1 Suppose $a, b > 0$.

Let $q_1 = 2a$ and $q_{n+1} = 2a + b^2/q_n$. Since $(q_n)_{n\in\mathbb{N}}$ converges, let $q_n q_{n+1} = 2aq_n + b^2$. Note that $L^2 - 2aL - b^2 = 0$ so $L = a \pm \sqrt{a^2+b^2}$. Notice that all terms in the sequence are positive so $L = a + \sqrt{a^2+b^2}$, otherwise $L = a - \sqrt{a^2+b^2} < 0$ which is impossible, as $a < \sqrt{a^2+b^2}$.

2 **(a)** Let $z_1 = 5 + \mathrm{i}$ and $z_2 = 239 + \mathrm{i}$. Then $|z_1| = \sqrt{26}$ and $|z_2| = \sqrt{57122}$. Also, $\arg(z_1) = \tan^{-1}(\frac{1}{5})$ and $\arg(z_2) = \tan^{-1}(\frac{1}{239})$.

(b) By considering the properties of $re^{\mathrm{i}\theta}$, see that $|z_4| = |z_1|^4/|z_2| = \frac{676}{\sqrt{57122}} = \sqrt{\frac{456976}{57122}} = 2\sqrt{2}$. Also, $\arg(z_4) = 4\arg(z_1) - \arg(z_2) = 4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239})$.

(c) We are to show $\arg(z_4) + \arg(z_3) = 4\arg(z_1)$, i.e. we must literally find $\arg(z_4)$. By GC,

$$\frac{z_1^4}{z_2} = \frac{(5+\mathrm{i})^4}{239+\mathrm{i}} = 2 + 2\mathrm{i}.$$

Hence $\arg(z_4) = \tan^{-1}(1) = \frac{1}{4}\pi$, and the conclusion follows.

3 **(a)** $x^2 + (y+a)^2 = r^2$.

(b) Write $y = \sqrt{r^2 - x^2} - a$. Set $y = 0$ so $x = \pm\sqrt{r^2 - a^2}$. Also note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{\sqrt{r^2 - x^2}}.$$

Hence

$$\begin{aligned}
&2\pi \int_{-\sqrt{r^2-a^2}}^{\sqrt{r^2-a^2}} (\sqrt{r^2-x^2} - a)\sqrt{1 + \left(\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,\mathrm{d}x \\
&= 2\pi \int_{-\sqrt{r^2-a^2}}^{\sqrt{r^2-a^2}} (\sqrt{r^2-x^2} - a)\sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x \\
&= 2\pi r \int_{-\sqrt{r^2-a^2}}^{\sqrt{r^2-a^2}} \left(1 - \frac{a}{\sqrt{r^2-x^2}}\right)\mathrm{d}x \\
&= 2\pi r \left[x - a\sin^{-1}\left(\frac{x}{r}\right)\right]_{-\sqrt{r^2-a^2}}^{\sqrt{r^2-a^2}} \\
&= 4\pi r \left(\sqrt{r^2-a^2} - a\sin^{-1}\left(\frac{\sqrt{r^2-a^2}}{r}\right)\right).
\end{aligned}$$

Here, $k = 4\pi$.

4 **(a)** $L \approx \pi\sqrt{2} + \frac{3\sqrt{2}}{2}\pi \approx 11.10721 \approx 11.1072$ (4dp).

(b) **(i)** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin(x) + x\cos(x)$ so $L = \displaystyle\int_0^{2\pi} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_0^{2\pi} \sqrt{1 + (\sin(x) + x\cos(x))^2}\,\mathrm{d}x$.

(ii) The size of each interval is $\frac{1}{4} \times 2\pi = \frac{\pi}{2}$. Hence, letting $f(x) = \sqrt{1 + (\sin(x) + x\cos(x))^2}$,

$$L \approx \frac{\pi}{4}\left(f(0) + 2f\left(\frac{\pi}{2}\right) + 2f(\pi) + 2f\left(\frac{3\pi}{2}\right) + f(2\pi)\right) \approx 15.40396 \approx 15.4040 \text{ (4dp)}.$$

(iii) $L \approx 15.37457 \approx 15.3746$ (4dp) by GC.

(c) The student's approximation is inaccurate because it is far from the true (GC) value, while the trapezium rule approximation is accurate because it is close to the true value.

5 **(a)** $z = f(x, y) = x\tan^2(y)$, so $\frac{\partial z}{\partial x} = \tan^2(y)$ and $\frac{\partial z}{\partial y} = 2x\tan(y)\sec^2(y)$. Hence

$$z = f\left(2, \frac{\pi}{4}\right) + (x-2)f_x\left(2, \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right) f_y\left(2, \frac{\pi}{4}\right) = 2 + (x-2)(1) + \left(y - \frac{\pi}{4}\right)(8)$$
$$\implies z = x + 8y - 2\pi \implies x + 8y - z = 2\pi.$$

Then $a = 1$, $b = 8$ and $c = -1$.

(b) A vector normal to the plane is $\begin{pmatrix} 1 \\ 8 \\ -1 \end{pmatrix}$.

(c) Calculate $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 8\mathbf{k}$. Then $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 8 \\ -1 \end{pmatrix}$ which is normal to the plane.

(d) Both $\mathbf{u}$ and $\mathbf{v}$ are direction vectors on P. $\mathbf{u}$ is the direction vector giving the directional derivative along the x-axis at the point P, while $\mathbf{v}$ is that along the y-axis.

6 **(a)** The *annual* interest rate is $(1.008^{12} - 1) \times 100\% \approx 1.10\%$ (3sf). F is the minimum amount the business repays per month, and G is the additional amount that the business pays per month as the number of months passed increases and more interest is charged monthly.

(b) Let

$$\overbrace{A \times 1.008^{n+1} + B(n+1) + C}^{M_{n+1}} = \overbrace{1.008(A \times 1.008^n + Bn + C)}^{1.008M_n} - F - Gn.$$

Then $Bn + B + C = 1.008Bn + 1.008C - F - Gn \implies 0.008Bn - B + 0.008C = F + Gn$. Hence $B = 125G$ and $-B + 0.008C = -125G + 0.008C = F \implies C = 125(F + 125G)$. It now remains to determine A. Since $M_0 = 10^6 = A + C$, $A = 10^6 - C = 10^6 - 125(F + 125G)$.

(c) Set $M_{60} = 0$. Then

$$(10^6 - 125(F + 125G)) \times 1.008^{60} + 60 \times 125G + 125(F + 125G) = 0$$

and we solve for G in terms of F. Indeed,

$$-15625 \times 1.008^{60} \times G + (10^6 - 125F) \times 1.008^{60} + 23125G + 125F = 0$$

$$G = \frac{(10^6 - 125F) \times 1.008^{60} + 125F}{15625 \times 1.008^{60} - 23125}.$$

Since G varies linearly with F and is decreasing, $\max\limits_{F \geq 2500} G \approx 684$ (3sf) occurs at $\min\limits_{F \geq 2500} F = 2500$.

7 **(a)** T admits two eigenvalues, one with multiplicity 2. By GC,

$$p_T(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 3 & 2-\lambda & -1 \\ 3 & -2 & 3-\lambda \end{vmatrix} = (\lambda+2)(\lambda-4)^2 = 0$$

implies $\lambda = -2$ or $\lambda = 4$.
If $\lambda = 4$, then

$$[T - \lambda \cdot \mathrm{id}]_\beta \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} -3 & 2 & 1 \\ 3 & -2 & -1 \\ 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} -3 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}.$$

Hence $z = 3x - 2y$, and the equation of Π is

$$\Pi : \mathbf{r} = a\begin{pmatrix}1\\0\\3\end{pmatrix} + b\begin{pmatrix}0\\1\\-2\end{pmatrix} \quad \forall a, b \in \mathbb{R}.$$

Else if $\lambda = -2$, then

$$\operatorname{rref}\begin{pmatrix}3 & 2 & 1\\3 & 4 & -1\\3 & -2 & 5\end{pmatrix} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix} \implies l : \mathbf{r} = a\begin{pmatrix}-1\\1\\1\end{pmatrix} \quad \forall a \in \mathbb{R}.$$

(b) **(i)** For no unique solutions, $\det(T - t \cdot \mathrm{id}) = 0$. Hence the values of t are the eigenvalues of T, i.e. $t = -2$ and $t = 4$.

(ii) If $t = -2$ then $(a, b, c) \in \mathbb{R}^3$ must be on l. Hence $-a = b = c$ for a solution to exist.
If $t = 4$ then (a, b, c) must lie on Π. Hence $c = 3a - 2b$.

8 **(a)** $u = x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y \implies \dfrac{\mathrm{d}u}{\mathrm{d}x} = x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} \implies \dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = x\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Hence $\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} + u = 2$.
The characteristic equation is $\lambda^2 + 1 = 0$ so $\lambda = \pm\mathrm{i}$. Hence a solution to the *complementary* equation in u is $u = C_1\sin(x) + C_2\cos(x)$ for all $C_1, C_2 \in \mathbb{R}$. Thus a solution to the nonhomogenous DE in u is $u = C_1\sin(x) + C_2\cos(x) + 2$.

Let $C_1\sin(x) + C_2\cos(x) + 2 = x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y$. Then let $v = xy$ so $\dfrac{\mathrm{d}v}{\mathrm{d}x} = x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = C_1\sin(x) + C_2\cos(x) + 2$. Hence

$$v = -C_1\cos(x) + C_2\sin(x) + 2x + C_3 \implies y = -\frac{C_1\cos(x)}{x} + \frac{C_2\sin(x)}{x} + 2 + \frac{C_3}{x}.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = C_1 \times \frac{x\sin(x) + \cos(x)}{x^2} + C_2 \times \frac{x\cos(x) - \sin(x)}{x^2} - \frac{C_3}{x^2}.$$

When $x = \dfrac{\pi}{2}$, $y = 2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2}{\pi}$:

$$\frac{2C_2}{\pi} + 2 + \frac{2C_3}{\pi} = 2 \implies C_2 = -C_3,$$

$$\frac{2C_1}{\pi} - \frac{4C_2 + 4C_3}{\pi^2} = -\frac{2}{\pi} \implies C_1 = -1.$$

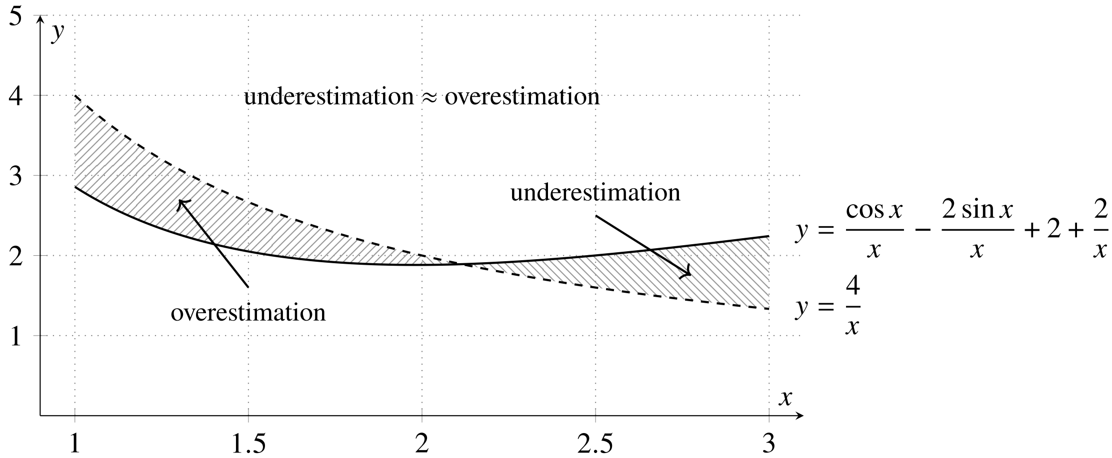
When $x = \pi$, $y = 2 + \dfrac{1}{\pi}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2}{\pi} - \dfrac{1}{\pi^2}$.

$$-\frac{1}{\pi} + 2 + \frac{C_3}{\pi} = 2 + \frac{1}{\pi} \implies C_3 = 2 \implies C_2 = -2.$$

It follows that

$$y = \frac{\cos(x)}{x} - \frac{2\sin(x)}{x} + 2 + \frac{2}{x}.$$

(b) By GC, $\displaystyle\int_1^3 \left(\frac{\cos(x)}{x} - \frac{2\sin(x)}{x} + 2 + \frac{2}{x}\right)\mathrm{d}x \approx 4.17431$ and $\displaystyle\int_1^3 \frac{4}{x}\,\mathrm{d}x \approx 4.39445$, both values to 5dp.
The relative error $\varepsilon = \left|\dfrac{4.17431 - 4.39445}{4.17431}\right| \times 100\% \approx 5.27\%$ to 3sf, which is a small error.
Considering the graphs:



the area of the overestimation is approximately the area of the underestimation, so $\int_1^3 y \, dx = \int_1^3 \frac{4}{x} \, dx -$ overestimation + underestimation $\approx \int_1^3 \frac{4}{x} \, dx$ and so $\int_1^3 \frac{4}{x} \, dx$ is suitable to approximate $\int_1^3 y \, dx$.

9 **(a)** Write $Q = \Sigma(y - ax^2 - bx)^2$. Then $(y - ax^2 - bx)^2 = y^2 + a^2x^4 + b^2x^2 - 2ax^2y - 2bxy + 2abx^3$. Hence $Q = \Sigma(y^2 + a^2x^4 + b^2x^2 - 2ax^2y - 2bxy + 2abx^3) = \Sigma(y^2) + a^2\Sigma(x^4) + b^2\Sigma(x^2) - 2a\Sigma(x^2y) - 2b\Sigma(xy) + 2ab\Sigma(x^3)$.

(b) $\frac{\partial Q}{\partial a} = 2a\Sigma(x^4) - 2\Sigma(x^2y) + 2b\Sigma(x^3)$ and $\frac{\partial Q}{\partial b} = 2b\Sigma(x^2) - 2\Sigma(xy) + 2a\Sigma(x^3)$.

(c) **(i)** Since Q is minimal, $\nabla Q = \mathbf{0}$. Hence

$$\begin{aligned} a\Sigma(x^4) + b\Sigma(x^3) &= \Sigma(x^2y) \\ a\Sigma(x^3) + b\Sigma(x^2) &= \Sigma(xy) \end{aligned}$$

and this is a linear system such that $\overbrace{\begin{pmatrix} \Sigma(x^4) & \Sigma(x^3) \\ \Sigma(x^3) & \Sigma(x^2) \end{pmatrix}}^{\mathbf{A}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \Sigma(x^2y) \\ \Sigma(xy) \end{pmatrix} = \mathbf{u}$.

(ii) Recall that if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Thus

$$\begin{pmatrix} \Sigma(x^4) & \Sigma(x^3) \\ \Sigma(x^3) & \Sigma(x^2) \end{pmatrix}^{-1} = \frac{1}{\Sigma(x^4)\Sigma(x^2) - (\Sigma(x^3))^2} \begin{pmatrix} \Sigma(x^2) & -\Sigma(x^3) \\ -\Sigma(x^3) & \Sigma(x^4) \end{pmatrix}.$$

Considering $\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{A}^{-1}\mathbf{u}$,

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{\Sigma(x^4)\Sigma(x^2) - (\Sigma(x^3))^2} \begin{pmatrix} \Sigma(x^2) & -\Sigma(x^3) \\ -\Sigma(x^3) & \Sigma(x^4) \end{pmatrix} \begin{pmatrix} \Sigma(x^2y) \\ \Sigma(xy) \end{pmatrix} \\ &= \frac{1}{\Sigma(x^4)\Sigma(x^2) - (\Sigma(x^3))^2} \begin{pmatrix} \Sigma(x^2)\Sigma(x^2y) - \Sigma(x^3)\Sigma(xy) \\ -\Sigma(x^3)\Sigma(x^2y) + \Sigma(x^4)\Sigma(xy) \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} a &= \frac{\Sigma(x^2)\Sigma(x^2y) - \Sigma(x^3)\Sigma(xy)}{\Sigma(x^4)\Sigma(x^2) - (\Sigma(x^3))^2} \\ b &= \frac{-\Sigma(x^3)\Sigma(x^2y) + \Sigma(x^4)\Sigma(xy)}{\Sigma(x^4)\Sigma(x^2) - (\Sigma(x^3))^2} \end{aligned}$$

(d) Q is a sum of squares, so it is bounded below. Hence the values of a and b in **(c)(i)**, which give a stationary value of Q, must correspond to a minimum value of Q.

(e) (We are asked to find *approximate* values of a and b.) Note that for all the points, $x \approx y$. Hence $\Sigma(x^2y) \approx \Sigma(x^3)$ and $\Sigma(xy) \approx \Sigma(x^2)$. Using the result in **(c)(ii)**, $a \approx 0$ and $b \approx 1$.

10 **(a)** **(i)** $z = \exp(\frac{2\pi k \mathrm{i}}{15})$ for $k = 0, \dots, 14$.

(ii) $z^{15} - 1 = (z^5)^3 - 1 = (z^5 - 1)(z^{10} + z^5 + 1)$. Let $z = \cos(\frac{2\pi}{15}) + \mathrm{i}\sin(\frac{2\pi}{15}) = \exp(\frac{2\pi \mathrm{i}}{15})$. Clearly $\exp(\frac{2\pi \mathrm{i}}{15})$ is not a solution to $z^5 - 1 = 0$ so it must be that $z^{10} + z^5 + 1 = 0$.

(iii) $z^n + z^{-n} = \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta) = 2\cos(n\theta)$.

(b) Let $z = \cos(\frac{2\pi}{15}) + \mathrm{i}\sin(\frac{2\pi}{15})$. Then $z^{10} + z^5 + 1 = 0$ so $z^5 + z^{-5} + 1 = 0$. See that $z^n + z^{-n} = 2\cos(\frac{2n\pi}{15})$ so $2\cos(5 \times \frac{2\pi}{15}) + 1 = 0$.

By De Moivre's theorem notice that

$$\begin{aligned}
\cos(5\theta) = \Re[(\cos(\theta) + \mathrm{i}\sin(\theta))^5] &= \cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos(\theta)\sin^4(\theta) \\
&= \cos^5(\theta) - 10\cos^3(\theta)(1 - \cos^2(\theta)) + 5\cos(\theta)(1 - \cos^2(\theta))^2 \\
&= 16\cos^5(\theta) - 20\cos^3(\theta) + 5\cos(\theta).
\end{aligned}$$

Hence $32\cos^5(\frac{2\pi}{15}) - 40\cos^3(\frac{2\pi}{15}) + 10\cos(\frac{2\pi}{15}) + 1 = 0$. Set $x = 2\cos(\frac{2\pi}{15})$ so $x^5 - 5x^3 + 5x + 1 = 0$. But then notice that $x^5 - 5x^3 + 5x + 1 = (x+1)(x^4 - x^3 - 4x^2 + 4x + 1)$. Clearly $2\cos(\frac{2\pi}{15}) + 1 \neq 0$ so it must be that $x^4 - x^3 - 4x^2 + 4x + 1 = 0$.

(c) We bound the value of $\cos(\frac{2\pi}{15})$. Recall that $\cos : (0, \pi) \to (-1, 1)$ is strictly decreasing, so $\frac{2}{15} = \frac{4}{30} < \frac{5}{30} = \frac{1}{6} \implies \cos(\frac{2\pi}{15}) > \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. We now have a reference point to compare the roots against.

Set $x = 2\cos(\frac{2\pi}{15})$. We first compare $x_1 = \frac{1}{4}(1 - \sqrt{5} + \sqrt{6(5+\sqrt{5})})$. Check that $\sqrt{5} > 1$ so

$$\frac{1}{4}\left(1 - \sqrt{5} + \sqrt{6(5+\sqrt{5})}\right) \leq \frac{1}{4}\sqrt{6(5+\sqrt{5})} \leq \frac{1}{4}\sqrt{6(5+3)} = \frac{\sqrt{48}}{2} = \sqrt{3} < 2\cos\left(\frac{2\pi}{15}\right)$$

noting that $5 \leq 9 \implies \sqrt{5} \leq 3$. Hence $x \neq x_1$, and $x = x_2 = \frac{1}{4}(1 + \sqrt{5} + \sqrt{6(5 - \sqrt{5})})$.