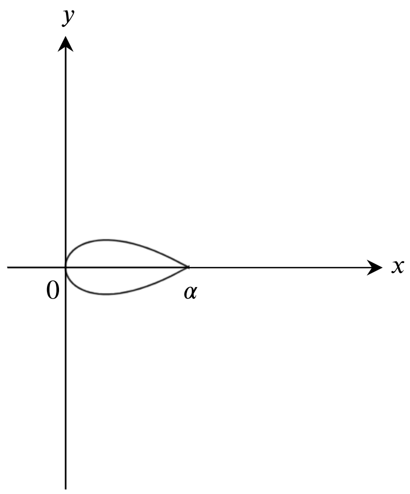
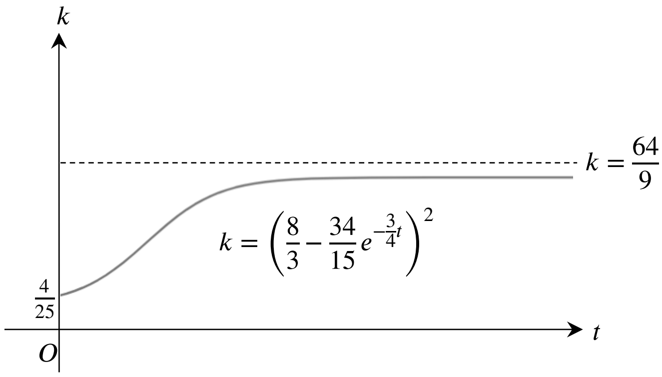
MATHEMATICS

Paper 9758/01
Set II – Paper 1

Topic identification and short answers

Qn	Topic(s)	Part	Answers
1	Graphs and transformations		Required sequence of transformations: 1st: Translation of $\frac{1}{2}\pi$ units to the positive x–direction 2nd: Scaling of scale factor $\frac{1}{2}$ parallel to the x–axis 3rd: Translation of 1 unit to the positive y–direction 4th: Scaling of scale factor $\frac{1}{2}$ parallel to the y–axis (Other possible sequences acceptable.)
2	Maclaurin series	(a)	$\cos(x+\sin x) \approx 1-2x^2+x^4$
		(b)	$\frac{\pi}{3}-\frac{\pi^3}{324}+\frac{\pi^5}{38880}$ or $2\sin\frac{1}{2}$
3	Sequences and series (arithmetic series)		$u_n = 5+(n-1)3$
4	Differentiation (concept); Maclaurin series (small angle approximation)	(a)	[shown]
		(b)	[shown]
5	Inequalities	(a)	[shown]
		(b)	$\frac{\pi}{4}<x\le\frac{\pi}{3}$ or $\frac{2\pi}{3}\le x<\frac{3\pi}{4}$
6	Sequences and series (geometric series); Complex numbers (modulus); Differentiation	(a)	[shown]
		(b)	$\lvert z\rvert = 1$
		(c)	[shown]
		(d)	[shown]
7	Integration techniques (by parts); Definite integrals (volume of revolution)	(a)	$I_1 = \tan^{-1} c$
		(b)	[shown]
		(c)	Volume $=\frac{37}{48}\pi-\frac{5}{64}\pi^2$ units^3

8	Functions (inverse, composite, domain restriction); Inequalities	**(a) (i)**	[shown]
		(a) (ii)	$\mathrm{R_h} = \left(-\infty, q^2 - 6\right]$ [shown]
		(a) (iii)	$\mathrm{gh}(x) = \dfrac{x^2 - 2qx + 13}{\lvert x^2 - 2qx + 16\rvert} = \dfrac{x^2 - 2qx + 13}{x^2 - 2qx + 16}$ $\mathrm{R_{gh}} = \left[\dfrac{13 - q^2}{\lvert 16 - q^2\rvert}, 1\right) = \left[\dfrac{13 - q^2}{16 - q^2}, 1\right)$ (Other equivalent forms acceptable.)
		(b) (i)	$\mathrm{XY}(2) = 8$ $\mathrm{YX}^{-1}(5) = 7$ $\mathrm{X}^{-1}\mathrm{X}^{-1}\mathrm{Y}(4) = 2$
		(b) (ii)	Both Y^{-1} and YX does not exist.
9	Graph (parametric); Inequality; Differentiation; Definite integrals	**(a)**	$3\alpha y^2 = x(x - \alpha)^2$ (Other equivalent forms acceptable.)
		(b)	
		(c)	Surface area $= \dfrac{\pi}{3}\alpha^2$ units^2

11	Vectors (tree dimensions, angle between two lines, cartesian equations, distances)	**(a)**	[shown]
		(b)	[shown] $\phi = \frac{\pi}{6}$
		(c)	The direction vector of m is not parallel to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$.
		(d)	$P\ :\ \left(2\cos\theta + 2\sqrt{3}, 2\sqrt{2}\sin\theta, 2\cos\theta - 2\sqrt{3}\right)$
		(e)	$C\ :\ \left(2\sqrt{3}, 0, -2\sqrt{3}\right)$ $\left\lvert\overrightarrow{CP}\right\rvert = 2\sqrt{2}$ units As θ varies, P is a circle with radius $2\sqrt{2}$ units centred at C.
		(f)	$2\sqrt{6}$ units
10	Differential equations; Differentiation (connected rates of change); Graph	**(a)**	[shown]
		(b)	$k = \left[\frac{sA}{\lambda} + \left({k_0}^{1-a} - \frac{sA}{\lambda}\right)e^{-\lambda(1-a)t}\right]^{\frac{1}{1-a}}$
		(c)	
		(d)	(Any of) increase s / A / a or decrease λ Assumption: other constants remain unchanged

Suggested solutions and post-mortem

Qn	Suggested Solutions	Post-mortem
1 **[4]**	Using trigonometric identities in MF26, $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2}\left(1+\sin\left(2x-\frac{\pi}{2}\right)\right)$ Transforming from $y = \sin x$, $\xrightarrow{1} y = \sin\left(x-\frac{\pi}{2}\right)$ $\xrightarrow{2} y = \sin\left(2x-\frac{\pi}{2}\right)$ $\xrightarrow{3} y = 1+\sin\left(2x-\frac{\pi}{2}\right)$ $\xrightarrow{4} y = \frac{1}{2}\left(1+\sin\left(2x-\frac{\pi}{2}\right)\right) = \sin^2 x$ Required sequence of transformations: 1st: Translation of $\frac{1}{2}\pi$ units to the positive x–direction (with y–axis invariant) 2nd: Scaling of scale factor $\frac{1}{2}$ parallel to the x–axis 3rd: Translation of 1 unit to the positive y–direction (with x–axis invariant) 4th: Scaling of scale factor $\frac{1}{2}$ parallel to the y–axis (Invariant axes for scaling are optional.)	While this question mainly tests on graph transformation, it also requires fluency in trigonometric manipulation. Valid sequences of transformations may vary up to of 5 or 6 steps, but a 4-step sequence is feasible, as hinted by the marks. Great care must be taken to avoid flipping signs or using an inappropriate negative scale factors and/or translation units. As a side, it is a practice in some JCs to include, on top of the scale factor and the parallel axis, the *invariant axis* as a required scaling property. The invariant axis is the reference line to which the distance from all points on the graph is being scaled. Although invariant axes are an optional detail for graph transformations in A–Levels, it must be noted that scaling can happen with respect to any line normal to the parallel axis.

2 (a) [3]	$\cos(x+\sin x)$ $= \cos\left(x + \left(x - \frac{1}{3!}x^3 + \cdots\right)\right)$ $= 1 - \frac{1}{2!}\left(2x - \frac{1}{6}x^3 + \cdots\right)^2 + \frac{1}{4!}(2x - \cdots)^4 - \cdots$ $= 1 - \frac{1}{2}\left(4x^2 - \frac{2}{3}x^4 + \cdots\right) + \frac{1}{24}\left(16x^4 - \cdots\right) - \cdots$ $= 1 - 2x^2 + \dfrac{1}{3}x^4 + \dfrac{2}{3}x^4 + \cdots$ $\approx 1 - 2x^2 + x^4$	Expansions for nested standard functions such as this one is most amiably done by expanding the inner function first (in this case, the sine function), followed by the outer function (in this case, the cosine function.) Where expansions are abridged, be reminded to use ellipses or approximation signs as appropriate.
2 (b) [3]	**<u>Method 1: Considering expansion</u>** $\cos(x+\sin x) + \cos(x - \sin x)$ $= 1 - 2x^2 + x^4 + \cos\left(x - \left(x - \frac{1}{6}x^3 + \cdots\right)\right)$ $= 1 - 2x^2 + x^4 + \cos\left(\frac{1}{6}x^3 - \cdots\right)$ $= 1 - 2x^2 + x^4 + 1 - \frac{1}{2}\left(\frac{1}{6}x^3 - \cdots\right)^2 + \cdots$ $\approx 2 - 2x^2 + x^4$ $\therefore \int_0^{\frac{\pi}{6}} [\cos(x+\sin x) + \cos(x-\sin x)]\,\mathrm{d}x \approx \int_0^{\frac{\pi}{6}} 2 - 2x^2 + x^4\,\mathrm{d}x$ $= \left[2x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^{\frac{\pi}{6}} = \frac{1}{3}\pi - \frac{1}{324}\pi^3 + \frac{1}{38880}\pi^5$ **<u>Method 2: Direct integration via angle formula</u>** $\int_0^{\frac{\pi}{6}} [\cos(x+\sin x) + \cos(x-\sin x)]\,\mathrm{d}x$ $= \int_0^{\frac{\pi}{6}} [\cos x\cos(\sin x) - \sin x\sin(\sin x) + \cos x\cos(\sin x) + \sin x\sin(\sin x)]\,\mathrm{d}x$ $= 2\int_0^{\frac{\pi}{6}} \cos x\cos(\sin x)\,\mathrm{d}x$ $= 2[\sin(\sin x)]_0^{\frac{\pi}{6}} = 2\sin\dfrac{1}{2}$	There are two ways to obtain the value required: by further expansion, or by direct integration. However, be wary of the fact that values obtained from abridged expansions, although expressed in exact form, are still approximations. Unless the exact form is obtained from direct integration, approximations require the use of the appropriate sign. As a side, integrals that evaluate to self-nested functions has appeared in past A–Levels and preliminary examinations, one example being $\int \frac{1}{x\ln x}\,\mathrm{d}x$, from which this question was inspired.

3
[6]

Sum of the arithmetic progression S_n is given by:

$$S_n = \frac{n}{2}(2a + (n-1)d) = an + \frac{dn}{2}(n-1) = an + \frac{1}{2}dn^2 - \frac{1}{2}dn$$

Using $n = p, 2p$ and $3p$, we have:

$$S_p = ap + \frac{1}{2}dp^2 - \frac{1}{2}dp = 185$$
$$S_{2p} = 2ap + 2dp^2 - dp = 670$$
$$S_{3p} = 3ap + \frac{9}{2}dp^2 - \frac{3}{2}dp = 1455$$

Solving the three equations simultaneously using GC,

$$ap = 35 + \frac{1}{2}dp, \qquad dp^2 = 300$$

Considering integer values of d and p such that $dp^2 = 300$,

d	300	75	12	**3**
p	1	2	5	**10**
	$d > p$	$d > p$	$d > p$	$\boldsymbol{d < p}$

$\therefore d = 3, p = 10$

$\therefore a = \frac{35}{10} + \frac{3}{2} = 5$

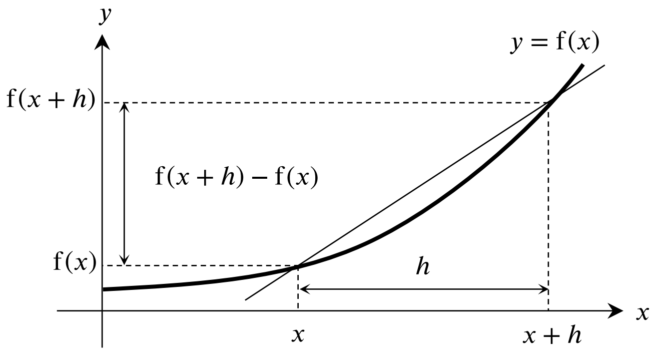
General term u_n of the arithmetic sequence is:

$u_n = 5 + (n-1)3$

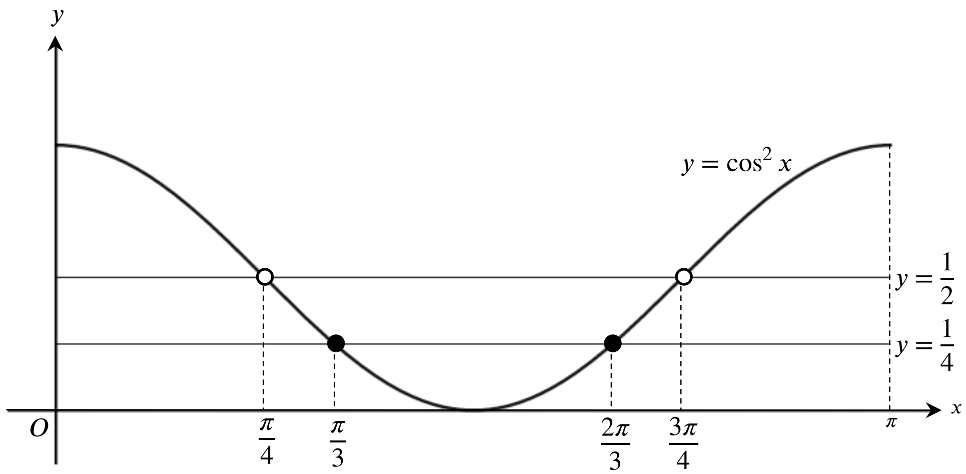
This question concerns the topic of arithmetic progression and systems of linear equations. With the formula for arithmetic series a system of linear equations in terms of (the products of) a, d and p, can be obtained. The remaining follows using GC, or through manual derivation by hand.

Since there are limited pairs of integers d and p such that $dp^2 = 300$, it is possible to proceed by simply listing out all possible cases to arrive at the final general formula.

As a side, this question serves as an example in which the system of linear equation formed does not directly yield the final values of the variables of interest. Such questions may expect candidates to use knowledge from other topics to find the required values.

4 **[2]**	 From the diagram, $\dfrac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}$ is the gradient of the line passing $\left(x, \mathrm{f}(x)\right)$ and $\left(x+h, \mathrm{f}(x+h)\right)$. As $h \to 0$, the point $\left(x+h, \mathrm{f}(x+h)\right)$ approaches $\left(x, \mathrm{f}(x)\right)$ such that the line becomes tangent to the curve at $\left(x, \mathrm{f}(x)\right)$. The gradient of this tangent corresponds to the derivative of f at that point x, namely $\mathrm{f}'(x)$.	This question concerns the concept of differentiation as a slope of the tangent, which is a prerequisite knowledge under O–Level Additional Mathematics. Candidates must show using the sketch that the expression will form a line which increasingly appears tangential as $h \to 0$.
[4]	When $\mathrm{f}(x) = \cos bx$, $\mathrm{f}'(x) = \lim\limits_{h\to 0} \dfrac{\cos b(x+h) - \cos bx}{h}$ $= \lim\limits_{h\to 0} \dfrac{\cos(bx)\cos(bh) - \sin(bx)\sin(bh) \; - \cos bx}{h}$ $= \lim\limits_{h\to 0} \dfrac{\cos(bx)\left[\cos(bh) - 1\right] - \sin(bx)\sin(bh)}{h}$ Since bh is small as $h \to 0$, using small angle approximation, $\mathrm{f}'(x) \; = \lim\limits_{h\to 0} \dfrac{\cos(bx)\left[1 - \dfrac{b^2h^2}{2} - 1\right] - bh\sin(bx)}{h}$ $= \lim\limits_{h\to 0} -\dfrac{b^2h}{2}\cos(bx) - b\sin(bx)$ $= -b\sin bx$	This question, if done properly, will summon the use of small angle approximations. Nonetheless, acceptable responses will still depend on the ability to approximate behaviour of trigonometric functions as $h \to 0$.

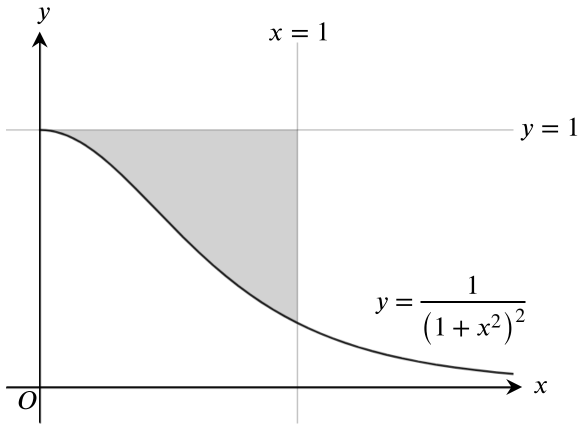
5 (a) [4]	Since the inequality reduces to a range of real values, $\rightarrow 2x+3 \geq 0 \rightarrow x \geq -\frac{3}{2}$ $\rightarrow 4x-1 \geq 0 \rightarrow x \geq \frac{1}{4}$ Since $\frac{1}{4} > -\frac{3}{2}$, the two inequalities reduce to $x \geq \frac{1}{4}$. Since both sides are positive, squaring both sides: $2x+3 > 1+2\sqrt{4x-1}+4x-1$ $-2x+3 > 2\sqrt{4x-1}$ Since $2\sqrt{4x-1} > 0, -2x+3 > 0 \rightarrow x < \frac{3}{2}$ Squaring both sides again, $4x^2-12x+9 > 4(4x-1)$ $4x^2-28x+13 > 0$ $(2x-1)(2x-13) > 0$ $\therefore x < \frac{1}{2}$ or $x > \frac{13}{2}$ $\left(\text{reject since } x < \frac{3}{2}\right)$ $\therefore \frac{1}{4} \leq x < \frac{1}{2}$	This question calls for the ability to solve square root inequalities. To begin, it must be understood that: • **unsigned square root functions are positive,** and • **since the final range to be shown is real, the value inside the square roots cannot be negative** (which otherwise would yield complex numbers). With these two understandings, the remaining steps follow with mathematical justification. The most obvious start is to square both sides of the inequality. This will yield another square function, which signals the need to square both sides another time. Throughout the intermediary steps, any spurious ranges can be rejected validly with deductions based on the two understandings above.

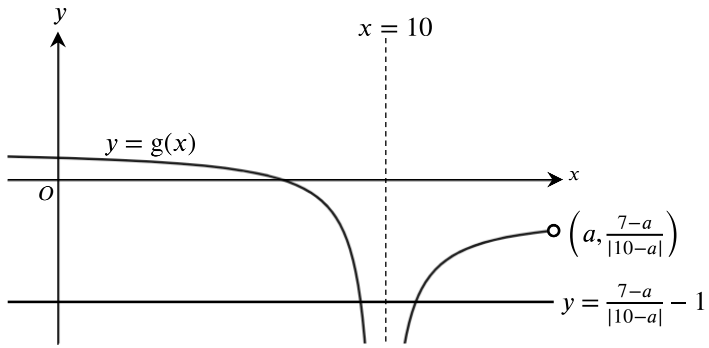
5 (b) [3]	$\sqrt{\cos 2x + 4} > 1 + \sqrt{2\cos 2x + 1}$ $\sqrt{2\cos^2 x - 1 + 4} > 1 + \sqrt{2(2\cos^2 x - 1) + 1}$ $\sqrt{2\cos^2 x + 3} > 1 + \sqrt{4\cos^2 x - 1}$ Using the result in **(i)**, replace x with $\cos^2 x$: $\frac{1}{4} \le \cos^2 x < \frac{1}{2}$ Using GC,  $\therefore \frac{\pi}{4} < x \le \frac{\pi}{3}$ or $\frac{2\pi}{3} \le x < \frac{3\pi}{4}$	With trigonometric manipulation, it will become apparent that the inequality can be expressed in the form similar to **(a)**. Do note the values that need to be included or excluded in the solution intervals. Note that, unlike the previous part, it is no longer required to find the result "algebraically" and thus candidates should not be discouraged from graphical methods, which would otherwise cause some solution interval to be omitted. Similarly, as there is no prohibition against the use of GC in this question, non-exact unsupported answers from GC are indeed allowed. Such approach does not even require the trigonometric manipulation at the start since candidates can directly sketch graphs and deduce the solution intervals accordingly to give non-exact intervals. This may be an appealing strategy for candidates who may not be able to properly see the required manipulation under time constraint.

6 (a) [1]	$\sum_{r=1}^{n} z^{2r-1}$ is a geometric series with initial term z and ratio z^2 $\therefore \sum_{r=1}^{n} z^{2r-1} = z\left(\frac{1-\left(z^2\right)^n}{1-z^2}\right) = \frac{z}{z}\left(\frac{1-z^{2n}}{z^{-1}-z}\right) = \frac{1-z^{2n}}{z^{-1}-z}$	This question aims to show that geometric series can be applied to complex numbers.
6 (b) [1]	$\lvert z\rvert = \left\lvert\sqrt{(\cos^2\theta+\sin^2\theta)}\right\rvert = 1$	This question tests candidates on the evaluation of modulus calculation given a non-constant complex number.
6 (c) [4]	Using de Moivre's theorem, $\rightarrow \sum_{r=1}^{n} \lvert z\rvert^{2r-1}\{\cos[(2r-1)\theta]+\mathrm{i}\sin[(2r-1)\theta]\} = \frac{1-\lvert z\rvert^{2n}(\cos 2n\theta+\mathrm{i}\sin 2n\theta)}{\lvert z\rvert^{-1}(\cos(-\theta)+\mathrm{i}\sin(-\theta)) - \lvert z\rvert(\cos\theta+\mathrm{i}\sin\theta)}$ $\rightarrow \sum_{r=1}^{n}\{\cos[(2r-1)\theta]+\mathrm{i}\sin[(2r-1)\theta]\} = \frac{1-(\cos 2n\theta+\mathrm{i}\sin 2n\theta)}{(\cos(-\theta)+\mathrm{i}\sin(-\theta)) - (\cos\theta+\mathrm{i}\sin\theta)}$ 6 $\therefore \sum_{r=1}^{n}\cos[(2r-1)\theta] + \mathrm{i}\sum_{r=1}^{n}\sin[(2r-1)\theta]$ $= \frac{1-\cos 2n\theta-\mathrm{i}\sin 2n\theta}{\cos\theta-\mathrm{i}\sin\theta-\cos\theta-\mathrm{i}\sin\theta}$ $= \frac{1-\cos 2n\theta - i\sin 2n\theta}{-2\mathrm{i}\sin\theta}\times\frac{\mathrm{i}}{\mathrm{i}}$ $= \frac{(1-\cos 2n\theta)\mathrm{i}+\sin 2n\theta}{2\sin\theta}$ $= \frac{\sin 2n\theta}{2\sin\theta} + \mathrm{i}\left(\frac{1-\cos 2n\theta}{2\sin\theta}\right)$ Comparing imaginary parts, $\sum_{r=1}^{n}\sin[(2r-1)\theta] = \frac{1-\cos 2n\theta}{2\sin\theta} = \frac{2\sin^2 n\theta}{2\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta}$	This question may prove challenging. *De Moivre's* theorem is a complex number theorem under H2 Further Mathematics. Using the theorem and the previous results, it can be deduced that $\sum_{r=1}^{n} z^{2r-1} = \sum_{r=1}^{n}\lvert z\rvert^{2r-1}\{\cos[(2r-1)\theta]+i\sin[(2r-1)\theta]\}$, or its corollary $\mathrm{Im}\left(\sum_{r=1}^{n} z^{2r-1}\right) = \sum_{r=1}^{n}\sin[(2r-1)\theta]$ which might be more obvious. Note that the theorem can (and should) be applied to the expression on the left-hand side of the equation, which also contains some z to a power. The remaining steps follow with trigonometric manipulation.

6 (d) [3]	Differentiating the result in **(ii)** with respect to θ, $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sum_{r=1}^{n}\sin[(2r-1)\theta]\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin^2 n\theta}{\sin\theta}\right)=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^2 n\theta\right)(\operatorname{cosec}\theta)$ $\sum_{r=1}^{n}(2r-1)\cos[(2r-1)\theta]=2\sin n\theta\,(n\cos n\theta)(\operatorname{cosec}\theta)+\sin^2 n\theta\,(-\operatorname{cosec}\theta\cot\theta)$ Substitute $\theta=\frac{\pi}{2n}\to n\theta=\frac{\pi}{2}$, $\sum_{r=1}^{n}(2r-1)\cos\left[\frac{(2r-1)\pi}{2n}\right]=2\sin\left(\frac{\pi}{2}\right)\left(n\cos\frac{\pi}{2}\right)\left(\operatorname{cosec}\frac{\pi}{2n}\right)+\sin^2\left(\frac{\pi}{2}\right)\left(-\operatorname{cosec}\frac{\pi}{2n}\cot\frac{\pi}{2n}\right)$ $=2(1)(0)\left(\operatorname{cosec}\frac{\pi}{2n}\right)+(1)^2\left(-\operatorname{cosec}\frac{\pi}{2n}\cot\frac{\pi}{2n}\right)$ $=-\operatorname{cosec}\frac{\pi}{2n}\cot\frac{\pi}{2n}$	It is apparent that the result calls for differentiation as $(2r-1)\cos[(2r-1)\theta]=\frac{\mathrm{d}}{\mathrm{d}\theta}\sin[(2r-1)\theta]$. The rest of the steps follow with proper substitution and trigonometric identities.

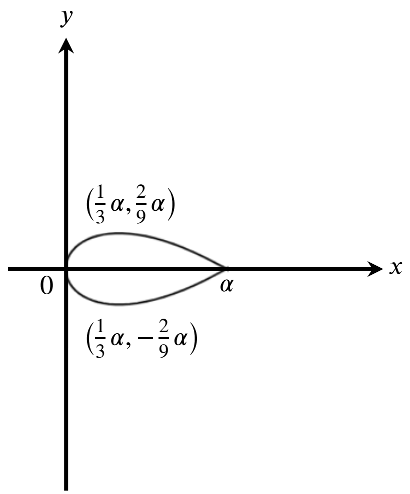
7 (a) [1]	$I_1 = \int_0^c \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^c = \tan^{-1} c$	This integration is fairly straightforward and can be easily deduced from the List of Formulae (MF27).
7 (b) [5]	Using integration by parts, I_n $= \left[(x)\left(\frac{1}{(1+x^2)^n}\right)\right]_0^c - \int_0^c (x)(-n)\left(\frac{1}{(1+x^2)^{n+1}}\right)(2x)\,\mathrm{d}x$ $= \frac{c}{(1+c^2)^n} + 2n\int_0^c \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x$ $= \frac{c}{(1+c^2)^n} + 2n\int_0^c \frac{1+x^2-1}{(1+x^2)^{n+1}}\,\mathrm{d}x$ $= \frac{c}{(1+c^2)^n} + 2n\int_0^c \frac{1+x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x - 2n\int_0^c \frac{1}{(1+x^2)^{n+1}}\,\mathrm{d}x$ $= \frac{c}{(1+c^2)^n} + 2n\int_0^c \frac{1}{(1+x^2)^{n}}\,\mathrm{d}x - 2n\int_0^c \frac{1}{(1+x^2)^{n+1}}\,\mathrm{d}x$ $= \frac{c}{(1+c^2)^n} + 2nI_n - 2nI_{n+1}$ $\rightarrow 2nI_{n+1} = \frac{c}{(1+c^2)^n} + (2n-1)I_n$ $\therefore I_{n+1} = \frac{c}{2n(1+c^2)^n} + \left(\frac{2n-1}{2n}\right)I_n$	Integral equations of this form are known as *reduction formula* of an integral. While reduction formula is not formally tested in A–Levels, it has often appeared in preliminary exams as an application of integration by parts. Observe that I_{n+1} and I_n are in the equation. This means that the integrand $\frac{1}{(1+x^2)^n}$ will undergo differentiation, signalling the need for integration by parts. The remaining steps follow with proper algebraic manipulation.

7 (c) [4]	Using GC to sketch the curve,  $\therefore$ Required volume $= \pi(1)^2(1) - \pi \int_0^1 \left(\frac{1}{(1+x^2)^2}\right)^2 \mathrm{d}x$ $= \pi - \pi \int_0^1 \frac{1}{(1+x^2)^4} \mathrm{d}x$ $= \pi\left[1 - I_4\right]$ $= \pi\left[1 - \left(\frac{1}{6(2)^3} + \frac{5}{6}I_3\right)\right]$ $= \pi\left[1 - \frac{1}{48} - \frac{5}{6}\left(\frac{1}{4(2)^2} + \frac{3}{4}I_2\right)\right]$ $= \pi\left[1 - \frac{1}{48} - \frac{5}{96} - \frac{5}{8}\left(\frac{1}{2(2)^1} + \frac{1}{2}I_1\right)\right]$ $= \pi\left[1 - \frac{1}{48} - \frac{5}{96} - \frac{5}{32} - \frac{5}{16}\left(\tan^{-1} 1\right)\right]$ $= \pi\left[\frac{37}{48} - \frac{5}{16}\left(\frac{\pi}{4}\right)\right]$ $= \frac{37}{48}\pi - \frac{5}{64}\pi^2$ units^3	This question concerns definite integrals for volume of revolution. It can be easily deduced, without a sketch, that $c = 1$. Using the previous reduction formula, the volume required involves slowly and carefully evaluating I_4 down to I_1. Do note that I_4 is not the final required volume, which is easily apparent through a sketch of the region of interest.

8 (a) (i) [2]	 **For any *a***, the line $y = \dfrac{7-a}{\lvert 10-a \rvert} - 1$ cuts the graph $y = \mathrm{g}(x)$ twice. $\therefore$ g is not a one $-$ one function. $\therefore$ g^{-1} does not exist when $a > 10$.	While the explanation for the existence of an inverse of a function is fairly standard, the explanation required in this part requires more care. The sketch to be drawn (and explained) not only must have the correct shape, but also: • end at some value to be excluded, to show awareness of the domain restriction, and • include a line of the equation $y = \mathrm{g}(a) - k$, where $k > 0$, to show the awareness that $y = \mathrm{g}(a)$ will not intersect the graph twice.
8 (a) (ii) [3]	**<u>Using completing the square</u>** $\mathrm{h}(x) = -x^2 + 2qx - 6 = -(x-q)^2 + q^2 - 6$ $\therefore \mathrm{R_h} = \left(-\infty, q^2 - 6\right]$ **<u>OR Using differentiation</u>** $\mathrm{h}'(x) = -2x + 2q, \qquad \mathrm{h}''(x) = -2$ The turning point is a maximum at $\mathrm{h}'(x) = 0 \rightarrow x = q$, $y = -q^2 + 2q^2 - 6 = q^2 - 6$ $\therefore \mathrm{R_h} = \left(-\infty, q^2 - 6\right]$ $\lvert q \rvert < 4$ $\rightarrow 0 < q^2 < 16$ $\rightarrow -6 < q^2 - 6 < 10$ Since $\mathrm{D_g} = (-\infty, 10)$, $\mathrm{R_h} \subseteq \mathrm{D_g}$ for any $q \rightarrow$ gh exists	Since h is a quadratic function, finding $\mathrm{R_h}$ can easily be done by completing the square or by differentiation. The remaining part on explaining why gh exists requires manipulating the restriction $\lvert q \rvert < 4$, using inequality techniques for absolute signs, to show that any q in this range will satisfy $\mathrm{R_h} \subseteq \mathrm{D_g}$. Do note that mathematical notations must be correctly used, which may otherwise be penalised.

8 (a) (iii) [3]	$\text{gh}(x) = \text{g}\left(-x^2 + 2qx - 6\right)$ $= \dfrac{7 - \left(-x^2 + 2qx - 6\right)}{\lvert 10 - (-x^2 + 2qx - 6)\rvert} = \dfrac{x^2 - 2qx + 13}{\lvert x^2 - 2qx + 16\rvert}$ $= \dfrac{7 - \left(-x^2 + 2qx - 6\right)}{10 - (-x^2 + 2qx - 6)} = \dfrac{x^2 - 2qx + 13}{x^2 - 2qx + 16} = 1 - \dfrac{3}{x^2 - 2qx + 16}$ **since $\mathbf{h(x) < 10 \rightarrow \left\lvert 10 - \left(-x^2 + 2qx - 6\right)\right\rvert = 10 - \left(-x^2 + 2qx - 6\right)}$.** $D_{gh} = D_h = (-\infty, \infty)$ As $x \rightarrow -\infty$, $\text{g}(x) \rightarrow 1$ At $x = q^2 - 6$, $\text{g}\left(q^2 - 6\right)$ $= \dfrac{7 - \left(q^2 - 6\right)}{\lvert 10 - (q^2 - 6)\rvert} = \dfrac{13 - q^2}{\lvert 16 - q^2\rvert}$ $= \dfrac{7 - \left(q^2 - 6\right)}{10 - (q^2 - 6)} = \dfrac{13 - q^2}{16 - q^2} = 1 - \dfrac{3}{16 - q^2}$ **since $\mathbf{0 < q^2 < 16 \rightarrow \left\lvert 16 - q^2\right\rvert > 0}$.** $\rightarrow R_{gh} = \left[\dfrac{13 - q^2}{\lvert 16 - q^2\rvert}, 1\right) = \left[\dfrac{13 - q^2}{16 - q^2}, 1\right)$ (Any equivalent form is acceptable.)	The part on finding an expression for $gh(x)$ and stating its domain may prove straightforward. Finding R_{gh} can be tricky; the use of GC with appropriate values of q may be considered. Technically, the absolute sign may be omitted since $h(x) < 10$, as found beforehand in **(a)(i)**. However, reasoning for this must be made explicit, or otherwise the unjustified omission may lead to penalty. Amidst producing highly technical answers, be reminded still of the use of proper mathematical notation.

8 (b) (i) [1]	$XY(2) = X(7) = 8$ $YX^{-1}(5) = Y(2) = 7$ $X^{-1}X^{-1}Y(4) = X^{-1}X^{-1}(6) = X^{-1}(5) = 2$	This question may prove to be straightforward. Great care must be taken, otherwise no marks are awarded for even one incorrect result.
8 (b) (ii) [2]	Both Y^{-1} and YX does not exist. Y^{-1} does not exist as Y is not a one-one function: $Y(1) = Y(4) = 6$ YX does not exist as $R_X \nsubseteq D_Y$: $8 \in R_X$ but $8 \notin D_Y$	It should be fairly easy to cite relevant conditions for the (non-)existence of both functions. However, examples where this condition fails must be included for the explanation to be complete. Otherwise, marks may not be awarded. Do note that **interval notation may be reserved for continuous real values and thus cannot be used to refer to a set of discrete integers**. As such, instead of writing "$R_X = [2,8]$", it is encouraged to write the lengthier but more appropriate "$R_X = \{2,3,4,5,6,7,8\}$".

9 (a) [4]	$y = \alpha t\left(3t^2 - 1\right) = t\left(3\alpha t^2 - 1\right) = t(x - \alpha)$ $y^2 = t^2(x-\alpha)^2$ $3\alpha y^2 = 3\alpha t^2(x-\alpha)^2 = x(x-\alpha)^2$ Given $-\frac{\sqrt{3}}{3} \le t \le \frac{\sqrt{3}}{3}$, $\rightarrow 0 \le t^2 \le \frac{1}{3}$ $\rightarrow 0 \le 3\alpha t^2 \le \alpha$ $\therefore 0 \le x \le a$ The range of values of y is defined by the turning points of the graph $y = \alpha t\left(3t^2 - 1\right)$. (Seen from GC) $\rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 9\alpha t^2 - \alpha = 0$ $\rightarrow t = \pm\frac{1}{3}$ $\therefore -\frac{2}{9}\alpha \le y \le \frac{2}{9}\alpha$	Finding cartesian equation of a curve defined parametrically is considered routine under H2 Mathematics. Responses that yield a similar equation with the suggested solution will be awarded accordingly. Stating the range of x can be directly achieved from the given range of t. Finding the range of y requires candidates to notice the turning points of the cubic curve $y = \mathrm{f}(t)$, which could be visualised via GC. Differentiating y with respect to t eventually reveals the parameters for the maximum and minimum values.
9 (b) [2]		Using GC, producing this sketch should be straightforward. Marks are awarded for a closely symmetrical shape about the x–axis, and all correct axial intercepts and turning points. All results in part **(a)** are relevant for the labels on this sketch. As the question does not explicitly require any label in the form of coordinates, labelling the axial intercepts with the axial values (as demonstrated in the suggested solution) is still acceptable in this case, though coordinates are also welcome. If the question explicitly demands for coordinates instead, it is imperative that candidates follow through.

9 **(c)** **[7]**	<u>**Using cartesian equation in (a)**</u> $3\alpha y^2 = x(x-\alpha)^2 \rightarrow y^2 = \frac{x}{3\alpha}(x-\alpha)^2$ Implicitly differentiating with respect to x, $6\alpha y\frac{\mathrm{d}y}{\mathrm{d}x} = (x-\alpha)^2 + 2x(x-\alpha) = (x-\alpha)(3x-\alpha)$ Squaring both sides, $(6\alpha y)^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = (x-\alpha)^2(3x-\alpha)^2$ $\frac{36\alpha^2 x(x-\alpha)^2}{3\alpha}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = (x-\alpha)^2(3x-\alpha)^2$ $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{(3x-\alpha)^2}{12\alpha x}$ $\therefore \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{\frac{12\alpha x + (3x-\alpha)^2}{12\alpha x}} = \sqrt{\frac{9x^2+6\alpha x+\alpha^2}{12\alpha x}} = \frac{3x+\alpha}{\sqrt{12\alpha x}}$ Required surface area $= 2\pi\int_0^\alpha \sqrt{\frac{x}{3\alpha}}(\alpha - x)\left(\frac{3x+\alpha}{\sqrt{12\alpha x}}\right)\mathrm{d}x$ $= 2\pi\int_0^\alpha \sqrt{\frac{x}{36\alpha^2 x}}(\alpha - x)(3x+\alpha)\,\mathrm{d}x$ $= 2\pi\int_0^\alpha \frac{1}{6\alpha}\left(\alpha^2 + 2\alpha x - 3x^2\right)\mathrm{d}x$ $= \frac{\pi}{3\alpha}\left[\alpha^2 x + \alpha x^2 - x^3\right]_0^\alpha$ $= \frac{\pi}{3\alpha}\left(\alpha^3\right) = \frac{\pi}{3}\alpha^2$ units^2	This question pertains definite integrals as applied to out-of-syllabus formulae, namely surface area of revolution. From 2025 onwards, this formula is made available in the list of formulae (MF27). As the question calls for finding an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$, there are two ways to approach this question: using implicit differentiation of the equation in **(a)**, or direct differentiation of the parametric equations. Either way, the eventual integral to be evaluated proves to be simple. For the approach using implicit differentiation, the correct expression for y must be obtained. Note that since the cartesian equation in **(a)** is written in the form $y^2 = \mathrm{f}(x)$, the correct square root must be chosen for the y expression to be used in the integral. Candidates may want to introduce an absolute sign to ensure that the value obtained for the surface area is positive.

[Continued]

Otherwise (using parametric equations)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha\left[\left(3t^2 - 1\right) + t(6t)\right] = \alpha\left(9t^2 - 1\right)$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6\alpha t$$
$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left[\frac{\alpha\left(9t^2 - 1\right)}{6\alpha t}\right]^2 = \left[\frac{9t^2 - 1}{6t}\right]^2 = \frac{81t^4 - 18t^2 + 1}{36t^2}$$
$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{\frac{36t^2 + 81t^4 - 18t^2 + 1}{36t^2}} = \sqrt{\frac{81t^4 + 18t^2 + 1}{36t^2}} = \frac{9t^2 + 1}{6t}$$

When $x = 0, t = 0$

When $x = \alpha, t = \pm\frac{\sqrt{3}}{3}$ (Either value for t works.)

Required surface area

$$= 2\pi \int_\beta^\gamma y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$
$$= 2\pi \int_0^{\frac{\sqrt{3}}{3}} \alpha t\left(3t^2 - 1\right)\left(\frac{9t^2 + 1}{6t}\right)(6\alpha t)\,\mathrm{d}t$$
$$= 2\pi\alpha^2 \int_0^{\frac{\sqrt{3}}{3}} t\left(3t^2 - 1\right)\left(9t^2 + 1\right)\mathrm{d}t$$
$$= 2\pi\alpha^2 \int_0^{\frac{\sqrt{3}}{3}} 27t^5 - 6t^3 - t\,\mathrm{d}t$$
$$= 2\pi\alpha^2 \left[\frac{9}{2}t^6 - \frac{3}{2}t^4 - \frac{1}{2}t^2\right]_0^{\frac{\sqrt{3}}{3}}$$
$$= 2\pi\alpha^2\left(\frac{1}{6}\right) = \frac{\pi}{3}\alpha^2 \text{ units}^2$$

Proceeding using parametric equations might be much more straightforward in this part. Do take note that there will be two values that may correspond to t. No matter the choice of t used, the boundary values of the integration must be consistent: from a smaller value to a larger value.

Take note to append the surface area with the appropriate unit. While such omission is trivial, a strict marking scheme would have penalised it otherwise.

10 (a) [3]	$\cos\phi = \dfrac{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \cos\theta + \sqrt{3} \\ \sqrt{2}\sin\theta \\ \cos\theta - \sqrt{3} \end{pmatrix}}{\left\lvert \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rvert \left\lvert \begin{pmatrix} \cos\theta + \sqrt{3} \\ \sqrt{2}\sin\theta \\ \cos\theta - \sqrt{3} \end{pmatrix} \right\rvert}$ $= \dfrac{a\left(\cos\theta + \sqrt{3}\right) + b\left(\sqrt{2}\sin\theta\right) + c\left(\cos\theta - \sqrt{3}\right)}{\sqrt{a^2+b^2+c^2}\sqrt{\cos^2\theta + 2\sqrt{3}\cos\theta + 3 + 2\sin^2\theta + \cos^2\theta - 2\sqrt{3}\cos\theta + 3}}$ $= \dfrac{a\cos\theta + \sqrt{3}a + \sqrt{2}b\sin\theta + c\cos\theta - \sqrt{3}c}{\sqrt{a^2+b^2+c^2}\sqrt{2(\cos^2\theta + \sin^2\theta) + 6}}$ $= \dfrac{(a+c)\cos\theta + \sqrt{2}b\sin\theta + (a-c)\sqrt{3}}{2\sqrt{2}\sqrt{a^2+b^2+c^2}}$	This question assesses candidates' recognition on using cross product to relate the angle between two lines given their direction vectors. The remaining intermediary steps follow, involving simplifying the denominator using trigonometric properties.
10 (b) [3]	Independent of θ $\rightarrow\ b = 0,$ $\rightarrow\ a + c = 0 \rightarrow c = -a$ $\rightarrow m\colon \boldsymbol{r} = \mu\begin{pmatrix} a \\ 0 \\ -a \end{pmatrix} = \mu a\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mu, a \in \mathbb{R}$ $\therefore$ Cartesian equation for m: $x = -z, y = 0$ Using **(i)** to find ϕ, $\cos\phi = \dfrac{2\sqrt{3}a}{2\sqrt{2}\left\lvert \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix} \right\rvert} = \dfrac{2\sqrt{3}a}{2\sqrt{2}\left(\sqrt{2}a\right)} = \dfrac{\sqrt{3}}{2}$ Since ϕ is acute, $\phi = \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{6}$	The way to proceed in this question is to notice that the constants a, b and c may contribute to the removal of θ, which then makes ϕ independent of θ. With the conditions on the three constants obtained, the remaining results follow.

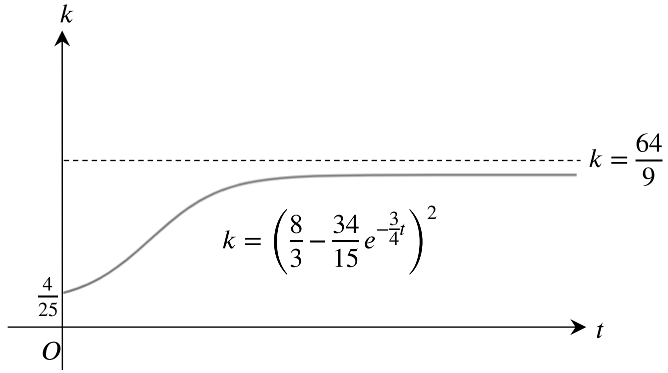
10 (c) [1]	The direction vector of m is not parallel to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$.	The previous part, **(b)**, hints to the fact that there is only one special line which makes an angle with l that is independent of θ. Therefore, if ϕ varies with θ instead, m is no longer the same as the line in **(b)**. This is as simple as saying line m is no longer parallel to the direction vector of the special line found in **(b)**.
10 (d) [2]	wall: $x - z = 4\sqrt{3} \to \boldsymbol{r}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} = 4\sqrt{3}$ $l\colon \boldsymbol{r} = \lambda\begin{pmatrix}\cos\theta+\sqrt{3}\\ \sqrt{2}\sin\theta\\ \cos\theta-\sqrt{3}\end{pmatrix},\quad \lambda\in\mathbb{R}$ To find $\boldsymbol{P}$, substitute l into the vector equation for the wall. $\left[\lambda\begin{pmatrix}\cos\theta+\sqrt{3}\\ \sqrt{2}\sin\theta\\ \cos\theta-\sqrt{3}\end{pmatrix}\right]\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} = 4\sqrt{3}$ $\lambda\left(\cos\theta+\sqrt{3}\right) - \lambda(\cos\theta - 3) = 4\sqrt{3}$ $2\sqrt{3}\lambda = 4\sqrt{3}\lambda = 2$ $\therefore$ $\boldsymbol{P}$ has coordinates $\left(2\cos\theta+2\sqrt{3},\ 2\sqrt{2}\sin\theta,\ 2\cos\theta-2\sqrt{3}\right)$	This question is simply asking for the coordinates of the intersection of a line and a wall, which can be done by solving the two equations as necessary. It must be noted that candidates may attempt this part onwards with the results shown in previous parts.

10 (e) [4]	$m: x = -z, \quad y = 0 \qquad \pi: x - z = 4\sqrt{3}$ To find C, substitute $x = -z$ into the equation of π $-z - z = -2z = 4\sqrt{3}$ $\rightarrow z = -2\sqrt{3}$ $\rightarrow x = -z = 2\sqrt{3}$ $\therefore$ C has coordinates $\left(2\sqrt{3}, 0, -2\sqrt{3}\right)$ $\therefore \left\lvert \overrightarrow{CP} \right\rvert$ $= \left\lvert \begin{pmatrix} 2\cos\theta + 2\sqrt{3} \\ 2\sqrt{2}\sin\theta \\ 2\cos\theta - 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 2\sqrt{3} \\ 0 \\ -2\sqrt{3} \end{pmatrix} \right\rvert = \left\lvert \begin{pmatrix} 2\cos\theta \\ 2\sqrt{2}\sin\theta \\ 2\cos\theta \end{pmatrix} \right\rvert$ $= \sqrt{4\cos^2\theta + 8\sin^2\theta + 4\cos^2\theta} = \sqrt{8(\cos^2\theta + \sin^2\theta)} = 2\sqrt{2}$ units, which is independent of θ. As θ varies, P traces a circle with constant radius $2\sqrt{2}$ units centred at $C\left(2\sqrt{3}, 0, -2\sqrt{3}\right)$	Proceeding in this question entails finding first the coordinates of C, which is then used to find $\overrightarrow{CP}$ and its modulus. With the constant distance, the geometrical behaviour of P as θ varies works out to be a circle, which can be defined with its radius and the coordinates of its centre. It must be noted that, in truth, there are infinitely many circles which fulfils these criteria – all of which lie on the surface of a sphere with the same centre and radius. A more complete description would include the normal of the plane on which the circle lies. In this context, P is already given to be on the wall, and as such the circle which P traces would also have the same normal as the wall, which is trivial to elucidate.

10 (f) [1]	**<u>Using $\lvert\overrightarrow{CP}\rvert$ and ϕ</u>** Let d be the required distance. $\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}=\frac{\lvert\overrightarrow{CP}\rvert}{d}$ $\rightarrow d=2\sqrt{2}\times\sqrt{3}=2\sqrt{6}$ units **<u>Using $\lvert\overrightarrow{CP}\rvert$ and $\lvert\overrightarrow{OP}\rvert$</u>** Distance required $=\sqrt{OP^2-CP^2}$ $=\sqrt{\left(2\cos\theta+2\sqrt{3}\right)^2+\left(2\sqrt{2}\sin\theta\right)^2+\left(2\cos\theta-2\sqrt{3}\right)^2-\left(2\sqrt{2}\right)^2}$ $=\sqrt{8\cos^2\theta+24+8\sin^2\theta-8}$ $=\sqrt{24}=2\sqrt{6}$ units **<u>Otherwise (using the equation of π)</u>** $\pi: \mathbf{r}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}=4\sqrt{3}$ $\therefore$ Distance from origin to π $=\frac{D}{\lvert\mathbf{n}\rvert}=\frac{4\sqrt{3}}{\left\lvert\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\rvert}=2\sqrt{6}$ units **<u>Otherwise (using $\lvert\overrightarrow{OC}\rvert$)</u>** Since m is perpendicular to π, $\therefore$ distance required $=\lvert\overrightarrow{OC}\rvert=\sqrt{\left(2\sqrt{3}\right)^2+0^2+\left(-2\sqrt{3}\right)^2}=\sqrt{24}=2\sqrt{6}$ units	There are numerous approaches to find the required distance. After the previous parts, it is most natural to arrive at the final answer using the fact that OCP is a right-angled triangle with one known side length, CP, and one known acute angle, ϕ. The result follows with simple trigonometry Without depending on any previous result, it is still feasible to calculate the distance using the given equation for π, as suggested in the third approach.

11 (a) [3]	From given information about labour, $\frac{\mathrm{d}L}{\mathrm{d}t} = \lambda L \to \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t} = \lambda$ Since k is capital-to-labour ratio, $k = \frac{K}{L}$ Differentiating k with respect to t, $\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{L\frac{\mathrm{d}K}{\mathrm{d}t} - K\frac{\mathrm{d}L}{\mathrm{d}t}}{L^2}$ $= \frac{1}{L}\left[sA(K^a)\left(L^{1-a}\right)\right] - \frac{K}{L^2}\left(\frac{\mathrm{d}L}{\mathrm{d}t}\right)$ $= sA\left(\frac{K}{L}\right)^a - \left(\frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}t}\right)\frac{K}{L}$ $= sAk^a - \lambda k$	This question requires candidates to deduce several equations and/or expressions from the contextual information given, particularly those concerning the capital-to-labour ratio as well as the amount of labour. With care, the remaining steps follow.

11 (b) [7]	Differentiating $y = Ak^{1-a}$ with respect to t, $\frac{\mathrm{d}y}{\mathrm{d}t} = (1-a)Ak^{-a}\frac{\mathrm{d}k}{\mathrm{d}t}$ $= (1-a)(Ak^{-a})(sAk^{a} - \lambda k)$ $= (1-a)\left(sA^2k^{a-a} - \lambda Ak^{1-a}\right)$ $= (1-a)\left(sA^2 - \lambda y\right)$ Solving the differential equation, $\int \frac{1}{sA^2 - \lambda y}\,\mathrm{d}y = \int (1-a)\,\mathrm{d}t$ $-\frac{1}{\lambda}\ln\left\|sA^2 - \lambda y\right\| = (1-a)t + c, \qquad c \in \mathbb{R}$ $sA^2 - \lambda y = Be^{-\lambda(1-a)t}, \quad B = \pm e^{-\lambda c}$ $y = \frac{sA^2 - Be^{-\lambda(1-a)t}}{\lambda}$ $Ak^{1-a} = \frac{sA^2}{\lambda} + De^{-\lambda(1-a)t}, D = -\frac{B}{\lambda}$ When $t = 0, k = k_0$ $\rightarrow Ak_0{}^{1-a} = \frac{sA^2}{\lambda} + D$ $\rightarrow D = Ak_0{}^{1-a} - \frac{sA^2}{\lambda}$ Substitute in D, $Ak^{1-a} = \frac{sA^2}{\lambda} + \left(Ak_0{}^{1-a} - \frac{sA^2}{\lambda}\right)e^{-\lambda(1-a)t}$ $k^{1-a} = \frac{sA}{\lambda} + \left(k_0{}^{1-a} - \frac{sA}{\lambda}\right)e^{-\lambda(1-a)t}$ $k = \left[\frac{sA}{\lambda} + \left(k_0{}^{1-a} - \frac{sA}{\lambda}\right)e^{-\lambda(1-a)t}\right]^{\frac{1}{1-a}}$	This part may prove challenging. While the many constants admittedly complicate the expression for k, the substituted differential equation is a rather standard solve. Approaching this question calls for connected rates of change, using which the substitute $y = Ak^{1-a}$ will reveal a relation between $\frac{\mathrm{d}y}{\mathrm{d}t}$ and $\frac{\mathrm{d}k}{\mathrm{d}t}$. The remaining terms in k can be substituted with y accordingly, and the result follows.

<table>
<tr>
<td>11
(c)
[2]</td>
<td>


</td>
<td>Marks from this graph sketching may be difficult to secure. Once the constants in the equation found in (b) is substituted with the given values, the resulting graph will have a point of inflexion. Using GC, it can be observed that $\frac{\mathrm{d}y}{\mathrm{d}x}$ values starting from $x = 0$ increases up to a certain point before it eventually decreases.</td>
</tr>
<tr>
<td>11
(d)
[2]</td>
<td>
$k = \left[\frac{sA}{\lambda} + \left({k_0}^{1-a} - \frac{sA}{\lambda}\right) e^{-\lambda(1-a)t}\right]^{\frac{1}{1-a}}$

As $t \to \infty$, $e^{-\lambda(1-a)t} \to 0$
$\therefore k \to \left(\frac{sA}{\lambda}\right)^{\frac{1}{1-a}}$

This limit can be increased by any one of the following:

<table>
<tr><td rowspan="3">Increasing</td><td>s
The rate of savings</td></tr>
<tr><td>A
Technological advancement</td></tr>
<tr><td>a
The exponent for capital-to-labour ratio</td></tr>
<tr><td>Decreasing</td><td>λ
The proportionality constant for labour growth</td></tr>
</table>
The suggestion will be effective assuming that other constants remain unchanged.
</td>
<td>While results from previous parts are relevant in answering this part, this part can still be answered by deducing from the initial differential equation. Note that $\frac{\mathrm{d}k}{\mathrm{d}t}$ may be increased by changing its related constants appropriately: s, A, a and λ.

Using the correct equation $k = \mathrm{f}(t)$ obtained in (b), candidates may discuss mathematically that the asymptote $k = \left(\frac{sA}{\lambda}\right)^{\frac{1}{1-a}}$ may be increased by changing any of its components appropriately.

The assumption required for this one change to be effective is that other constants must remain unchanged. (This is a tiny nod to the concept of ceteris paribus for candidates studying Economics.)</td>
</tr>
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