X JUNIOR COLLEGE
YEAR 6 PRELIMINARY EXAMINATION
Mock Arrangement
in preparation for Candidates' Examination
Higher 2

# MATHEMATICS 9758/01

Paper 1 **Set II**

**3 hours**

Additional Materials: Answer Paper
Graph paper
List of Formulae and Results (MF27)

**READ THESE INSTRUCTIONS FIRST**

Write your name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

**ABOUT THIS PAPER**

*X Junior College* (XJC) is an unofficial initiative aimed at preparing pre-university and/or junior college students in Singapore for school-level and/or national-level H2 Mathematics examinations through self-prepared mock papers. It has no affiliation with any existing institution in Singapore or worldwide.

This mock paper follows closely the 9758 H2 Mathematics GCE Advanced Level syllabus, most suitable for preparation towards preliminary examinations and A–Levels. The paper intends to explore the unconventional ways and/or applications in which topics within the syllabus can be tested, which may affect the difficulty of this paper to varying degrees. **While it is ideal to attempt this paper under examination constraints, prospective candidates are reminded not to use this potentially non-conforming paper as a definitive gauge for actual performance.**

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This document consists of **5** printed pages and **3** blank pages.

**1** Describe a sequence of transformations that transform the graph of $y = \sin x$ onto the graph of $y = \sin^2 x$. [4]

**2** **(a)** Using standard series from the list of formulae, expand $\cos(x + \sin x)$ as far as the term in $x^4$. [3]

**(b)** By expanding $\cos(x + \sin x) + \cos(x - \sin x)$ as far as the term in $x^4$, or otherwise, evaluate

$$\int_0^{\frac{\pi}{6}} [\cos(x + \sin x) + \cos(x - \sin x)]\,dx,$$

giving your answer in exact form. [3]

**3** An arithmetic series has an integer common difference $d$. The sum of the first $p$, $2p$ and $3p$ terms of the series are 185, 670, and 1455 respectively. Given that $0 < d < p$, find a general formula for the $n$th term of this series. [6]

**4** Given a continuous function f, explain, with the aid of a sketch, why the expression

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

approximates to $f'(x)$. [2]

Given that $f(x) = \cos bx$, where $b > 0$, use the expression above and suitable small angle approximations to show that $f'(x) = -b \sin bx$. [4]

**5** **(a)** Show algebraically that the inequality $\sqrt{2x + 3} > 1 + \sqrt{4x - 1}$ can be reduced to $\frac{1}{4} \leq x < \frac{1}{2}$. [4]

**(b)** Find the range(s) of values of $x$, where $0 \leq x \leq \pi$, such that $\sqrt{\cos 2x + 4} > 1 + \sqrt{2\cos 2x + 1}$. [3]

**6** **(a)** Show that $\displaystyle\sum_{r=1}^{n} z^{2r-1} = \frac{1 - z^{2n}}{z^{-1} - z}$. [1]

It is given that $z = \cos\theta + \mathrm{i}\sin\theta$, where $\sin\theta \neq 0$.

**(b)** Evaluate $|z|$. [1]

**(c)** *De Moivre's* theorem states that for any complex $\omega$ and rational $q$, $\omega^q = |\omega|^q(\cos q\theta + \mathrm{i}\sin q\theta)$. Use this theorem and the results in **(a)** and **(b)** to show that

$$\sum_{r=1}^{n} \sin[(2r - 1)\theta] = \frac{\sin^2 n\theta}{\sin\theta}.$$ [4]

**(d)** Hence, deduce that $\displaystyle\sum_{r=1}^{n} (2r - 1)\cos\left[\frac{(2r - 1)\pi}{2n}\right] = -\operatorname{cosec}\left(\frac{\pi}{2n}\right)\cot\left(\frac{\pi}{2n}\right)$. [3]

**7** The integral $I_n$ where $n = 1, 2, 3, \dots$ is given by

$$I_n = \int_0^c \frac{1}{(1+x^2)^n}\,\mathrm{d}x, \quad c > 0.$$

**(a)** Find $I_1$ in terms of $c$. [1]

**(b)** Show that $I_{n+1} = \dfrac{c}{2n(1+c^2)^n} + \left(\dfrac{2n-1}{2n}\right) I_n$ for $n > 1$. [5]

The region enclosed by the curve $y = \dfrac{1}{(1+x^2)^2}$ and the lines $x = 1$ and $y = 1$ is denoted by $R$.

**(c)** Find the exact volume of the solid formed by rotating $R$ through $2\pi$ radians about the $x$–axis. [4]

**8** **(a)** The functions g and h are given such that

$$\mathrm{g}(x) = \frac{7-x}{|10-x|}, \qquad x \in \mathbb{R},\ \ x < a,$$

$$\mathrm{h}(x) = -x^2 + 2qx - 6, \qquad x \in \mathbb{R},$$

where $a$ and $q$ are real, and $q$ is restricted such that $|q| < 4$.

**(i)** Explain using a suitable sketch whether the inverse of g exists for values $a > 10$. [2]

For the rest of part **(a)**, $a = 10$.

**(ii)** Find the range of h in terms of $q$. Hence, show that gh exists for any $q$ in the given restriction. [3]

**(iii)** Write down, in terms of $q$, an expression for $\mathrm{gh}(x)$ including its domain and range. [3]

**(b)** Refer to the following table.

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| X($r$) | 3 | 5 | 4 | 7 | 6 | 2 | 8 |
| Y($r$) | 6 | 7 | 3 | 6 | 4 | 5 | 2 |

Given that X and Y are functions,

**(i)** evaluate XY(2), $\mathrm{YX}^{-1}(5)$ and $\mathrm{X}^{-1}\mathrm{X}^{-1}\mathrm{Y}(4)$. [1]

**(ii)** do functions $\mathrm{Y}^{-1}$ and YX exist? Explain your reasoning briefly. [2]

**9** A curve $C$ has parametric equations

$$x = 3\alpha t^2, \quad y = \alpha t\left(3t^2 - 1\right), \quad -\frac{\sqrt{3}}{3} \le t \le \frac{\sqrt{3}}{3},$$

for some constant $\alpha > 0$.

**(a)** Find the cartesian equation of $C$, stating exactly any restrictions on the value of $x$ and $y$. [4]

**(b)** Sketch $C$, indicating exactly the axial intercepts and turning points. [2]

**(c)** Using suitable integral from the list of formulae, find the exact surface area obtained when $C$ is completely rotated about the $x$–axis. [7]

**10** A line $l$ has equation $\mathbf{r} = \lambda\mathbf{s}$, where $\lambda$ is a scalar and $\mathbf{s} = (\cos\theta + \sqrt{3})\,\mathbf{i} + \sqrt{2}\sin\theta\;\mathbf{j} + (\cos\theta - \sqrt{3})\,\mathbf{k}$, for real values of $\theta$. Another line $m$ meets $l$ at the origin and is parallel to the direction vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, for some real constants $a$, $b$ and $c$. The angle between $l$ and $m$ is $\phi$, and the two lines intersect a plane $\pi$ at two distinct points.

**(a)** Show that, for any $\theta$,

$$\cos\phi = \frac{(a+c)\cos\theta + \sqrt{2}b\sin\theta + (a-c)\sqrt{3}}{2\sqrt{2}\,\sqrt{(a^2+b^2+c^2)}}.$$ [3]

It is given that $\phi$ is acute and independent of $\theta$.

**(b)** Use the result in **(a)** to deduce the values of $a$, $b$ and $c$ for the case of line $m$, and hence show that line $m$ has cartesian equation $x = -z$, $y = 0$. State also the value of $\phi$ in this case. [3]

**(c)** If $\phi$ is instead not independent of $\theta$, what can be said about line $m$? [1]

Plane $\pi$ has cartesian equation $x - z = 4\sqrt{3}$. The lines $l$ and $m$ intersect $\pi$ at $P$ and $C$ respectively.

**(d)** Find the coordinates of $P$ in terms of $\theta$. [2]

**(e)** Find the shortest distance between $C$ and $P$ and show that it is independent of $\theta$. Describe geometrically the curve traced by of $P$ as $\theta$ varies. [4]

**(f)** Hence, or otherwise, find the shortest distance between $\pi$ and the origin. [1]

**11** In Economics, economic growth can be determined using *Cobb-Douglas* production function and *Solow* model. An economy with capital $K$ and labour $L$ that both vary with time has production function given by $A(K^a)(L^{1-a})$, where $A$ represents technological advancement and $0 < a < 1$. Given the rate of savings $s$ of the economy, where $0 < s < 1$, its growth according to the model satisfies the differential equation

$$\frac{\mathrm{d}K}{\mathrm{d}t} = sA(K^a)(L^{1-a}).$$

An economy has capital-to-labour ratio $k$. It is known that the amount of labour in this economy increases at a rate proportional to the amount of labour, with some proportionality constant $\lambda$.

**(a)** Show that $\dfrac{\mathrm{d}k}{\mathrm{d}t} = sAk^a - \lambda k$. [3]

**(b)** It is given that $k = k_0$ at $t = 0$. By substituting $y = Ak^{1-a}$, find $k$ in terms of $s$, $A$, $a$, $\lambda$, $k_0$ and $t$. [7]

**(c)** The following values are further given for the economy:

$$\frac{sA}{\lambda} = \frac{8}{3}; \qquad \lambda = \frac{3}{2}; \qquad k_0 = \frac{4}{25}; \qquad a = \frac{1}{2}.$$

Sketch the graph of $k$ against $t$, indicating clearly any main features. [2]

The economy eventually experiences slower growth as $k$ approaches its 'maximum capacity' after a long time. Economic policies may be put in place to increase the theoretical maximum capacity of the economy.

**(d)** With reference to the constants in the model in **(b)**, suggest one way to increase the maximum capacity of $k$. State a necessary assumption for this suggestion to be effective. [2]

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