X JUNIOR COLLEGE
YEAR 6 PRELIMINARY EXAMINATION
Mock Arrangement
in preparation for Candidates' Examination
Higher 2

MATHEMATICS 9758/01

Paper 1 **Set III**

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Write your name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

ABOUT THIS PAPER

X Junior College (XJC) is an unofficial initiative aimed at preparing pre-university and/or junior college students in Singapore for school-level and/or national-level H2 Mathematics examinations through self-prepared mock papers. It has no affiliation with any existing institution in Singapore or worldwide.

This mock paper follows closely the 9758 H2 Mathematics GCE Advanced Level syllabus, most suitable for preparation towards preliminary examinations and A–Levels. The paper intends to explore the unconventional ways and/or applications in which topics within the syllabus can be tested, which may affect the difficulty of this paper to varying degrees. **While it is ideal to attempt this paper under examination constraints, prospective candidates are reminded not to use this potentially non-conforming paper as a definitive gauge for actual performance.**

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This document consists of **5** printed pages and **3** blank pages.

1 Show that the y-coordinates of the turning points of $y = e^x \cos x$ in increasing x-coordinate values follow a geometric progression, stating its common ratio in exact form. [3]

2 A non-constant arithmetic sequence $u_1, u_2, u_3, \ldots$ with first term 1 and common difference d is such that the value of $(u_{n+1} + u_{n+2} + \cdots + u_{2n})$ is a constant multiple of the value of $(u_1 + u_2 + \cdots + u_n)$, for all integers $n \geq 1$.

(a) Show that $\dfrac{2 + (3n-1)d}{2 + (n-1)d}$ is constant. [2]

(b) Hence, find the value of d. [2]

3 **(a)** On the same axes, sketch the curves $y^2 = ax$ and $ay = x^2 - 2ax$, where a is a positive constant. Indicate, for both curves, the point where $y = -a$. [2]

(b) Hence, using a suitable value for a, solve exactly the inequality $\sqrt{x} \leq |x^2 - 2x|$. [3]

4 A curve has equation $y = f(x)$. It is known that the curve has asymptotes at $x = 0$ and $x = 1$, and that it has exactly three stationary points.

(a) Given that $f(x) = f(1-x)$, show that one of the stationary points has an x–coordinate of $\frac{1}{2}$. [2]

(b) Given further that $f(x) = f\left(\frac{1}{x}\right)$, find the x–coordinate of the other two stationary points. [4]

5 It is given that $f(x) = \ln(\ln x)$ and $g(x) = \dfrac{1}{\ln x}$. Using a suitable integration by parts, find

(a) $\displaystyle\int \{f(x) + g(x)\}\,dx$. [2]

(b) $\displaystyle\int_{e}^{e^2} \left\{f(x) + \left(g(x)\right)^2\right\}\,dx$. Give your answer in exact form. [4]

6 A curve C has an equation $y^2 = x^k f(x)$, where k is a constant and f is a function in x. It is given that C passes through the point $(1, 1)$ and that it also satisfies the differential equation

$$2xy\frac{dy}{dx} = y^2 + x^2 - 2.$$

(a) Using calculus, find k and $f(x)$. [4]

(b) Sketch C, labelling exactly the coordinates of any stationary point and any point where the curve meets the axes, as well as the equation of any asymptotes. [3]

7 A sequence $\{u_n\}$ for integers $n \geq 1$ is given by $u_n = \frac{5}{2}(1+2\mathrm{i})^n + \frac{5}{2}(1-2\mathrm{i})^n$.

(a) Show that $u_{n+2} = 2u_{n+1} - 5u_n$, stating the values of the two base cases for this recursion. Hence, explain why u_n is real for all n. [4]

It is given that $\dfrac{u_{n+1}}{u_n} \to r$ as $n \to \infty$.

(b) Show that $r^2 + ar + b = 0$, where a and b are constants to be determined. [2]

(c) Let the roots of the equation in **(b)** be α and β. Show that, for a positive integer m, $\alpha^m + \beta^m$ can be expressed in terms of u_m. Hence, or otherwise, find the least m such that $\alpha^m + \beta^m < 0$, stating this value. [2]

8 The roots of $z^5 = 1$ in increasing arguments are z_1, z_2, z_3, z_4 and z_5, which are represented in an Argand diagram by points Z_1, Z_2, Z_3, Z_4 and Z_5 respectively. It is known that the five points are vertices of a regular polygon.

(a) Explain why you can expect the polygon to be symmetrical about the real axis. [1]

(b) Determine which of the five roots is real and state its value. Hence, explain why $\arg(z_4) = \frac{2}{5}\pi$. [2]

It is given that $z_4 = \frac{1}{4}(-1+\sqrt{5}) + \frac{1}{4}\sqrt{(10+2\sqrt{5})}\mathrm{i}$.

(c) Find $|z_4 - z_3|$, giving your answer in the form $\sqrt{k}$, where k is an exact constant to be determined. [2]

(d) Find the exact area of the regular polygon. [2]

(e) The points Z_1, Z_2, Z_3, Z_4 and Z_5 in the Argand diagram are transformed such that the complex numbers they represent are now roots of $z^5 = -243\mathrm{i}$. Plot these transformed points on an Argand diagram, indicating the moduli of the transformed complex numbers. [2]

9 A curve C has parametric equations

$$x = 2\cos\theta + \cos 2\theta, \quad y = 2\sin\theta - \sin 2\theta, \quad \text{for } 0 \leq \theta \leq 2\pi.$$

(a) Express $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\dfrac{\cos\theta + a}{\sin\theta}$, where a is a constant to be determined. [3]

(b) Find the equation of the tangent to C at the point with parameter θ and show that it can be written as $y\sin\theta + \cos 2\theta = x(\cos\theta - 1) + \cos\theta$. [2]

Points P and Q on C have parameters p and q respectively. The tangent at P meets the tangent at Q at the point R. It is given that the two tangents are perpendicular to one another.

(c) Verify that $q = p + \pi$. Hence, find the x-coordinate of R in the form $k\cos 2p$, where k is a constant to be determined, and find similarly the y-coordinate of R in terms of p. [4]

(d) Show that R traces a circle as p varies from 0 to 2π, stating its radius and centre, as well as the direction in which R traces this circle (clockwise or counterclockwise). [2]

10 The variables x and y are given such that their rates of change with respect to time t satisfies the equation

$$\frac{dx}{dt} = 2y - 2x \quad \text{and} \quad \frac{dy}{dt} = -\frac{dx}{dt} - 3y.$$

It is also given that initially $x = 5$ and $y = 0$.

(a) Show that $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$, where a and b are constants to be determined. [3]

(b) Given that $x = Ae^{kt} + Be^{mt}$, where A, B, k and m are constants and $k \neq m$, find x and y in terms of t. [6]

(c) Sketch a graph of y against x, indicating the coordinates of the endpoints and the turning point. [4]

11 Functions f and h are given by

$$f(x) = 1 - |2x - 1|, \qquad \text{for } 0 \le x \le 1,$$

$$h(x) = 2\cos(\pi x), \qquad \text{for } 0 \le x \le 1.$$

(a) Explain why the following functions exist.

(i) h^{-1} [1]

(ii) fh^{-1} [1]

(iii) hfh^{-1} [2]

A function g is given such that f, g and h satisfy the relationship

$$g(x) = hfh^{-1}(x).$$

Under this relationship, f and g is said to be *topologically conjugate* as there exists a function h which conjugates f to g, and vice versa. Scientists and mathematicians examine functions which exhibit topological conjugacy in the study of dynamical systems to predict the long-term behaviour of a system after multiple iterations.

(b) By considering $f(x)$ as two **non-modulus** functions in the domain $0 \le x \le \frac{1}{2}$ and $\frac{1}{2} \le x \le 1$ respectively, show that

$$g(x) = x^2 - 2, \qquad \text{for } -2 \le x \le 2,$$

explaining how the domain of g is obtained. [5]

(c) A function P is given by $P(x) = x^4 - 4x^2 + 2$, where $-2 \le x \le 2$. Another function Q is such that P and Q are topologically conjugate. Find Q, giving your answer in a similar form. [4]

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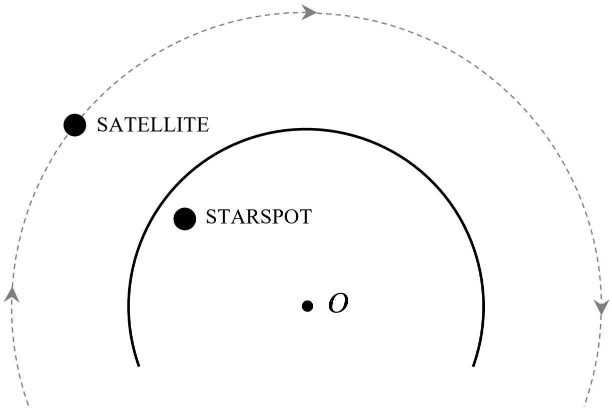

Astronomers use satellites to take photographs in outer space. For a particular observation, a satellite is deployed to circularly orbit a celestial body with a spot on its surface, called a *starspot*, where solar activities are being detected. It is known that the starspot and the satellite each follows a contrasting circular orbit centred at the origin $O(0, 0, 0)$, where the centre of the star is (see diagram). The location of the starspot and the satellite at orbital time unit t, where $0 \leq t \leq 2\pi$, is modelled by position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively given by

$$\mathbf{p} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} \quad \text{and} \quad \mathbf{q} = \frac{3}{2}\cos\left(t + \frac{1}{4}\pi\right)\mathbf{i} + 3\sin\left(t + \frac{1}{4}\pi\right)\mathbf{j} + \frac{3\sqrt{3}}{2}\cos\left(t + \frac{1}{4}\pi\right)\mathbf{k}.$$

(a) By considering two distinct points on the satellite orbit, find the normal of the satellite orbit $\mathbf{q}$. [2]

(b) Write down the normal of the starspot orbit $\mathbf{p}$. Hence, explain the contrast between the two orbits. [2]

Given their contrasting orbits, the satellite is not always ideally positioned to detect solar activities at the starspot. The effectiveness of the detection is affected by the angle θ between $\mathbf{p}$ and $\mathbf{q}$, where $0 \leq \theta \leq \pi$.

(c) Show that $\cos\theta = \dfrac{3\sqrt{2}}{8} - \dfrac{1}{4}\cos\left(2t + \dfrac{1}{4}\pi\right)$. [3]

[You may use the result $\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$ without proof.]

(d) Detection is effective when $\theta \leq \dfrac{1}{4}\pi$. Find the exact range(s) of values of t during which this occurs. [3]

(e) Detection is the most ideal when θ is minimum. Using calculus, find the exact value(s) of t at which this occurs. Hence, calculate the distance between the satellite and the starspot at this instance. [4]

(f) Read the following statement about the perfect condition for detection of solar activities.

The perfect condition for detection of solar activities is when the satellite is directly above the starspot. In such condition, a straight line can be drawn from the centre of the star such that it intersects the two orbits simultaneously. Where the line intersects the orbits correspond to the positions of the respective two objects at said perfect condition.

Justify whether the perfect condition described above is feasible. [1]

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