X JUNIOR COLLEGE
YEAR 6 PRELIMINARY EXAMINATION
Mock Arrangement
in preparation for Candidates' Examination
Higher 2

MATHEMATICS 9758/02

Paper 2 **Set III**

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Write your name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

ABOUT THIS PAPER

X Junior College (XJC) is an unofficial initiative aimed at preparing pre-university and/or junior college students in Singapore for school-level and/or national-level H2 Mathematics examinations through self-prepared mock papers. It has no affiliation with any existing institution in Singapore or worldwide.

This mock paper follows closely the 9758 H2 Mathematics GCE Advanced Level syllabus, most suitable for preparation towards preliminary examinations and A–Levels. The paper intends to explore the unconventional ways and/or applications in which topics within the syllabus can be tested, which may affect the difficulty of this paper to varying degrees. **While it is ideal to attempt this paper under examination constraints, prospective candidates are reminded not to use this potentially non-conforming paper as a definitive gauge for actual performance.**

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This document consists of **6** printed pages and **2** blank pages.

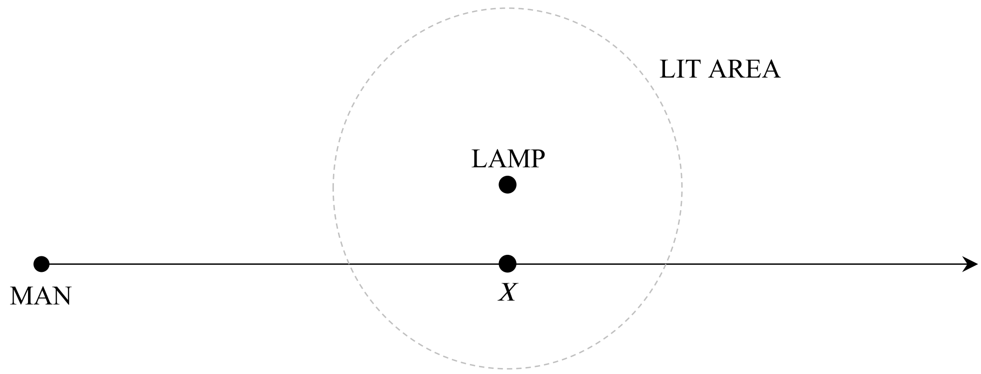
Section A: Pure Mathematics [40 marks]

1 The complex numbers u, w and z satisfy the equations

$$u - w + z = -6 + 6\mathrm{i},$$
$$3u + \mathrm{i}w - 2\mathrm{i}z = 16 + 19\mathrm{i},$$
$$\mathrm{i}u^* - 2w^* - 3z = 4 - 8\mathrm{i}.$$

Find u, w and z, giving your answers in the form $a + \mathrm{i}b$, where a and b are real constants. [5]

2

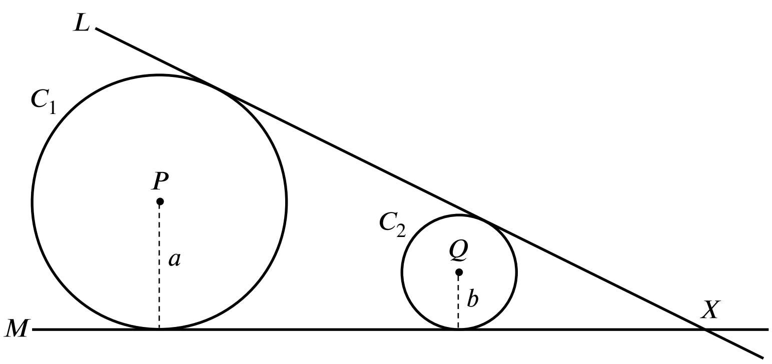


A 1.8-metre man walks on a straight path at a constant speed of 2 ms^{-1}. Nearby, a 9-metre lamp is positioned 4 metres away from a point X on the path which is nearest to the lamp. The lamp creates a lit area which illuminates part of the man's path (see diagram). When the man walks in the lit area, a shadow of s metres is cast onto the ground when he is x metres away from X. At one point during the walk in the lit area, the length of the shadow decreases at a rate of 0.3 ms^{-1}, and T seconds later increases at a rate of 0.4 ms^{-1}. Find a relationship between s and x and use it to find the exact value of T. [6]

3 **(a)** Use the substitution $x = \sin^2\theta$ to find $\displaystyle\int \sqrt{\left(\frac{1-x}{x}\right)}\,\mathrm{d}x$. [4]

(b) Find the exact volume of the solid formed when the region bounded by the curve $xy^4 + x = 1$ and the line $x = \frac{3}{4}$ is rotated completely about the x–axis. [3]

4



Two non-overlapping circles C_1 and C_2 have radius a and b respectively, where $a \neq b$, and centre P and Q with position vector $\mathbf{p}$ and $\mathbf{q}$ respectively, with reference to origin O. The lines L and M are outer common tangents of C_1 and C_2. The point where the tangents intersect, denoted by X, is called the *focus* of C_1 and C_2 (see diagram).

(a) Show by considering similar triangles that $\overrightarrow{OX} = \left(\dfrac{a}{a-b}\right)\mathbf{q} - \left(\dfrac{b}{a-b}\right)\mathbf{p}$. [2]

Another circle C_3 has radius c, where $a \neq c$ and $b \neq c$, and centre R with position vector $\mathbf{r}$ such that C_3 does not overlap either C_1 or C_2. The focus of C_1 and C_3 is Y and the focus of C_2 and C_3 is Z.

(b) Write down the position vectors of Y and Z in terms of a, b, c, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ where applicable. Hence, show that X, Y and Z are collinear. [5]

(c) Given that Y is the midpoint of XZ, find an expression for c in terms of a and b. [2]

5 The variables x and y are related by the recurrent differential equation

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+1)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + (4k^2 - n^2)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0, \qquad n = 0, 1, 2, 3, \ldots,$$

where $k > 0$. It is given that the line $y = 1$ is tangent to the graph of y against x at its y-intercept.

(a) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4k^2$ at $x = 0$. [2]

(b) Use the recurrent differential equation to find, in terms of k, the Maclaurin expansion of y in ascending powers of x up to and including the term in x^4. [3]

It is also known that x and y satisfy a second differential equation given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2m\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}},$$

where $m > 0$.

(c) Solve the second differential equation and show that $y = \cos(2m\mathrm{f}(x))$, where f is an inverse trigonometric function to be determined. [3]

(d) Deduce the relationship between k and m. [You may use expansions from the list of formulae.] [5]

Section B: Probability and Statistics [60 marks]

6 A disc is placed on an 8×8 square grid at the upper-left corner. It can be moved one square at a time downwards or rightwards. Find the number of different ways it can reach the square at the lower-right corner without making three or more consecutive downward moves. [4]

7 A company owns a social media platform where users can upload videos of varying durations. Past records suggest that the average duration of uploaded videos is μ_0 seconds. For research purposes, a hypothesis test is conducted on the durations, t seconds, of a random large sample of uploaded videos, which has been summarised as follows.

$$\Sigma(t - \mu_0) = -38.5 \qquad \Sigma(t - \mu_0)^2 = 362.869375$$

The test is done with appropriate hypotheses at α% significance level, for some integer α, such that the test does not reject having μ_0 as the current average duration of uploaded videos if the sample mean $\bar{t}$ is within the range $29.6325 < \bar{t} < 30.3675$. It is known that the sample gives a p-value of 0.0400404.

(a) Give a general definition for the p-value of a hypothesis test. [1]

(b) Deduce the hypotheses and the size of the sample used in the test and use them to determine the conclusion of the test in context, where values relevant to the hypotheses and conclusion are to be found. [6]

(c) It is found that the durations of uploaded videos have a variance of 1.875^2 seconds^2. With this information, the company repeats the test above on a new sample and achieves a different conclusion from **(b)**. What can you deduce about this new sample? [1]

8 A game is played with a fair coin and an unbiased cubical die with faces labelled 1, 1, 1, 2, 2, and 3. In one round of the game, the coin is tossed once and the die is rolled once. The outcome of the round, X, is determined by the result of the coin toss and the number d shown on the die as follows.

$$X = \begin{cases} |ad - 1|, & \text{if the result of the toss is head,} \\ |a - d|, & \text{if the result of the toss is tail,} \end{cases}$$

where a is a known single-digit integer. It is known that $\mathrm{E}(X^2)$ and $\mathrm{E}(X)$ are integers.

(a) Show that $\mathrm{E}(X^2) = a + \frac{13}{6}(a-1)^2$. [3]

(b) By considering $\mathrm{E}(X)$ in terms of a, deduce the value of a. [2]

The game is played in a carnival. Players pay \$1 to play the game, which consists of two rounds. The sum of the outcomes of the two rounds is recorded and denoted by S. Players are paid back \$$y$ if $S > 15$, or \$0 otherwise.

(c) Find the least integer y for which players profit from the game in the long run. [5]

9 A continuous random variable X is normally distributed with mean μ. The distribution curve of X can be modelled by the equation $y = \mathrm{f}(x)$, where f is the *probability density function* of X. The table below shows the values of y up to 4 decimal places at distinct values of x. (You may assume that there are no erroneous values in the table.)

x	44	48	52	56	59	61	64
y	0.0079	0.0150	0.0242	0.0333	0.0381	0.0396	0.0391

(a) Sketch a scatter diagram of these data. [1]

(b) Use your scatter diagram to explain why $59 < \mu < 64$. [1]

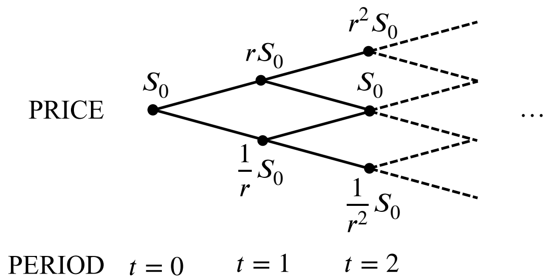
It is known that the data fits the model $y = \mathrm{f}(x)$ with $\mathrm{f}(x) = ae^{b(x-\mu)^2}$, where μ is a suitable integer constant.

(c) By calculating relevant product moment correlation coefficients, determine the most appropriate value of μ. Explain how you decide which value of μ is the most appropriate, and calculate a and b in this case. [4]

(d) Use the model to estimate two possible values of x at which $y = 0.0100$. Explain whether one estimate is more reliable than the other. [3]

(e) Another data point is added to the table above, and the model $y = \mathrm{f}(x)$ derived using these 8 data points is found to be identical to the one found in **(c)**. Find two possible such data points. [2]

10 In finance, valuation of options or stock prices can be simplified using the *binomial pricing model*. Under this model, prices either rise or drop by a fixed rate at fixed periods, creating a binomial tree of possible price movements.



A holder of a particular option uses this model to predict when to sell his option, which is valued at S_0 currently at period $t = 0$. For this option, it is known that the rate is fixed at r, where $r > 1$, and the stock price S at any period will either rise to rS or drop to $\frac{1}{r}S$ at the next period with equal probability (see diagram).

(a) The holder knows that the price changes of this option can be simplified using this model because it also exhibits a characteristic other than those mentioned above. State what this characteristic is. [1]

The holder intends to sell at a high enough price to reap profits, but not too high to avoid unpredictable outcomes. The holder does this by selling at an ideal price of r^nS_0, where n is either 2, 3, 4, or 5.

(b) Show that the probability of selling at an ideal price at period 10 is $\frac{165}{512}$. [2]

(c) Find an expression each for the probability of selling at an ideal price at period $2k$ and $(2k+1)$ respectively, for some integer $k > 0$. (You need not simplify your answers.) [4]

(d) Let t_{odd} and t_{even} be the period with the maximum probability of selling at an ideal price at an odd period and an even period respectively. By setting up suitable inequalities using the results in **(c)**, find the possible value(s) of t_{odd} and t_{even}. Hence, suggest with explanation when the holder should sell his option. [5]

11 In this question you should state the parameters of any distributions that you use.

A fruit vendor sells apples and pears. The masses, in grams, of apples and pears that are sold by the vendor are known to follow the distribution $\mathrm{N}(110, 5^2)$ and $\mathrm{N}(120, 8^2)$ respectively.

(a) On the same diagram, sketch the distribution of the masses of apples and pears respectively, indicating each mean value clearly. [2]

(b) The vendor picks 6 apples and 4 pears, and then calculates the mean mass of these fruits in two ways:

- finding the mean mass of all picked fruits combined,
- finding the mean mass of all picked apples and pears separately, then averaging the two means.

Find the probability that the two mean values differ by more than 2 grams. [4]

As part of a marketing strategy, the vendor organises a publicity stunt in which he works out the fruits that customers put on his digital scale only by being told how many fruits are on the scale and what the scale reads. (You may assume that the scale is accurate.)

(c) One fruit is on the scale. The vendor can identify that the fruit is more likely fruit X than fruit Y if the scale exceeds m grams. Given that the masses, in grams, of fruit X and fruit Y follow the distribution $\mathrm{N}(\mu_1, \sigma_1^2)$ and $\mathrm{N}(\mu_2, \sigma_2^2)$ respectively, show that

$$\frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1 - \sigma_2} < m.$$

Hence, find the least integer scale reading m to be exceeded such that the fruit is more likely a pear than an apple. Explain why the vendor may use a larger value of m for this stunt than the one you found. [5]

(d) Three fruits are on the scale. If the scale reads at most k grams, for some integer k, there is more than 75% probability that there are more apples than pears on it. Find k, showing your working clearly. [4]

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