NAME:		( )		CLASS:	

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YISHUN TOWN SECONDARY SCHOOL

PRELIMINARY EXAMINATION 2024
SECONDARY 4 EXPRESS / 5 NORMAL ACADEMIC
ADDITIONAL MATHEMATICS PAPER 2 (4049/02)

DATE : 23 August 2024 **DAY : Friday**

DURATION : 2h 15min **MARKS : 90**

READ THESE INSTRUCTIONS FIRST

Do not turn over the cover page until you are told to do so.
Write your name, class and class index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total marks for this paper is 90.

	MARKS	
	OBTAINED	FULL
1		6
2		8
3		7
4		10
5		10
6		8
7		9
8		10
9		10
10		12
TOTAL		90

This question paper consists of **21** printed pages and **1** blank page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for Δ *ABC*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

Answer all questions.

1 **(a)** Given that the curve $y=(4-k)x^2+6kx-4k$ has a minimum value, find the range of values of k for which the line $y=kx$ meets the curve. [5]

$(4-k)x^2+6kx-4k=kx$

$(4-k)x^2+5kx-4k=0$

For line to meet curve, $D\geq 0$.

$(5k)^2-4(4-k)(-4k)\geq 0$

$25k^2-16k^2+64k\geq 0$

$9k^2+64k\geq 0$

$k(9k+64)\geq 0$

$k\leq -\frac{64}{9}$ OR $k\geq 0$

Since curve has a maximum value,

$4-k>0$

$k<4$

$\therefore k\leq -\frac{64}{9}$ OR $0\leq k<4$

(b) State the largest possible integer value of k. [1]

Largest possible integer value of $k=3$

2 Given the following equation $4(9^x) = 15 + 4(3^{x+1}) + 4\left(3^{-x}\right)$,

(a) Using the substitution $u = 3^x$, show that the equation can be written as $4u^3 - 12u^2 - 15u + k = 0$, where k is a constant. [2]

$$4(9^x) = 15 + 4(3^{x+1}) + 4\left(3^{-x}\right)$$
$$4(3^{2x}) = 15 + 4(3 \cdot 3^x) + 4\left(\frac{1}{3^x}\right)$$
$$4(3^{3x}) = 15 \cdot 3^x + 12 \cdot 3^{2x} + 4$$
$$4(3^{3x}) - 12 \cdot 3^{2x} - 15 \cdot 3^x - 4 = 0$$
$$4u^3 - 12u^2 - 15u - 4 = 0$$

(b) Using the value of k found in **(a)**, show that $2u + 1$ is a factor of $4u^3 - 12u^2 - 15u + k$. [1]

Let $f(u) = 4u^3 - 12u^2 - 15u - 4$.

$$f(-\frac{1}{2}) = 4(-\frac{1}{2})^3 - 12(-\frac{1}{2})^2 - 15(-\frac{1}{2}) - 4$$
$$f(-\frac{1}{2}) = 0$$

By factor theorem, $2u + 1$ is a factor of $f(u)$.

(c) Hence, find the value(s) of x. [5]

From (ii), $f(u) = (2u+1)(au^2+bu+c)$.
By division algorithm/ comparing coefficients,
$f(u) = (2u+1)(2u^2-7u-4)$
$f(u) = (2u+1)^2(u-4)$
When $f(u) = 0$, $(2u+1)^2(u-4) = 0$

$u = -\frac{1}{2}$ OR $u = 4$

$3^x = -\frac{1}{2}$ (no solution) $3^x = 4$

$x \lg 3 = \lg 4$
$x = 1.26$ (3s.f.)

3 It is given that the equation of a curve is $y = e^x \sin x$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$. [3]

$$\frac{dy}{dx} = e^x \cos x + e^x \sin x$$
$$\frac{d^2 y}{dx^2} = e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x$$
$$\frac{d^2 y}{dx^2} = 2e^x \cos x$$

(b) Hence, find $\int \mathrm{e}^x \cos x \,\mathrm{d}x$. [2]

$$\int 2e^x \cos x \,\mathrm{d}x = e^x \cos x + e^x \sin x + c$$
$$\int e^x \cos x \,\mathrm{d}x = \frac{1}{2}\left(e^x \cos x + e^x \sin x + c\right)$$

(c) Using the results in **(a)** and **(b)**, find $\int \mathrm{e}^{x} \sin x \, \mathrm{d}x$. [2]

$$\int e^{x} \sin x + e^{x} \cos x \ dx = e^{x} \sin x + c$$

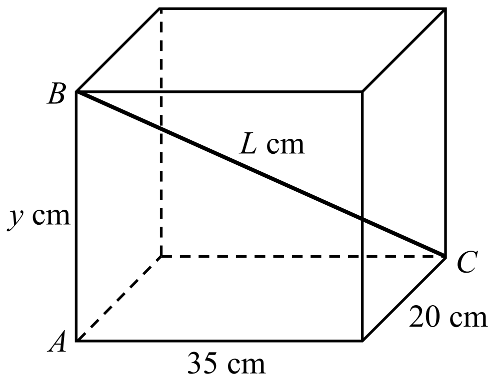
$$\int e^{x} \sin x \ dx + \int e^{x} \cos x \ dx = e^{x} \sin x + c$$

$$\int e^{x} \sin x dx = -\frac{1}{2}\left(\cos x e^{x} + e^{x} \sin x\right) + e^{x} \sin x + c_{2}$$

$$\int e^{x} \sin x dx = -\frac{1}{2}\cos x e^{x} - \frac{1}{2} e^{x} \sin x + e^{x} \sin x + c_{2}$$

$$\int e^{x} \sin x dx = -\frac{1}{2}\cos x e^{x} + \frac{1}{2} e^{x} \sin x + c_{2}$$

4 **(a)** A box with length 35 cm, breadth 20 cm and height y cm is being compressed from the top at a constant rate of 0.005 cm per second. The initial height of the box is 30 cm.



(i) Find the height of the box after one minute. [2]

Decrease in height = 60(0.005) = 0.3 cm
Height of the box = $30-0.3=29.7$ cm

(ii) The length of the diagonal BC is L cm. Show that $L=\sqrt{y^2+k}$, where k is an integer to be found. [1]

$$AC^2=35^2+20^2$$
$$=1625$$

$L=\sqrt{y^2+1625}$ (shown)

(iii) Assuming that the shape of the box does not change under the compression, find the rate at which BC is decreasing after one minute. [3]

$$L = \sqrt{y^2 + 1625}$$

$$\frac{\mathrm{d}L}{\mathrm{d}y} = \frac{1}{2}(y^2 + 1625)^{-\frac{1}{2}}(2y)$$

$$= \frac{y}{\sqrt{y^2 + 1625}}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}t}$$

$$= \frac{y}{\sqrt{y^2 + 1625}} \times \frac{\mathrm{d}y}{\mathrm{d}t}$$

After one minute, $y = 29.7$, $\frac{\mathrm{d}y}{\mathrm{d}t} = -0.005$

$$\therefore \quad \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{29.7}{\sqrt{29.7^2 + 1625}}(-0.005)$$

$$= -0.00297 \text{ (exact)}$$

The rate of decrease of AC is 0.00297 cm per second.

(b) The density D of the box is related to its volume V by the formula $DV = m$, where m is a constant. Given that when $D = 920$, $V = 0.0245$, find the rate at which the density is changing with respect to V when $V = 0.0196$. [4]

$$920\,(0.0245) = m$$

$$m = 22.54$$

$$\therefore DV = 22.54$$

$$D = \frac{22.54}{V}$$

$$\frac{\mathrm{d}D}{\mathrm{d}V} = -\frac{22.54}{V^2}$$

When $V = 0.0196$, $\frac{\mathrm{d}D}{\mathrm{d}V} = -\frac{22.54}{0.0196^2}$

$$\approx -58673.47$$

$$= -58700 \text{ g/cm}^6 \text{ (3 s.f.)}$$

5 **(a)** Explain why there are only odd powers of x in the expansion $\left(3x^3 - \frac{1}{kx}\right)^9$, where $k > 0$. [3]

$$T_{r+1} = \binom{9}{r}\left(3x^3\right)^{9-r}\left(-\frac{1}{kx}\right)^r$$

$$T_{r+1} = \binom{9}{r}3^{9-r} \cdot x^{27-3r}\left(-\frac{1}{k}\right)^r \cdot x^{-r}$$

$$T_{r+1} = \binom{9}{r}3^{9-r}\left(-\frac{1}{k}\right)^r \cdot x^{27-4r}$$

Since $4r$ is always even, $27-4r$ is always odd since the addition of an even number and an odd number, 27, is always odd.
Hence, x only has odd powers in this expansion.

(b) Given that the coefficient of x^3 in the expansion of $\left(3x^3 - \frac{1}{kx}\right)^9$ is $\frac{567}{16}$, show that the value of k is 2. [4]

Using the general term formula, to find x^3 term,
Let $27 - 4r = 3$
$r = 6$

$$T_7 = \binom{9}{6} \cdot 3^3 \cdot \left(-\frac{1}{k}\right)^6 \cdot x^3$$

$$T_7 = \frac{2268}{k^6}x^3$$

$$\frac{2268}{k^6} = \frac{567}{16}$$

$k^6 = 64$
$k = 2$ (rej. -2 as $k > 0$)

(c) Hence, find the term independent of x in the expansion $(1-4x^5)\left(3x^3-\frac{1}{kx}\right)^9$. [3]

To consider x^{-5} in the expansion of $\left(3x^3-\frac{1}{kx}\right)^9$.

Let $27-4r=-5$.

$r=8$

$T_9=\binom{9}{8}\cdot 3^1\cdot\left(-\frac{1}{2}\right)^8\cdot x^{-5}$

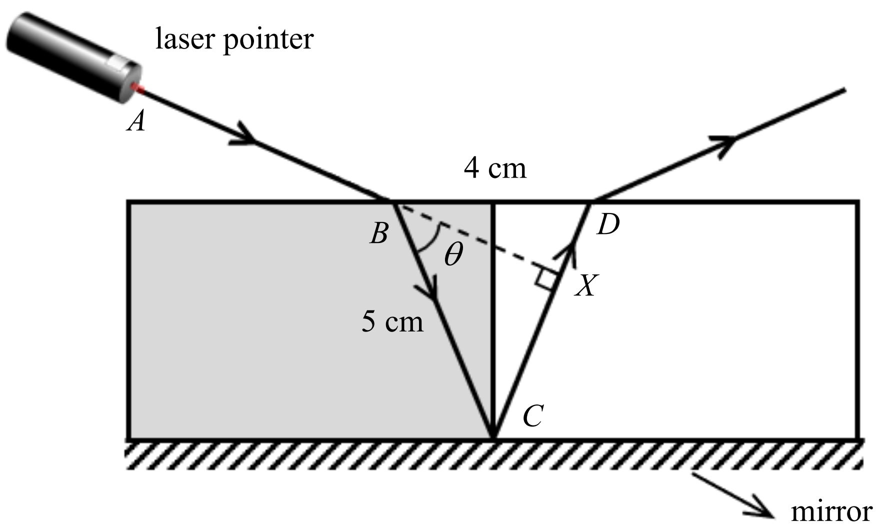
$T_9=\frac{27}{256}x^{-5}$

Constant term in $(1-4x^5)\left(3x^3-\frac{1}{kx}\right)^9$

$=-4\left(\frac{27}{256}\right)$

$=-\frac{27}{64}$

6



Two transparent rectangular blocks made of different materials are placed beside each other on a mirror. A laser pointer is pointed at the blocks and the diagram shows the path of the light ray from the laser pointer. The light ray *AB* refracts and travels through the path *BC*. It is then reflected and travels through the path *CD*, before getting refracted while leaving the prism at *D*. It is given that angle *BCD* is equal to angle *XBD*, and angle *XBC* is defined as θ, the angle in which the light ray deviates from its original path.

(a) Show that S cm, the distance travelled by the light ray in the rectangular prism, can be expressed in the form $p+q\sin\theta+r\cos\theta$, where p, q and r are constants to be found. [2]

In triangle BXC, $CX = 5\sin\theta$

In triangle BXD, $DX = 4\cos\theta$

Since $S = BC + CX + DX$,

$S = 5 + 5\sin\theta + 4\cos\theta$ (Shown)

(b) Express S in the form $R\sin(\theta+\alpha)+k$, where $R>0$, $0° \le \alpha \le 90°$ and k is an integer. [4]

Let $4\cos\theta + 5\sin\theta = R\sin(\theta+\alpha)$.

$$4\cos\theta + 5\sin\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$$

$$4 = R\sin\alpha$$

$$5 = R\cos\alpha$$

$$\tan\alpha = \frac{4}{5}$$

$$\alpha = \tan^{-1}\left(\frac{4}{5}\right);\ \alpha = 38.7°$$

$$R = \sqrt{4^2+5^2};\ R = \sqrt{41}$$

Hence, $S = \sqrt{41}\sin(\theta + 38.7) + 5$.

(c) Find the value of S and the corresponding value of θ if the light ray is to travel the maximum possible distance within the rectangular prism. [2]

Since $-1 \le \sin(\theta + 38.7°) \le 1$,

$-\sqrt{41} + 5 \le \sqrt{41}\sin(\theta + 38.7°) + 5 \le \sqrt{41} + 5$

Maximum $S = \sqrt{41} + 5$

Max. $S = 11.4cm$ (3s.f.)

For max. S, let $\sqrt{41}\sin(\theta + 38.7°) + 5 = \sqrt{41} + 5$.

$\sin(\theta + 38.7°) = 1$

$\theta + 38.7° = 90°$

Hence, corresponding $\theta = 90° - 38.7° = 51.3°$

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7 **(a)** Explain how a straight-line graph can be drawn to represent the equation $y = px^q$. [2]

$\lg y = q \lg x + \lg p$ (or equivalent form)

Plot a graph of $\lg y$ against $\lg x$.

(b) The following table shows experimental values of x and y with the equation $y = px^q$. An error has been made in recording **one** of the values of y.

x	2.82	7.94	14.1	63.1	126
y	50.1	25.1	7.08	6.31	3.98

(i) On the grid in the next page, draw a straight-line graph based on the experimental values in the table. [3]

(ii) Hence, calculate the values of p and q. [3]

$\lg p = 2$

$p = 10^2$

$p = 100 \pm 6$

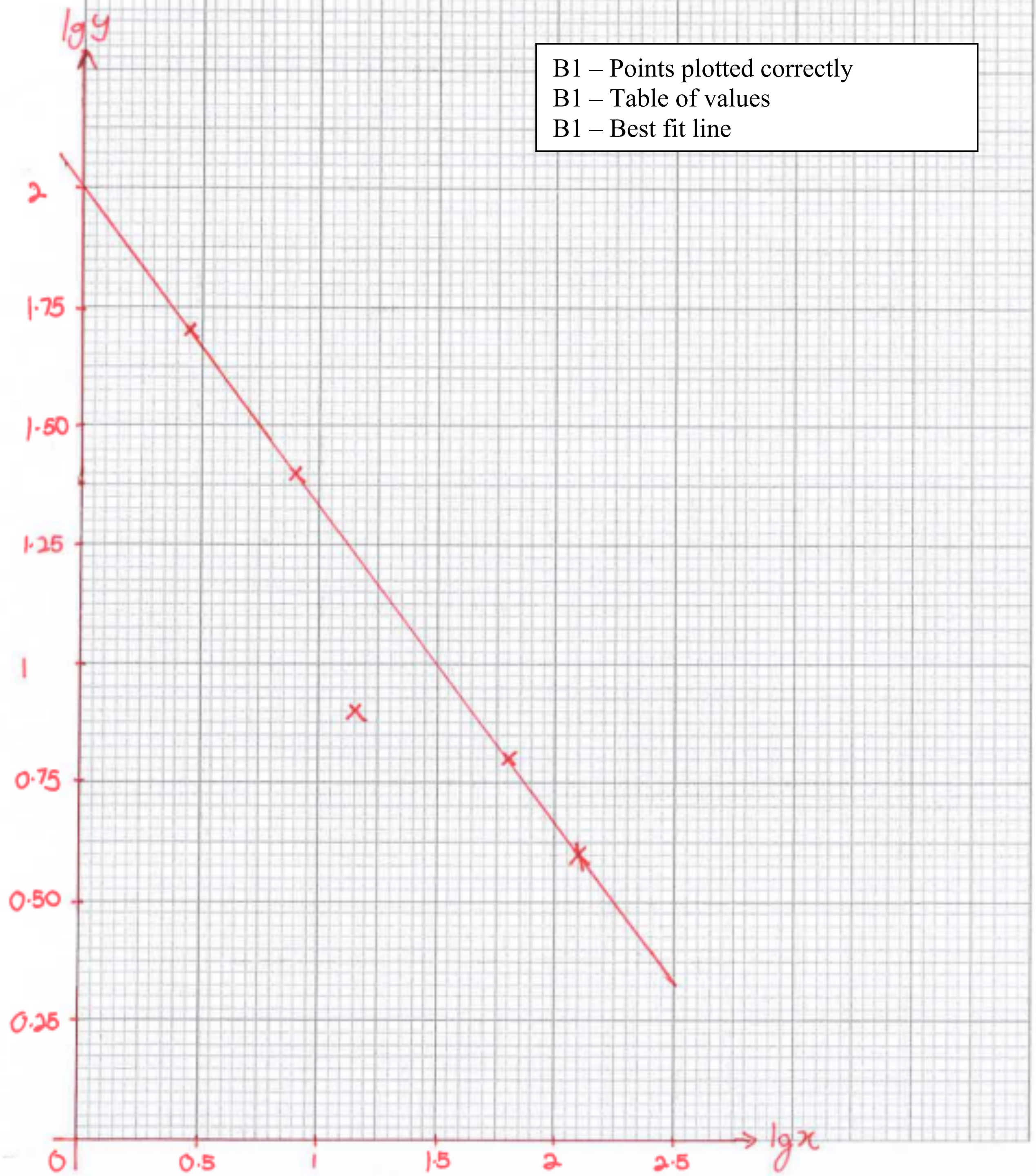
$q = \dfrac{1.7 - 0.5}{0.45 - 2.25}$

$q = -0.667 \pm 0.1$

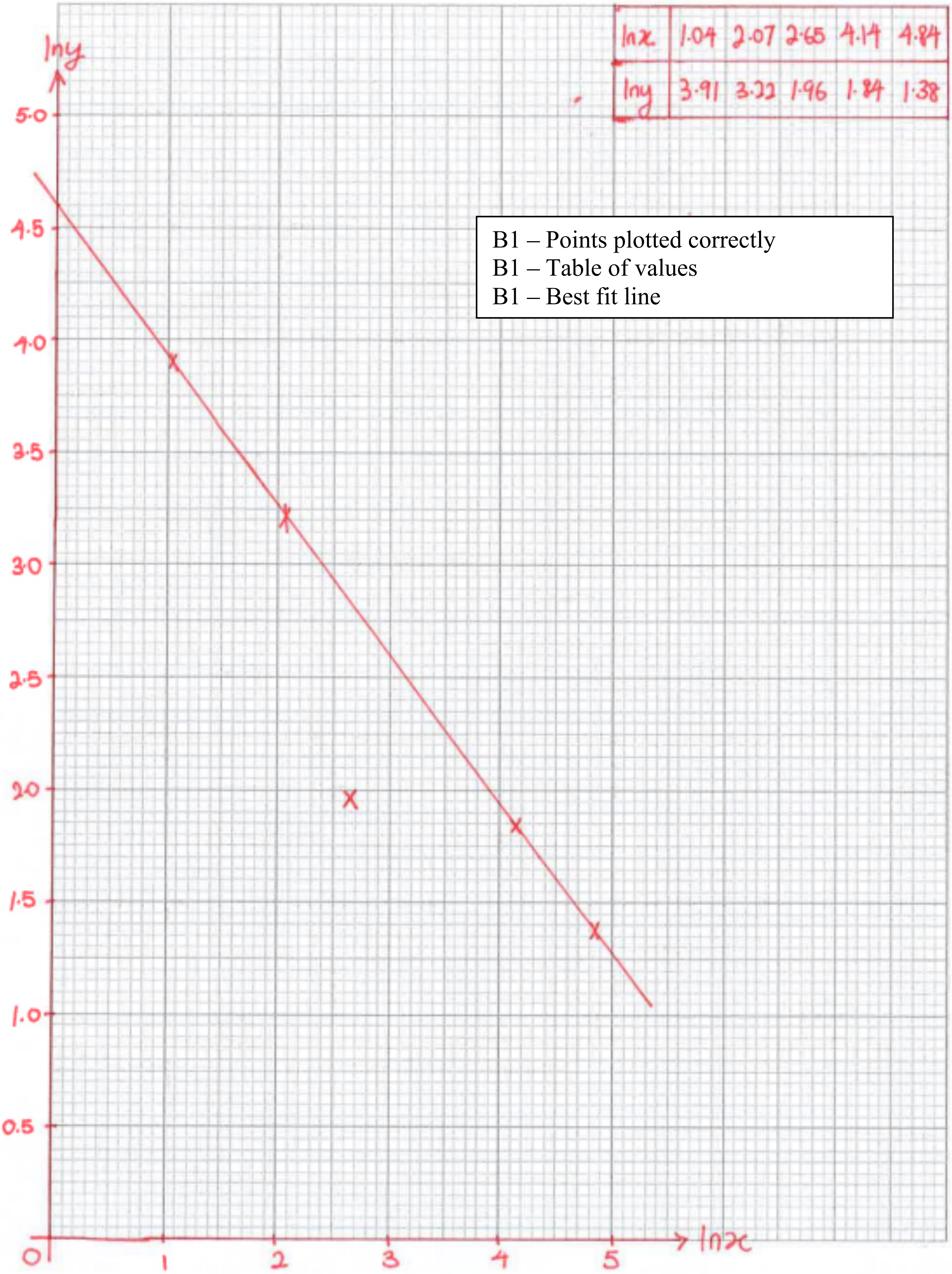
(iii) Use the straight line graph drawn in **(i)** to estimate a value of y to replace the incorrect recording of y. [1]

From the graph, we can see that the correct reading should be $(1.15, 1.23)$.
Hence, y-value of correct point
$= 10^{1.23}$
$= 17.0 \pm 1$ (3s.f.)

$\lg x$	0.450	0.900	1.15	1.80	2.10
$\lg y$	1.70	1.40	0.850	0.800	0.600



B1 – Points plotted correctly
B1 – Table of values
B1 – Best fit line



ln x	1.04	2.07	2.65	4.14	4.84
ln y	3.91	3.22	1.96	1.84	1.38

B1 – Points plotted correctly
B1 – Table of values
B1 – Best fit line

8 A particle travelling in a straight line passes through a fixed point O with its velocity, v m/s, given as $v = e^{2t} - 10e^{t} + 21$, where t is the time in seconds after it passes O.

(a) Find the acceleration of the particle when it first comes to instantaneous rest. [4]

When $v = 0$,
$e^{2t} - 10e^{t} + 21 = 0$
$(e^{t} - 3)(e^{t} - 7) = 0$
$e^{t} = 3$ OR $e^{t} = 7$
$t = \ln 3$ OR $t = \ln 7$

$a = 2e^{2t} - 10e^{t}$

Sub. $t = \ln 3$,
$a = 2e^{2\ln 3} - 10e^{\ln 3}$
$a = -12$ m/s^2

(b) Find the distance travelled by the particle when the particle reaches minimum velocity. [6]

$s = \int e^{2t} - 10e^{t} + 21 \, \mathrm{d}t$

$s = \frac{e^{2t}}{2} - 10e^{t} + 21t + c$

When $t = 0$, $s = 0$.

Hence, $c = \frac{19}{2}$.

$s = \frac{e^{2t}}{2} - 10e^{t} + 21t + \frac{19}{2}$

When $a = 0$,

$2e^{2t} - 10e^{t} = 0$

$2e^{t}\left(e^{t} - 5\right) = 0$

$t = 0$ OR $t = \ln 5$

When $t = \ln 3$,

$s = \frac{e^{2\ln 3}}{2} - 10e^{\ln 3} + 21\ln 3 + \frac{19}{2}$

$s = 7.0709$ m

When $t = \ln 5$,

$s = \frac{e^{2\ln 5}}{2} - 10e^{\ln 5} + 21\ln 5 + \frac{19}{2}$

$s = 5.7982$ m

Total distance travelled

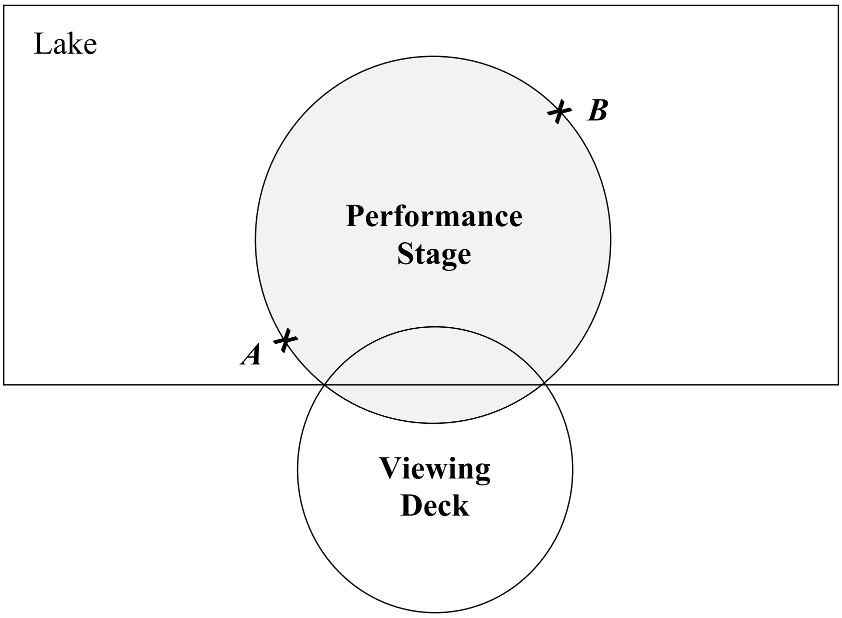
$= 7.0709 + \left(7.0709 - 5.7982\right)$

$= 8.34$ m (3s.f.)

9 At the Botanic Gardens, a team of landscape architects is embarking on a project to enhance tourism by constructing a performance area on a lake.

The team envisions a design of the performance area that features two overlapping circles. One circle represents the designated performance stage, where artists and performers will showcase their talents against the backdrop of water and greenery. The other circle serves as a viewing deck, offering spectators a vantage point to enjoy the performances.

Below is a simplified sketch of the plan.



The performance stage can be modelled by the equation of a circle given by $x^2 + y^2 - 14x - 20y + 124 = 0$ on the cartesian plane.

(a) Show that this model suggests that the radius of the performance stage is 5 m. [2]

$x^2 + y^2 - 14x - 20y + 124 = 0$
$(x-7)^2 + (y-10)^2 = 25$
Hence, radius = 5 m.

(b) A floor marker is to be placed at the point $(6,6)$. Explain why any straight line passing through the floor marker will intersect the circumference of the performance stage at two points. [3]

Length between point and centre of circle

$=\sqrt{(7-6)^2+(10-6)^2}$

$=\sqrt{17}<5$ m

The point lies inside the performance area, any line passing through the point will intersect the circumference at 2 points.

(c) Two spotlights at $A(4,6)$, and $B(10,14)$, are to be positioned on the circumference of the performance stage such that they form the endpoints of a diameter of the stage. It is observed that placing additional spotlights on the stage equidistant from these 2 spotlights is effective at increasing aesthetic appeal of performances.

Find the equation of line AB and hence determine whether a spotlight placed at $(11,7)$ is effective at increasing the aesthetic appeal of performances. [5]

$m_{AB}=\frac{6-14}{4-10}$

$m_{AB}=\frac{4}{3}$

Equation of line AB:

$\frac{y-6}{x-4}=\frac{4}{3}$

$y=\frac{4}{3}x+\frac{2}{3}$

Hence, $m_{\perp AB}=-\frac{3}{4}$.

Equation of perpendicular bisector of AB:

$\frac{y-10}{x-7}=-\frac{3}{4}$

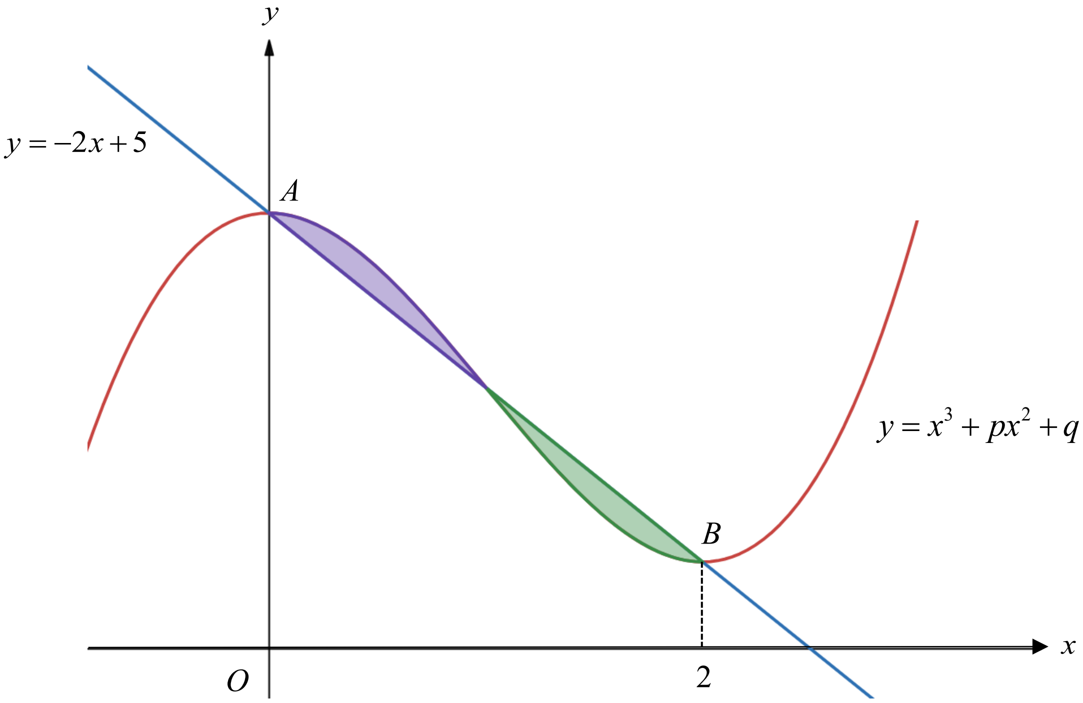
$y=-\frac{3}{4}x+\frac{61}{4}$

Sub. $x=11$ into equation of perpendicular bisector of AB:

$y=-\frac{3}{4}(11)+\frac{61}{4}$

$y=7$ (Verified, hence it is effective)

10



The diagram shows the line $y=-2x+5$ and the curve $y=x^3+px^2+q$, where p and q are constants. A and B are stationary points of the curve and they lie on the line. It is given that one of the stationary points of the curve occurs at $x=2$.

Find the total area of the shaded regions. [12]

$\frac{dy}{dx}=3x^2+2px$

When $\frac{dy}{dx}=0$,

$3x^2+2px=0$

$x(3x+2p)=0$

$x=0$ OR $3x+2p=0$

Since $x=2$ at stationary point,

$6+2p=0$

$p=-3$

When $x=2$,

$y=-2(2)+5$

$y=1$

Sub. $x=2$ and $y=1$ into equation of curve.

$1=2^3-3(2)^2+q$

$q=5$

To find all points of intersection,

$x^3-3x^2+5=-2x+5$

$x^3-3x^2+2x=0$

$x(x-1)(x-2)=0$

$x=0$ OR $x=1$ OR $x=2$

Continuation of working space for question 10.

Area of shaded region 1

$= \int_0^1 x^3 - 3x^2 + 5 - (-2x+5) \text{ d}x$

$= \int_0^1 x^3 - 3x^2 + 2x \text{ d}x$

$= \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1$

$= \frac{1}{4} - 1 + 1 - (0)$

$= \frac{1}{4}$ units^2

Area of shaded region 2

$= \int_1^2 -2x + 5 - (x^3 - 3x^2 + 5) \text{ d}x$

$= \int_1^2 -x^3 + 3x^2 - 2x \text{ d}x$

$= \left[-\frac{x^4}{4} + x^3 - x^2 \right]_1^2$

$= -4 + 8 - 4 - \left(-\frac{1}{4} + 1 - 1 \right)$

$= \frac{1}{4}$ units^2

Total area of shaded regions

$= 0.25 + 0.25$

$= 0.50$ units^2

End of Paper

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Answers Key

Qn	Answer	Qn	Answer
1	(a) $k \le -\frac{64}{9}$ OR $0 \le k < 4$ (b) $k = 3$	**6**	(a) $S = 5 + 5\sin\theta + 4\cos\theta$ (b) $S = \sqrt{41}\sin(\theta + 38.7°) + 5$ (c) Max. value of S = 11.4 cm, where $\theta =$ 51.3°
2	(a) $4u^3 - 12u^2 - 15u - 4 = 0$ (c) $x = 1.26$	**7**	(a) $\lg y = q\lg x + \lg p$ / $\ln y = q\ln x + \ln p$. Plot a graph of lgy/ lny against lgx/ lnx. (b)(ii) $p = 100 \pm 6$, $q = -0.667 \pm 0.1$ (b)(iii) $y = 17 \pm 1$
3	(a) $\frac{dy}{dx} = e^x \cos x + e^x \sin x$ $\frac{d^2y}{dx^2} = 2e^x \cos x$ (b) $\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + c_1$ (c) $\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + c_2$	**8**	(a) $a = -12$ m/s^2 (b) Distance = 8.34 m
4	(a)(i) 29.7 cm (a)(ii) $L = \sqrt{y^2 + 1625}$ (a)(iii) Rate of decrease = 0.00297 cm/s (b) $\frac{dD}{dV} = -58700\ g/cm^6$	**9**	(b) Length between point and centre of circle = $\sqrt{17} < 5$ (radius of circle). Hence, point is inside circle and any line passing through the point will intersect circumference at 2 points. (c) Equation of line AB: $y = \frac{4}{3}x + \frac{2}{3}$ Yes, it is effective to place spotlight.
5	(c) $-\frac{27}{64}$	**10**	0.50 units^2