\ln x}{x^4}$$

$$= \frac{2x - 4x\ln x}{x^4}$$

$$= \frac{2 - 4\ln x}{x^3} \quad \text{(Shown)}$$

(b) Find the range of values of $\ln x$ for the graph to be decreasing. [3]

For decreasing graph, $\dfrac{dy}{dx} < 0$

$$\frac{2 - 4\ln x}{x^3} < 0$$

Since $x > 0$, $x^3 > 0$.

Hence $2 - 4\ln x < 0$

$4\ln x > 2$

$\ln x > \dfrac{1}{2}$

(c) The curve passes through the x-axis at point A. Find the equation of the normal to the curve at A. [3]

At A, $y = 0$, $\dfrac{\ln x^2}{x^2} = 0$

$$\ln x^2 = 0$$

$$x = 1 \text{ (since } x > 0)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2-0}{1}$$

$$= 2$$

Equation of normal is $y = -\dfrac{1}{2}(x-1)$.

8 **(a)** Express $\dfrac{1-3x+10x^2}{x(1-2x)^2}$ in partial fractions. [4]

Let $\dfrac{1-3x+10x^2}{x(1-2x)^2}=\dfrac{A}{x}+\dfrac{B}{1-2x}+\dfrac{C}{(1-2x)^2}$

$1-3x+10x^2 = A(1-2x)^2 + Bx(1-2x) + Cx$

When $x=0$, $A=1$

When $x=\dfrac{1}{2}$, $1-\dfrac{3}{2}+10\left(\dfrac{1}{4}\right)=\dfrac{1}{2}C$

$C=4$

When $x=1$, $1-3+10=A-B+C$

$8=1-B+4$

$B=-3$

$$\frac{1-3x+10x^2}{x(1-2x)^2}=\frac{1}{x}-\frac{3}{1-2x}+\frac{4}{(1-2x)^2}$$

(b) Hence, find the integral of $\dfrac{1-3x+10x^2}{x(1-2x)^2}$. [3]

$$\int \frac{1-3x+10x^2}{x(1-2x)^2}\,\mathrm{d}x=\int\left[\frac{1}{x}-\frac{3}{1-2x}+\frac{4}{(1-2x)^2}\right]\mathrm{d}x$$

$$=\ln x+\frac{3}{2}\ln(1-2x)+\frac{2}{1-2x}+c$$

9 **(a)** Show that the solution of the equation $8^x = 3(16^{x+1})$ is $-\dfrac{\ln 48}{\ln 2}$. [4]

$$8^x = 3(16^{x+1})$$
$$x\ln 8 = \ln 3 + (x+1)\ln 16$$
$$x(\ln 8 - \ln 16) = \ln 3 + \ln 16$$
$$x = \frac{\ln 48}{\ln\left(\frac{8}{16}\right)}$$
$$= \frac{\ln 48}{\ln\left(\frac{1}{2}\right)}$$
$$= -\frac{\ln 48}{\ln 2} \quad \text{(Shown)}$$

Alternative 1,

$$8^x = 3(16^{x+1})$$
$$8^x = 3(16 \times 16^x)$$
$$\frac{8^x}{16^x} = 48$$
$$\left(\frac{8}{16}\right)^x = 48$$
$$x\ln\left(\frac{1}{2}\right) = \ln 48$$
$$x = \frac{\ln 48}{\ln\left(\frac{1}{2}\right)}$$
$$= -\frac{\ln 48}{\ln 2} \quad \text{(Shown)}$$

Alternative 2,

$$8^x = 3(16^{x+1})$$
$$2^{3x} = 3(2^{4x+4})$$
$$\frac{2^{3x}}{2^{4x+4}} = 3$$
$$2^{-x-4} = 3$$
$$-(x+4)\ln 2 = \ln 3$$
$$x + 4 = -\frac{\ln 3}{\ln 2}$$
$$x = -\frac{\ln 3}{\ln 2} - 4$$
$$= -\left(\frac{\ln 3}{\ln 2} + 4\right)$$
$$= -\left(\frac{\ln 3 + 4\ln 2}{\ln 2}\right)$$
$$= -\left(\frac{\ln\left(3 \times 2^4\right)}{\ln 2}\right)$$
$$= -\frac{\ln 48}{\ln 2} \quad \text{(Shown)}$$

(b) Express the equation $\log_2 \sqrt{x} + \log_{16}(x-1) = 1$ as a cubic equation in x. [4]

$$\log_2 \sqrt{x} + \log_{16}(x-1) = 1$$
$$\log_2 x^{\frac{1}{2}} + \frac{\log_2(x-1)}{\log_2 16} = 1$$
$$\frac{1}{2}\log_2 x + \frac{\log_2(x-1)}{4} = 1$$
$$2\log_2 x + \log_2(x-1) = 4$$
$$\log_2\left[x^2(x-1)\right] = 4$$
$$x^2(x-1) = 16 \quad \text{(accept)}$$
$$x^3 - x^2 - 16 = 0$$

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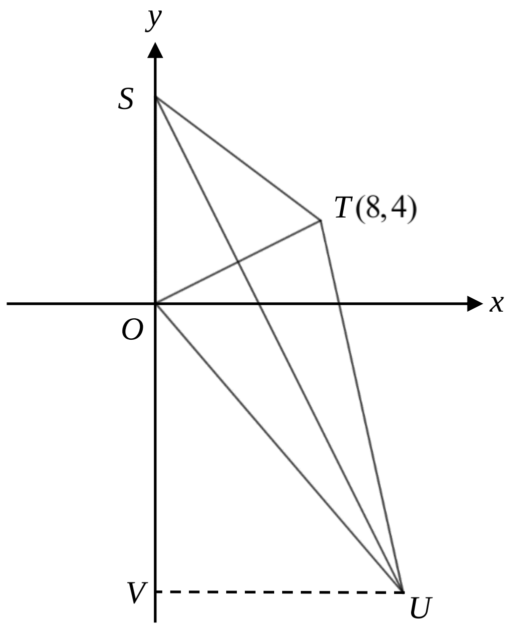


The diagram shows a kite *OSTU* in which $OS = TS$ and $OU = TU$. Point *S* lies on the *y*-axis, point *T* is $(8, 4)$ and *VU* is parallel to the *x*-axis.

(a) Find the equation of line *SU*. [3]

Midpoint of *OT* is $\left(\frac{0+8}{2}, \frac{0+4}{2}\right) = (4, 2)$.

Gradient of $OT = \frac{4}{8}$

$= \frac{1}{2}$

Gradient of $SU = -2$

Equation of line *SU* is $y - 2 = -2(x - 4)$

$y = -2x + 10$

(b) It is given that the coordinates of U are (h, k).

(i) Express k in terms of h. [1]

Since U lies on the line SU, $k = -2h + 10$.

(ii) Find the distance between S and U in terms of h. Give your answer in the form of $h\sqrt{a}$, where a is an integer. [3]

Coordinates of S are $(0,10)$.

$$
\begin{aligned}
SU &= \sqrt{(h-0)^2 + (-2h+10-10)^2} \\
&= \sqrt{h^2 + 4h^2} \\
&= h\sqrt{5} \quad \text{units}
\end{aligned}
$$

(c) In the case where $h = 12$, find the area of the quadrilateral $STUV$. [2]

$S(0,10),\ T(8,4),\ U(12,-14),\ V(0,-14)$

$$
\begin{aligned}
\text{Area of the quadrilateral } &= \frac{1}{2}\begin{vmatrix} 0 & 0 & 12 & 8 & 0 \\ 10 & -14 & -14 & 4 & 10 \end{vmatrix} \\
&= \frac{1}{2}\left| \ 0+0+48+80 \ - \ (0-168-112+0) \right| \\
&= \frac{1}{2}(408) \\
&= 204 \text{ units}^2
\end{aligned}
$$

Alternative:

$$
\begin{aligned}
\text{Area of the quadrilateral } &= \frac{1}{2}(10)(12) + \frac{1}{2}(12)(24) \\
&= 204 \text{ units}^2
\end{aligned}
$$

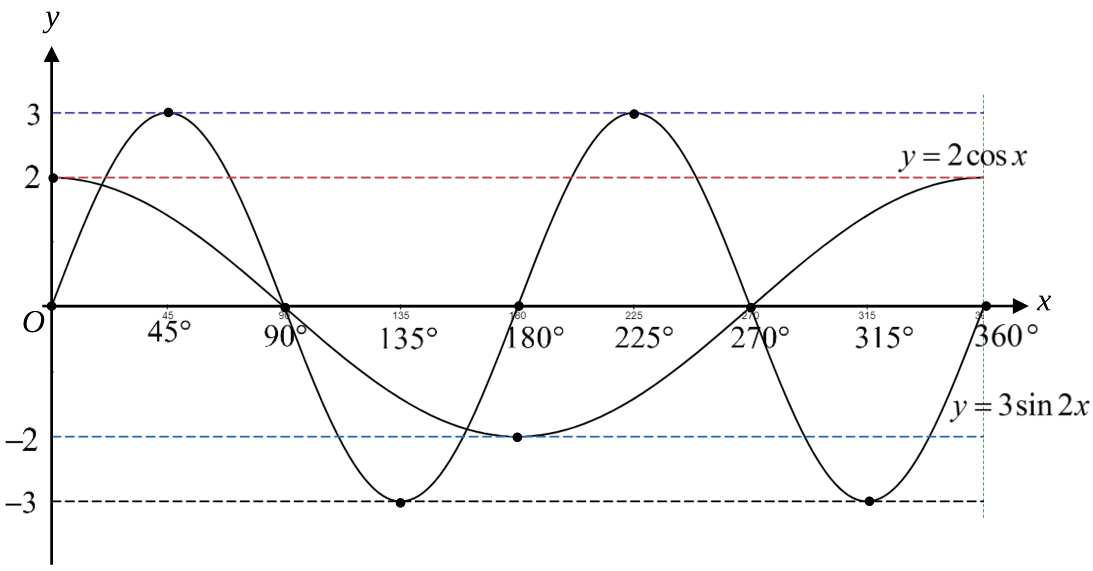
11 **(a)** Find the value of k such that $\int_0^{\frac{\pi}{3}} \left(2\sin^2\theta - \sin\theta\cos\theta - 1\right) \mathrm{d}\theta = -\dfrac{2\sqrt{3}+3}{k}$. [5]

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \left(2\sin^2\theta - \sin\theta\cos\theta - 1\right) \mathrm{d}\theta &= \int_0^{\frac{\pi}{3}} \left(-\cos 2\theta - \frac{\sin 2\theta}{2}\right) \mathrm{d}\theta \\ &= \left[-\frac{\sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{4}\left[-2\sin 2\theta + \cos 2\theta\right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{4}\left[-2\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2} - (0+1)\right] \\ &= \frac{1}{4}\left[-\sqrt{3} - \frac{3}{2}\right] \\ &= -\frac{2\sqrt{3}+3}{8}\end{aligned}$$

Therefore $k = 8$.

(b) The function $f(x)$ and $g(x)$ are given by $f(x)=3\sin 2x$ and $g(x)=2\cos x$.

(i) On the same axes, sketch the graphs of $y=f(x)$ and $y=g(x)$ for $0\leq x\leq 360°$. [4]



(ii) Hence, find the number of solutions of the equation $\sin 2x-\frac{2}{3}\cos x=0$ for $0\leq x\leq 360°$. [1]

$\sin 2x=\frac{2}{3}\cos x$

$3\sin 2x=2\cos x$

Hence, 4 intersections on graph implies 4 solutions.

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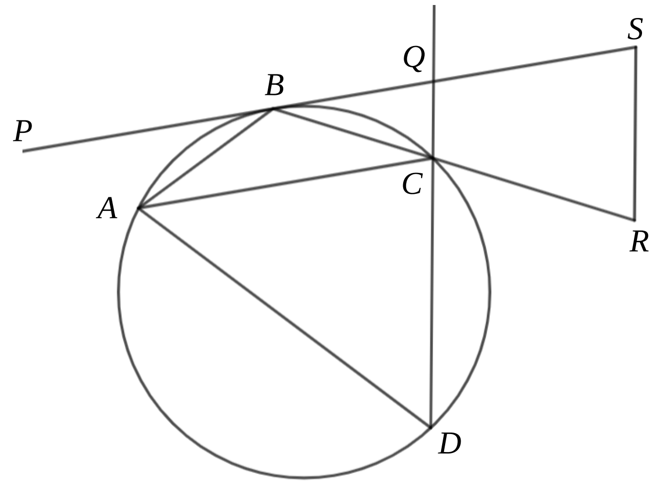


In the diagram, A, B, C and D lie on a circle. PQS is the tangent to the circle at B. DQ and BR are straight lines that meet at C. SR is parallel to QC. Angle ADC is twice the size of angle QBC.

(a) Prove that $AB = BC$. [4]

Let $\angle QBC = \theta$

$\angle ADC = 2\angle QBC = 2\theta$ (given)

$\angle BAC = \angle QBC = \theta$. (Alt. Segment Theorem)

$\angle ABC = 180° - \angle ADC$ ($\angle$s in the opp. segment)

$= 180° - 2\theta$

$\angle BCA = 180° - \angle ABC - \angle BAC$

$= 180° - (180° - 2\theta) - \theta$ ($\angle$ sum of Δ)

$= \theta$

$= \angle BAC$

Since $\angle BAC = \angle BCA = \theta$, ΔABC is an isosceles triangle.

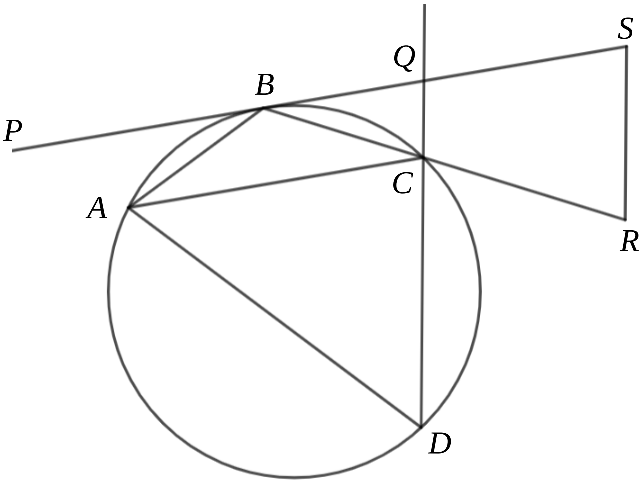
Hence $AB = BC$. (Proven)

(b) Prove that $AB \times SR = BR \times QC$. [4]

$\angle BQC = \angle BSR$ (corr. $\angle$s, $QC // SR$)
$\angle QBC = \angle SBR$ (common $\angle$)

Therefore, ΔBQC is similar to ΔBSR. (AA similarity test)

$\dfrac{BC}{BR} = \dfrac{QC}{SR}$ (corr. sides of similar triangles)
$BC \times SR = BR \times QC$
$AB \times SR = BR \times QC$ (since $AB = BC$, from part (a))
(Shown)



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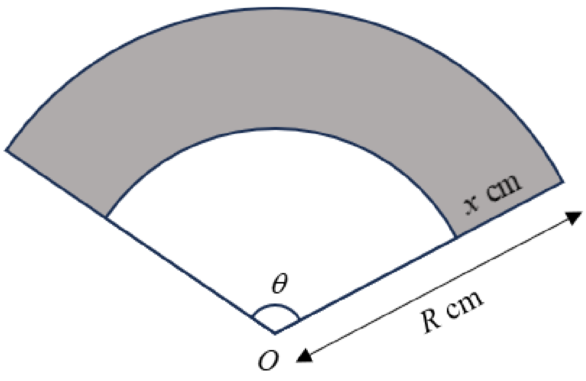

[Area of sector $= \dfrac{1}{2} r^2 \theta$]

Mr Rich owns a restaurant. His intention is to create a fan-shaped design for his restaurant, which can be modelled by a sector, centre *O*, with radius *R* cm. Part of the fan is to be painted gold, leaving an unpainted sector, also centre *O*, as shown in the diagram.
The width of the strip of gold paint is *x* cm.

(a) Show that the area, *A* cm^2, of the gold-painted region is given by $A = \dfrac{\theta}{2}\left(2Rx - x^2\right)$. [2]

$$\begin{aligned} A &= \frac{1}{2}R^2\theta - \frac{1}{2}(R-x)^2\theta \\ &= \frac{\theta}{2}\left[R^2 - (R^2 - 2Rx + x^2)\right] \\ &= \frac{\theta}{2}\left(2Rx - x^2\right) \quad \text{(shown)} \end{aligned}$$

It has been decided that the area of the gold-painted region is to be 880 cm^2.

(b) By making θ the subject, find x in terms of R which gives a stationary value of θ and determine its nature. [6]

$$880 = \frac{\theta}{2}\left(2Rx - x^2\right)$$

$$\theta = \frac{1760}{2Rx - x^2}$$

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}x} &= -1760\,(2Rx - x^2)^{-2}(2R - 2x)\\ &= -\frac{3520\,(R - x)}{(2Rx - x^2)^2}\end{aligned}$$

For stationary value of θ, $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0$

$$-\frac{3520\,(R - x)}{(2Rx - x^2)^2} = 0$$

$$x = R$$

First derivative test:

x	R^-	R	R^+
$\dfrac{\mathrm{d}\theta}{\mathrm{d}x}$	$^-$ve	0	$^+$ve
Tangent	$\diagdown$	—	$\diagup$

Therefore, θ is minimum when $x = R$.

Alternative (second derivative test):

$$\frac{\mathrm{d}^2\theta}{} = -3520\left[\frac{-(2Rx - x^2)^2 - (R - x)\cdot 2(2Rx - x^2)\cdot(2R - 2x)}{}\right]$$

(c) Explain why the value of x obtained in part **(b)** fails to achieve the intended effect of the design of the fan. [1]

The value of x implies **that the entire fan is to be painted with gold**, leaving no space for unpainted sector. Therefore, it fails to achieve the intended effect of the original design.

End of Paper