ACJC Mathematics Department
2025 H2 Mathematics Promotional Examinations Solutions

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<tr><td colspan="3">General comments:

If additional pages are used, students should alert the marker by indicating clearly in the original space for the question in the answer booklet.
If the additional pages at the end of this booklet are used, the question number must be clearly indicated.
Please do not change the order of the question number indicated in the answer booklet by cancelling off the original indicated question number. Your answers must be written only on the correct allocated space indicated by the question number in the answer booklet.
</td></tr>
<tr><td>1</td><td>$y^{\cos x} = x^{\sin 2x}$
$\ln(y^{\cos x}) = \ln(x^{\sin 2x})$
$(\cos x)\ln y = (\sin 2x)\ln x$
Differentiate both sides w.r.t x
$(-\sin x)(\ln y) + (\cos x)(\frac{1}{y})\frac{\mathrm{d}y}{\mathrm{d}x} = (2\cos 2x)(\ln x) + \frac{\sin 2x}{x}$ ---- (1)
Substitute $x = \pi$ into $y^{\cos x} = x^{\sin 2x}$,
$y^{\cos \pi} = x^{\sin 2\pi}$
$y^{-1} = 1$
Hence $y = 1$
From (1) $0 + (\cos \pi)(1)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln \pi + 0$
$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\ln \pi$</td><td>As the base is not a constant, students should not apply the formula
$\frac{d}{dx}a^{\mathrm{f}(x)} = \mathrm{f}'(x)a^{\mathrm{f}(x)}\ln a$

Students are expected to leave answers in exact form and not 3s.f.</td></tr>
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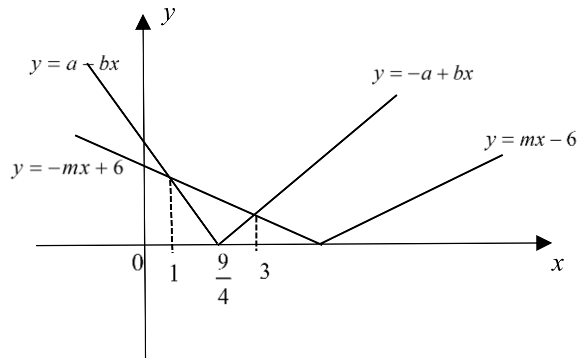
Qn	Solutions	Remarks
2	$y=\sqrt{1+\tan^{-1}2x} \qquad -\frac{1}{2}<x<\frac{1}{2}$ $y^2=1+\tan^{-1}2x$ Differentiate w.r.t x $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{1+4x^2}$ $y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+4x^2}=\left(1+4x^2\right)^{-1}$ $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\left(1+4x^2\right)^{-2}8x$ $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{8x}{(1+4x^2)^2}=0 \quad \text{(shown)}$	Students should consider implicit differentiation instead of differentiating it explicity and then trying to substitute in the expressions to prove LHS = RHS.
	Differentiate w.r.t x, $y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\frac{\mathrm{d}y}{\mathrm{d}x}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8x\left[-2(1+4x^2)^{-3}8x\right]+8(1+4x^2)^{-2}=0$ when $x=0$, $y=\sqrt{1+\tan^{-1}0}=1$ $\frac{\mathrm{d}y}{\mathrm{d}x}=1$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-8+2+1=-5$ Maclaurin series for y: $y\approx 1+x-\frac{1}{2!}x^2-\frac{5}{3!}x^3=1+x-\frac{1}{2}x^2-\frac{5}{6}x^3$	Most students were able to differentiate to find the third derivative but made careless mistakes.

Qn	Solutions	Remarks
3(a)	Given that $\sum_{r=1}^{n}\frac{1}{r(r+1)}=\frac{n}{n+1}$, $\sum_{r=2}^{n}\frac{1}{r(r-1)}$ [Note: replace r by $r+1$] $=\sum_{r=1}^{n-1}\frac{1}{r(r+1)}$ $=\frac{n-1}{n-1+1}$ $=\frac{n-1}{n}$ $=1-\frac{1}{n}$ (shown)	Some students who have identified the correct replacement fail to write down the correct start and end values for *r*.
(b)	$\sum_{r=2}^{\infty}\left(\frac{1}{2^r}+\frac{1}{r(r-1)}\right)$ $=\sum_{r=2}^{\infty}\left(\frac{1}{2^r}\right)+\sum_{r=2}^{\infty}\left(\frac{1}{r(r-1)}\right)$ $=\frac{\frac{1}{2^2}}{1-\frac{1}{2}}+\lim_{n\to\infty}\left(1-\frac{1}{n}\right)$ since as $n\to\infty, \frac{1}{n}\to 0,$ $=\frac{1}{2}+1$ $=\frac{3}{2}$	Many students wrongly wrote $\frac{1}{\infty}$ instead of " as $n\to\infty,\frac{1}{n}\to 0$. " Students should list $\sum_{r=2}^{\infty}\left(\frac{1}{2^r}\right)=\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$ to observe that it is the sum to infinity of a GP.

Qn	Solutions	Remarks
4(a)	 For $y=\|a-bx\|$ $when\ y=0, x=\dfrac{9}{4}$ $\left\|a-\dfrac{9b}{4}\right\|=0$ $4a-9b=0$ ....(1) At $x=1$, $a-b=-m+6$ $a-b+m=6$ ....(2) At $x=3$, $-a+3b=-3m+6$ $-a+3b+3m=6$ ....(3) From GC: $a=9, b=4, m=1$	Most students could substitute (9/4, 0) to form equation 1. Students who squared both sides of the inequality ended up with lengthy calculations that did not lead to a clear conclusion. Some students incorrectly compared the coefficients of $\|mx-6\|^2=\|a-bx\|^2$ and $(x-1)(x-3)=0$

	Alternative Solution: $\lvert mx-6\rvert < \lvert a-bx\rvert$ $(mx-6)^2 < (a-bx)^2$ $(mx-6)^2 - (a-bx)^2 < 0$ $\left((mx-6)+(a-bx)\right)\left((mx-6)-(a-bx)\right) < 0$ $(mx-6)+(a-bx)=0 \Rightarrow x=\dfrac{6-a}{m-b}$ $(mx-6)-(a-bx)=0 \Rightarrow x=\dfrac{6+a}{m+b}$ Solution given as $\{x : x \in \mathbb{R}, x<1 \text{ or } x>3\}$ $\dfrac{6-a}{m-b}=1 \quad \Rightarrow a-b+m=6$ $\dfrac{6+a}{m+b}=3 \quad \Rightarrow -a+3b+3m=6$	

Qn	Solutions	Remarks
4 (b)	$\frac{kx^2+1}{x^2+(k+1)x+k} > -1$ $\frac{kx^2+1}{x^2+(k+1)x+k}+1 > 0$ $\frac{kx^2+1+x^2+(k+1)x+k}{x^2+(k+1)x+k} > 0$ $\frac{(k+1)x^2+(k+1)x+\left(k+1\right)}{x^2+(k+1)x+k} > 0$ $\frac{x^2+x+1}{x^2+(k+1)x+k} > 0$ since $k+1>0$ $\frac{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}{x^2+(k+1)x+k} > 0$ $\frac{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}{x^2+(k+1)x+k} > 0$ OR $\frac{x^2+x+1}{x^2+(k+1)x+k} > 0$ For x^2+x+1, discriminant $=1-4(1)=-3<0$ and coefficient of $x^2>0$ Therefore, numerator is positive for all values of *x*. $\therefore x^2+(k+1)x+k>0$ $\left(x+1\right)\left(x+k\right)>0$ $x<-k \text{ or } x>-1$	Most students who attempted this question are aware that they should not cross multiply. Full credit is not awarded to students who obtain the correct answer but did not explain why $(k+1)x^2+(k+1)x+\left(k+1\right)>0$. All necessary working must be shown clearly. Quite a number of students assume that $x^2+x+1>0$ because discriminant is < 0. They need to include coefficient $x^2>0$. Quite a number of students used the quadratic formula to factorise $x^2+(k+1)x+k$. They did not realise that $x^2+(k+1)x+k=(x+1)(x+k)$ Quite a number who used the quadratic formula did not simplify the expression found and end up with a messy square root expression.

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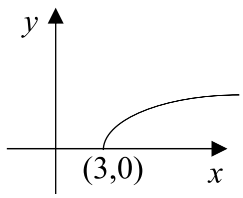
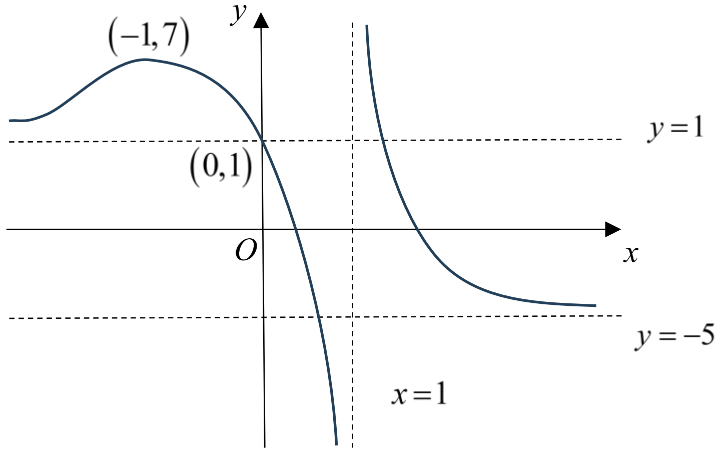
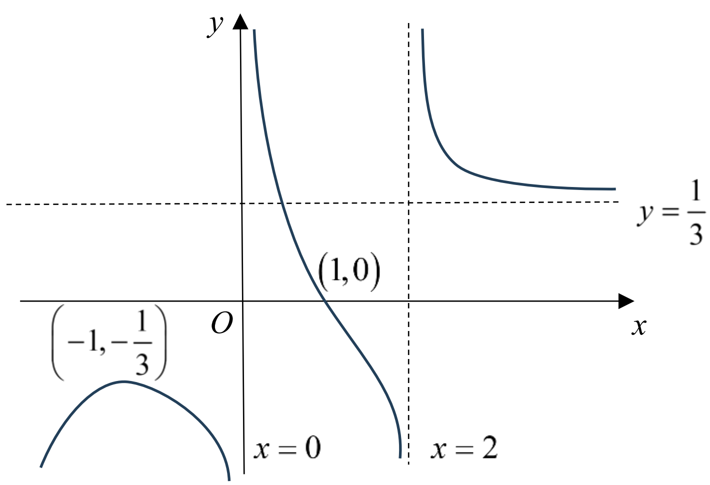
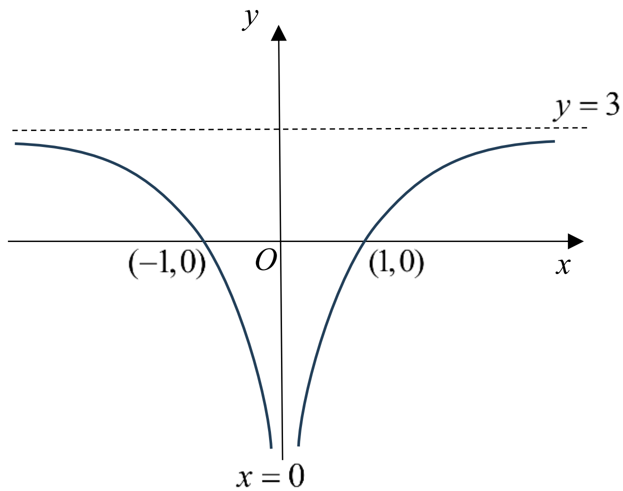
Qn	Solutions	Remarks
5(a)		Some students drew a parabola because they sketch for $0 \le \theta < 2\pi$ instead of $0 \le \theta < \frac{\pi}{2}$. Some wrote the other endpoint as $(3\sec^2\frac{\pi}{2}, 2\tan\frac{\pi}{2})$
(b)	$x = 3\sec^2\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 6\sec\theta(\sec\theta\tan\theta)$ $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\sec^2\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{1}{3\tan\theta}$	Generally well attempted except for some students who do not know how to differentiate or use the wrong notation $\frac{\mathrm{d}x}{\mathrm{d}t}$ or $\frac{\mathrm{d}y}{\mathrm{d}t}$ instead of using $\frac{\mathrm{d}x}{\mathrm{d}\theta}$ and $\frac{\mathrm{d}y}{\mathrm{d}\theta}$.
(c)(i)	**At the point** where the tangent L_1 is parallel to y axis, $3\tan\theta = 0 \Rightarrow \theta = 0$ and $x = 3\sec^2\theta = 3$. Hence the equation of tangent L_1 parallel to y axis is $x = 3$.	Students need to answer the question by stating clearly that the equation of the tangent L_1 is $x = 3$. Students who merely state $x = 3$ will not be given full credit as $x = 3$ may also represent the x-coordinate of a point.
(c)(ii)	At (3.75, 1), $2\tan\theta = 1 \Rightarrow \tan\theta = \frac{1}{2}$ $\frac{dy}{dx} = \frac{1}{3(\frac{1}{2})} = \frac{2}{3}$ Gradient of L_2 is $\frac{2}{3}$	Students need to state clearly that $\frac{1}{3(\frac{1}{2})} = \frac{2}{3}$ is the gradient or $\frac{dy}{dx}$
(d)	$\tan(\frac{\pi}{2} - \beta) = \frac{2}{3}$ Hence $\tan\beta = \frac{3}{2}$	Some students did not read the question carefully and find angle β instead of $\tan\beta$.

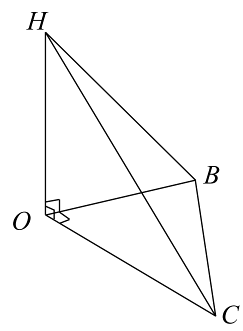
(e)	$y = 2\tan(\pi - \theta) = -2\tan\theta$ C_2 is a reflection of C_1 in the x-axis. Note: You can also use a GC to sketch the curve to infer the transformation .	Some students thought that transformation involves translation followed by reflection even though the question mention "**a** transformation." Students must realise that even though $\tan(\pi - \theta)$ is seen, however, it does not involve a translation because we are plotting y against x not y against θ.

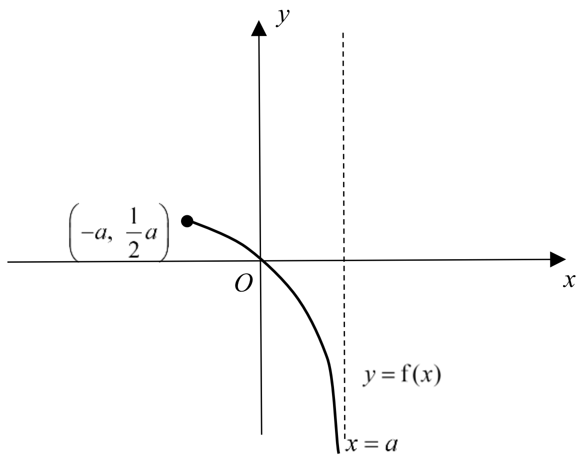
Qn	Solutions	Remarks
6(a) (i)	$y=-2\mathrm{f}(x)+1$ 	Many students failed to notice the existence of two horizontal asymptotes. Curve should be drawn such that it is tending towards the asymptotes.
6(a) (ii)	$y=\dfrac{1}{\mathrm{f}(x)}$ 	Some students labelled "y = 0" when the asymptote does not exist in the graph. Students should take note of the shape and position of the graph.

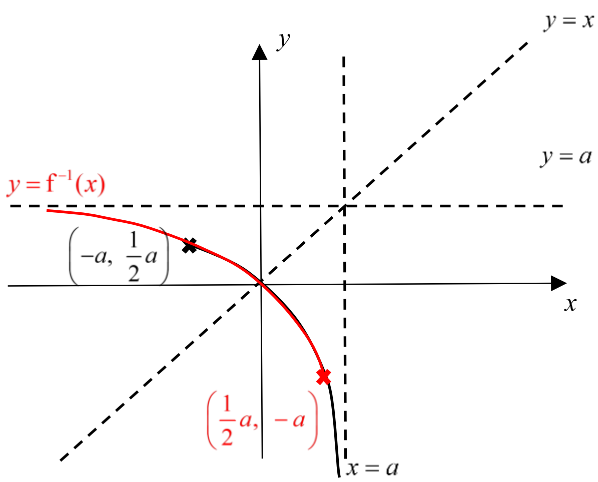
6(a) (iii)	$y=\mathrm{f}\left(\lvert x\rvert+1\right)$ 	Many students carry out the transformation in the **wrong order** by applying the modulus of the function first before translation. The **correct order** should be translation of 1 unit in the negative x-direction followed by applying modulus to the translation.
6(b)	**Method 1** $y=4x^2-4x+1=(2x-1)^2$ (1) Translate 1 unit in the positive x-direction (2) Scale by factor $\frac{1}{2}$ parallel to the x-axis **Method 2** $y=4x^2-4x+1=4\left(x-\frac{1}{2}\right)^2$ (1) Translate $\frac{1}{2}$ unit in the positive x-direction (2) Scale by factor 4 parallel to the y-axis	Many students attempted to translate by "x" units. The correct transformation should involve scaling and translation of a constant. For students who choose to scale by factor $\frac{1}{2}$ parallel to the x-axis **first**, the next step should be "translate $\frac{1}{2}$ units in the positive x-direction".s

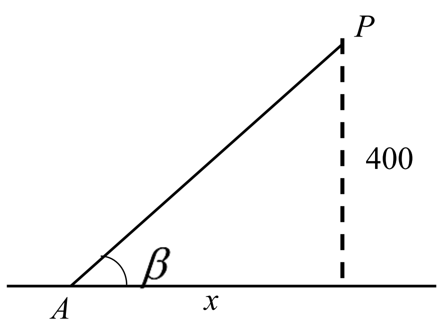
Qn	Solutions	Remarks
7 (a)	Since $\mathbf{a}+\mathbf{b}-\mathbf{c}$ and $\mathbf{a}-\mathbf{b}+\mathbf{c}$ are parallel, $(\mathbf{a}+\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{b}+\mathbf{c})=\mathbf{0}$ $(\mathbf{a}-(\mathbf{c}-\mathbf{b}))\times(\mathbf{a}+(\mathbf{c}-\mathbf{b}))=\mathbf{0}$ $\left(\mathbf{a}-\overrightarrow{\mathbf{BC}}\right)\times\left(\mathbf{a}+\overrightarrow{\mathbf{BC}}\right)=\mathbf{0}$ $(\mathbf{a}\times\mathbf{a})+\left(\mathbf{a}\times\overrightarrow{\mathbf{BC}}\right)+\left(-\overrightarrow{\mathbf{BC}}\times\mathbf{a}\right)+\left(-\overrightarrow{\mathbf{BC}}\times\overrightarrow{\mathbf{BC}}\right)=\mathbf{0}$ $\left(\mathbf{a}\times\overrightarrow{\mathbf{BC}}\right)+\left(-\overrightarrow{\mathbf{BC}}\times\mathbf{a}\right)=\mathbf{0}$ $\left(\mathbf{a}\times\overrightarrow{\mathbf{BC}}\right)+\left(\mathbf{a}\times\overrightarrow{\mathbf{BC}}\right)=\mathbf{0}$ $2\left(\mathbf{a}\times\overrightarrow{\mathbf{BC}}\right)=\mathbf{0}$ $\mathbf{a}\times\overrightarrow{\mathbf{BC}}=\mathbf{0}$ $\overrightarrow{OA}\times\overrightarrow{BC}=\mathbf{0}$ $\therefore \overrightarrow{OA}$ and $\overrightarrow{BC}$ are parallel. (Shown) <u>Alternative method</u> Since $\mathbf{a}+\mathbf{b}-\mathbf{c}$ and $\mathbf{a}-\mathbf{b}+\mathbf{c}$ are parallel, $(\mathbf{a}+\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{b}+\mathbf{c})=\mathbf{0}$ $(\mathbf{a}\times\mathbf{a})+(\mathbf{a}\times-\mathbf{b})+(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times-\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(-\mathbf{c}\times\mathbf{a})+(-\mathbf{c}\times-\mathbf{b})+(-\mathbf{c}\times\mathbf{c})=\mathbf{0}$ $\mathbf{0}+(\mathbf{a}\times-\mathbf{b})+(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{a})+\mathbf{0}+(\mathbf{b}\times\mathbf{c})+(-\mathbf{c}\times\mathbf{a})+(-\mathbf{c}\times-\mathbf{b})+\mathbf{0}=\mathbf{0}$ $\mathbf{0}+(\mathbf{b}\times\mathbf{a})+(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{a})+\mathbf{0}+(\mathbf{b}\times\mathbf{c})+(\mathbf{a}\times\mathbf{c})+(\mathbf{c}\times\mathbf{b})+\mathbf{0}=\mathbf{0}$ $2(\mathbf{b}\times\mathbf{a})+2(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})+(-\mathbf{b}\times\mathbf{c})=\mathbf{0}$ $2(\mathbf{b}\times\mathbf{a})+2(\mathbf{a}\times\mathbf{c})+\mathbf{0}=\mathbf{0}$ $(\mathbf{b}\times\mathbf{a})+(\mathbf{a}\times\mathbf{c})=\mathbf{0}$ $(\mathbf{a}\times-\mathbf{b})+(\mathbf{a}\times\mathbf{c})=\mathbf{0}$ $\mathbf{a}\times(-\mathbf{b}+\mathbf{c})=\mathbf{0}$ $\mathbf{a}\times(\mathbf{c}-\mathbf{b})=\mathbf{0}$ $\overrightarrow{OA}\times\overrightarrow{BC}=\mathbf{0}$ $\therefore \overrightarrow{OA}$ and $\overrightarrow{BC}$ are parallel. (Shown)	Inappropriate use of vector notations is commonly seen. Many students do not differentiate between a vector and a scalar. Note that $\underset{\sim}{a}\neq a$. Note that $\mathbf{a}+\mathbf{b}-\mathbf{c}$ and $\mathbf{a}-\mathbf{b}+\mathbf{c}$ are parallel **does not imply** they are equal vectors. Many students did not use $(\mathbf{a}+\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{b}+\mathbf{c})=\mathbf{0}$. Instead, many started with $(\mathbf{a}+\mathbf{b}-\mathbf{c})=k(\mathbf{a}-\mathbf{b}+\mathbf{c})$ and proceed to show that $\overrightarrow{OA}$ and $\overrightarrow{BC}$ are parallel without vector product which is required by the question. Most students are able to recall correctly that $(\mathbf{a}\times\mathbf{a})=0$ and $(\mathbf{a}\times-\mathbf{b})=(\mathbf{b}\times\mathbf{a})$. For students who found success in this question, the alternative method is the most commonly used. There are also quite a number of students who could not proceed after reaching

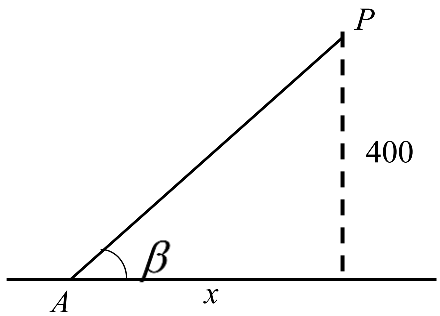
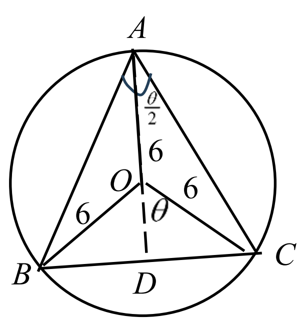
		$(\mathbf{a}\times-\mathbf{b})+(\mathbf{a}\times\mathbf{c})=\mathbf{0}$. They do not know that it is equivalent to $\mathbf{a}\times(-\mathbf{b}+\mathbf{c})=\mathbf{0}$.
7 (b)	Since $\angle OBC=60^\circ$, $\overrightarrow{BO}\bullet\overrightarrow{BC}=\left\lvert\overrightarrow{BO}\right\rvert\left\lvert\overrightarrow{BC}\right\rvert\cos 60^\circ$ $\overrightarrow{BO}\bullet\overrightarrow{BC}=\left\lvert\overrightarrow{BO}\right\rvert\left[2\left\lvert\overrightarrow{OA}\right\rvert\right]\cos 60^\circ$ $-\mathbf{b}\bullet(\mathbf{c}-\mathbf{b})=(1)(2)(1)(0.5)$ $-\mathbf{b}\bullet\mathbf{c}+\mathbf{b}\bullet\mathbf{b}=1$ $-\mathbf{b}\bullet\mathbf{c}+\lvert\mathbf{b}\rvert^2=1$ $-\mathbf{b}\bullet\mathbf{c}+1^2=1$ $\mathbf{b}\bullet\mathbf{c}=0$ $\therefore\overrightarrow{OB}$ and $\overrightarrow{OC}$ are perpendicular. (Shown)	The most common <u>mistake</u> seen is $\overrightarrow{OB}\bullet\overrightarrow{BC}=\left\lvert\overrightarrow{OB}\right\rvert\left\lvert\overrightarrow{BC}\right\rvert\cos 60^\circ$. This is incorrect as the angle between $\overrightarrow{OB}$ & $\overrightarrow{BC}$ is 120º.
	<u>Alternative method</u> $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$ Since $\overrightarrow{BC}=2\mathbf{a}$, $2\mathbf{a}=\mathbf{c}-\mathbf{b}$ $\mathbf{c}=\mathbf{b}+2\mathbf{a}$ Considering $\mathbf{b}\bullet\mathbf{c}=\mathbf{b}\bullet(\mathbf{b}+2\mathbf{a})$ $=\mathbf{b}\bullet\mathbf{b}+2\mathbf{b}\bullet\mathbf{a}$ $=\lvert\mathbf{b}\rvert^2+2\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\cos 120^\circ$ $=1^2+2(1)(1)\left(-\frac{1}{2}\right)$ $=0$ $\therefore\overrightarrow{OB}$ and $\overrightarrow{OC}$ are perpendicular. (Shown)	Many students tried to use the alternative method but few found success. The most common mistake seen is using 60° as the angle between $\overrightarrow{OA}$ & $\overrightarrow{OB}$ which is incorrect. The correct angle between $\overrightarrow{OA}$ & $\overrightarrow{OB}$ is 120°.

7 (c)	 $\mathbf{h}=\mathbf{b}\times\mathbf{c}\Rightarrow\mathbf{h}\perp\mathbf{b}$ and $\mathbf{h}\perp\mathbf{c}$ $\left\|\overrightarrow{OH}\right\|$ is the height of the pyramid *HOBC*. $\|\mathbf{h}\|=\|\mathbf{b}\times\mathbf{c}\|$ $=\|\mathbf{b}\|\|\mathbf{c}\|\sin 90^\circ=(1)\left(\sqrt{3}\right)(1)=\sqrt{3}\quad\left(\angle BOC=90^\circ\ \textit{since}\ \mathbf{b}\bullet\mathbf{c}=0\right)$ Area of $\Delta OBC=\frac{1}{2}\left(\|\mathbf{b}\times\mathbf{c}\|\right)=\frac{1}{2}\left(\|\mathbf{h}\|\right)=\frac{\sqrt{3}}{2}$ Volume of pyramid *HOBC* $=\left(\frac{1}{3}\right)(\text{Area of }\Delta OBC)\left(\left\|\overrightarrow{OH}\right\|\right)=\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{3}\right)=\frac{1}{2}$ units^2	Some students could not infer that *OH* is perpendicular to *OB* and *OC* since $\mathbf{h}=\mathbf{b}\times\mathbf{c}$. Hence they could not form the pyramid *HOBC*. The most common <u>mistake</u> seen when finding $\|\mathbf{h}\|$ is using $\|\mathbf{b}\times\mathbf{c}\|=\|\mathbf{b}\|\|\mathbf{c}\|\sin 60^\circ$. This is <u>incorrect</u> because the angle between $\mathbf{b}\ \&\ \mathbf{c}$ is 90° since $\mathbf{b}\bullet\mathbf{c}=0$ is already shown in part **(b)**. Hence $\|\mathbf{b}\times\mathbf{c}\|=\|\mathbf{b}\|\|\mathbf{c}\|\sin 90^\circ$. Note also that Area of ΔOBC $\neq\|\mathbf{b}\times\mathbf{c}\|$. Many did not realise that ΔOBC is a right angle triangle. Hence Area of ΔOBC $=\frac{1}{2}\|\mathbf{b}\|\|\mathbf{c}\|$.

Qn	Solutions	Remarks
8 (a)	$\mathrm{f} : x \mapsto \dfrac{ax}{x-a}$, for $x \in \mathbb{R},\ -a \le x < a$ $y = \mathrm{f}(x) = \dfrac{ax}{x-a} = a + \dfrac{a^2}{x-a}$ When $y = 0,\ x = 0$ Asymptotes: $y = a,\ x = a$ 	Curve $y = \mathrm{f}(x)$ should be sketched only for the domain of the function which is $-a \le x < a$. There should be a shaded dot at $x = -a$ and coordinates of the endpoint should be indicated with the y-coordinate simplified to its lowest term. Curve should pass through the origin O. There is an asymptote at $x = a$, hence there is NO endpoint there.
	The horizontal line $y = k$, for **all** $k \in \mathbb{R}$, cuts the graph of $y = \mathrm{f}(x)$ **at most once. Therefore,** f is a **one-to-one** function, and the inverse of f exists. <u>Alternative phrasing</u> The horizontal line $y = k$, for **all** $k \in$ range of function f, cuts the graph of $y = \mathrm{f}(x)$ **exactly once. Therefore,** f is a **one-to-one** function, and the inverse of f exists.	The reason should be accurately worded. Refer to the key words in bold. The fact that function f is a one-to-one function for this domain should also be mentioned before concluding that inverse of f exists.

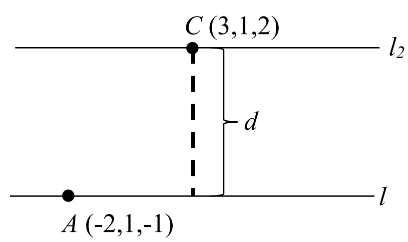
(b)	$\text{Let } y=\dfrac{ax}{x-a}$ $xy-ay=ax$ $x(y-a)=ay$ $x=\dfrac{ay}{y-a}$ $\mathrm{f}^{-1}(x)=\dfrac{ax}{x-a}$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\left(-\infty,\dfrac{1}{2}a\right]$	The rule for the inverse function of f i.e. $\mathrm{f}^{-1}(x)=\dfrac{ax}{x-a}$ should be written out. Do not stop at $x=\dfrac{ay}{y-a}$.
(c)	Graph of $y=\mathrm{f}^{-1}(x)$ in red  $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for $-a\le x\le\dfrac{1}{2}a$ or $\left[-a,\dfrac{1}{2}a\right]$ i.e Intersection of domain of f and domain of f^{-1}.	It is NOT necessary to sketch this diagram. This is how the two functions f and f^{-1} curves look like. Since they have the same rule, they intersect for a range of values of x and not just at one value of x. So, using the method of intersection of $y=\mathrm{f}(x)$ and $y=x$ will not work here. $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for the intersection of domain of f and domain of f^{-1} i.e intersection of $-a\le x<a$ **and** $\left(-\infty,\dfrac{1}{2}a\right]$
(d)	Since $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for $-a\le x\le\dfrac{1}{2}a$, $\mathrm{f}^2(x)=x$ for $-a\le x\le\dfrac{1}{2}a$ $\therefore \mathrm{f}^{2k+1}\left(-\dfrac{1}{2}a\right)=\mathrm{f}\left(-\dfrac{1}{2}a\right)=\dfrac{-\frac{1}{2}a^2}{-\frac{1}{2}a-a}=\dfrac{1}{3}a$	Note that since $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ for $-a\le x\le\dfrac{1}{2}a$ only, $\mathrm{f}^2(x)=x$ **for values of** $\boldsymbol{x}$ **in this domain only.** $x=-\dfrac{1}{2}a$ is in this interval, therefore,

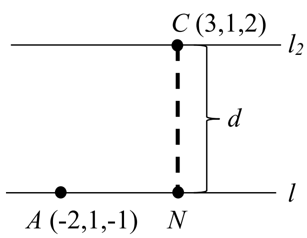
		$\mathrm{f}^2\left(-\frac{1}{2}a\right)=\mathrm{ff}^{-1}\left(-\frac{1}{2}a\right)=-\frac{1}{2}a.$ Also $\mathrm{f}^{2k}\left(-\frac{1}{2}a\right)=-\frac{1}{2}a$ since *2k* is an even number and $\mathrm{f}^{2k+1}\left(-\frac{1}{2}a\right)=\mathrm{f}\left(-\frac{1}{2}a\right)$ as $2k+1$ is an odd number.
(e)	$\mathrm{fg}(x)=x+a$ for $x\in\mathbb{R},\ x\le -a$ $\mathrm{g}(x)=\mathrm{f}^{-1}(x+a)$ $=\dfrac{a(x+a)}{x+a-a}$ $=\dfrac{ax+a^2}{x}=a+\dfrac{a^2}{x}$ for $x\le -a$ Alternative Method : $\mathrm{f\,g}(x)=\mathrm{f}(\mathrm{g}(x))=\dfrac{a\mathrm{g}(x)}{\mathrm{g}(x)-a}=x+a$ $ag(x)=\left(\mathrm{g}(x)-a\right)(x+a)$ $ag(x)=xg(x)+ag(x)-ax-a^2$ $\mathrm{g}(x)=\dfrac{ax+a^2}{x}=a+\dfrac{a^2}{x}$ for $x\le -a$	Note that $D_{fg}=D_g$.
9 (a)	<u>**Method 1**</u> $\tan\beta=\dfrac{400}{x}$ $\beta=\tan^{-1}\left(\dfrac{400}{x}\right)$  $\dfrac{\mathrm{d}\beta}{\mathrm{d}x}=\dfrac{1}{1+\left(\dfrac{400}{x}\right)^2}\left(-\dfrac{400}{x^2}\right)$	Note that the rate of change of β should be a negative value, as it is obvious that β is decreasing.

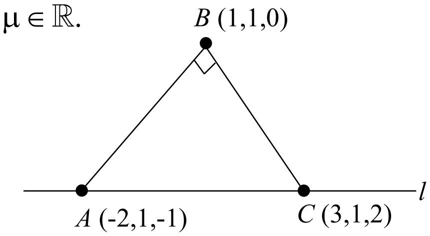
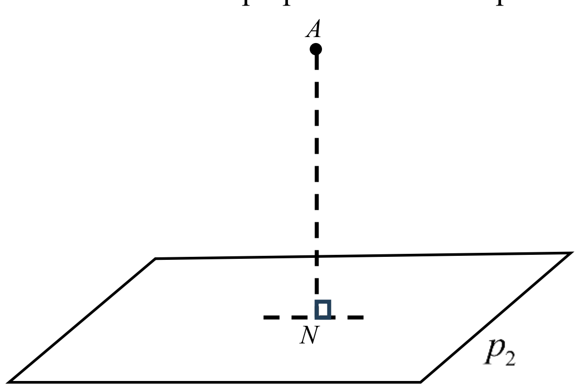
	When $x = 100$, $\frac{d\beta}{dx} = \frac{1}{1+\left(\frac{400}{100}\right)^2}\left(-\frac{400}{100^2}\right) = \frac{1}{1+16}\left(-\frac{4}{100}\right) = -\left(\frac{1}{17(25)}\right)$ $\frac{d\beta}{dt} = \left(\frac{d\beta}{dx}\right)\frac{dx}{dt} = -\left(\frac{1}{17(25)}\right)(10) = -\frac{2}{85}$ Rate of change of β when $x = 100$ m is $-\frac{2}{85}$ rads^{-1}	Answer should be given as an exact value as required in the question. For all of differentiation, angles should be in radians and not in degrees. So, the unit for the final answer is radians per second and not degrees per second.
	Method 2 $\tan\beta = \frac{400}{x}$ Diff w.r.t to t $\sec^2\beta\frac{d\beta}{dt} = -\frac{400}{x^2}\frac{dx}{dt}$  $\frac{d\beta}{dt} = -\frac{400}{x^2(\sec^2\beta)}\frac{dx}{dt}$ When $x = 100$, $\tan\beta = \frac{400}{100} = 4$ $\sec^2\beta = 1+\tan^2\beta = 1+4^2 = 17$ $\frac{d\beta}{dt} = -\frac{400}{100^2(17)}(10) = -\frac{2}{85}$ Rate of change of β when $x = 100$ m is $-\frac{2}{85}$ rads^{-1}	
9 (b)	To find Area of $\Delta ABC = A$ **Method 1** Area $\Delta ABC =$ $A = \frac{1}{2}(BC)(AD)$ $= \frac{1}{2}(2)(6\sin\theta)(6+6\cos\theta)$ $= 36(\sin\theta)(1+\cos\theta)$ 	Note that triangle ABC is **NOT** an equilateral triangle, thus it is incorrect to find the area of triangle ABC using 3 times of area of triangle ABO.

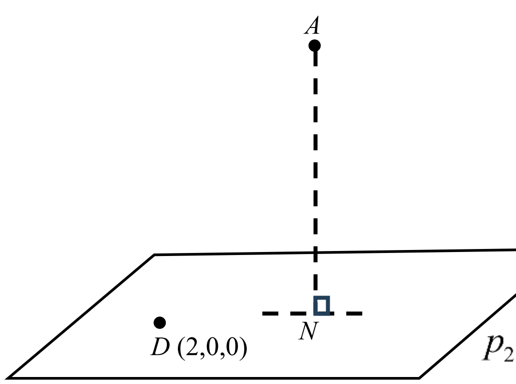
	Method 2 Area $\Delta ABC = A$ $= 2\times \text{Area}\Delta AOC + \text{Area}\Delta BOC$ $= 2\times[\frac{1}{2}(OC)(OA)\sin(\pi-\theta)] + \frac{1}{2}(OB)(OC)\sin(2\theta)$ $= (6)(6)\sin\theta + \frac{1}{2}(6)(6)(2\sin\theta\cos\theta)$ $= 36\left(\sin\theta\right)(1+\cos\theta)$ **Method 3** $AC = 2(6\cos\frac{\theta}{2}) = 12\cos\frac{\theta}{2}$ Area $\Delta ABC = A = \frac{1}{2}(AB)(AC)\sin\theta$ $= \frac{1}{2}(12\cos\frac{\theta}{2})(12\cos\frac{\theta}{2})\sin\theta$ $= 72\cos^2\frac{\theta}{2}\sin\theta$ $= 72\left[\frac{\cos\theta+1}{2}\right]\sin\theta = 36\left(\sin\theta\right)(1+\cos\theta)$	
	$\frac{\mathrm{d}A}{\mathrm{d}\theta} = 36\cos\theta(1+\cos\theta) + 36\sin\theta(-\sin\theta)$ $= 36\cos\theta + 36\cos^2\theta) - 36\sin^2\theta$ $= 36[\cos\theta + \cos^2\theta) - (1-\cos^2\theta)]$ $= 36[2\cos^2\theta + \cos\theta - 1] = 0$ Hence $2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$ $\cos\theta = \frac{1}{2}$ or -1 $\cos\theta \neq -1$ because θ is acute. Hence $\cos\theta = \frac{1}{2}$ $\quad\therefore\ \theta = \frac{\pi}{3}$	
	$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = 36[4\cos\theta(-\sin\theta) - (\sin\theta)]$ $=36[-4\cos\frac{\pi}{3}\sin\frac{\pi}{3} - \sin\frac{\pi}{3}]$ $=36[-4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)]$ $= 36[-3\left(\frac{\sqrt{3}}{2}\right)] = -54\sqrt{3} < 0$	

	Hence when $\theta = \frac{\pi}{3}$, A is a maximum Maximum area A = $36\sin\frac{\pi}{3}(1+\cos\frac{\pi}{3})$ $= 36\left(\frac{\sqrt{3}}{2}\right)(1+\frac{1}{2}) = 27\sqrt{3}$	Finding the value of area *A* is often missed out. Many students simply stopped after finding the 2nd derivative. Expressing *A* exactly is seldom seen.

Qn	Solutions	Remarks
10 (a)	Let θ the acute angle between planes p_1 and p_2. $\cos\theta = \dfrac{\begin{pmatrix}1\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|}$ $\cos\theta = \dfrac{1-1+1}{\sqrt{3}\sqrt{3}}$ $\cos\theta = \dfrac{1}{3}$ $\theta = 70.5^\circ$	Angle must be in degrees, to 1 decimal place. Some students made the mistake of using $\sin\theta$ instead of $\cos\theta$.
10 (b)	$p_1:\ x-y+z=0$ --- (1) $p_2:\ x+y+z=2$ --- (2) Using GC to solve (1) and (2): Line of intersection of planes p_1 and p_2: $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}, \lambda \in \mathbb{R}.$	Some students did not realise that they could use the GC to solve the equations.
10 (c)	**Method 1:** Line l: $\mathbf{r} = \begin{pmatrix}-2\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\-1\end{pmatrix}, \mu \in \mathbb{R}.$ Let $l_2:\ x-3=-z+2,\ y=1$ $x = 3+\beta$ $y = 1+0\beta$ $z = 2-\beta$ $l_2:\ \boldsymbol{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\0\\-1\end{pmatrix}, \beta \in \mathbb{R}.$  $\overrightarrow{AC} = \begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}-2\\1\\-1\end{pmatrix} = \begin{pmatrix}5\\0\\3\end{pmatrix}$	Students need to realise that the direction vectors of l and l_2 are the same and hence they are parallel. Some converted $l_2:\ x-3=-z+2,\ y=1$ wrongly and resulted in using a wrong position vector of a point on l_2. Line l_2 should be $\boldsymbol{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\0\\-1\end{pmatrix}, \beta \in \mathbb{R}.$

	Let d be the shortest distance. $d=\left\|\overrightarrow{AC}\times\frac{\begin{pmatrix}1\\0\\-1\end{pmatrix}}{\left\|\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\|}\right\| = \frac{\left\|\begin{pmatrix}5\\0\\3\end{pmatrix}\times\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\|}{\sqrt{2}}=\frac{\left\|\begin{pmatrix}0\\8\\0\end{pmatrix}\right\|}{\sqrt{2}}=\frac{8}{\sqrt{2}}=4\sqrt{2}.$	Some students used dot product instead of cross product.
	Method 2: Line l_2: $\boldsymbol{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\beta\begin{pmatrix}1\\0\\-1\end{pmatrix}, \beta\in\mathbb{R}.$ Line l: $\mathbf{r}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-1\end{pmatrix}, \mu\in\mathbb{R}.$  Let N be the foot of perpendicular from point C to line l. $\overrightarrow{ON}=\begin{pmatrix}-2+\mu\\1\\-1-\mu\end{pmatrix}$ Therefore $\overrightarrow{CN}=\overrightarrow{ON}-\overrightarrow{OC}=\begin{pmatrix}-2+\mu\\1\\-1-\mu\end{pmatrix}-\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}\mu-5\\0\\-3-\mu\end{pmatrix}$ Since $\overrightarrow{CN}$ is perpendicular to lines, $\overrightarrow{CN}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}=0$ $\begin{pmatrix}\mu-5\\0\\-3-\mu\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}=0$ $\mu-5+3+\mu=0$ $\mu=1$ $\overrightarrow{CN}=\begin{pmatrix}1-5\\0\\-3-1\end{pmatrix}=\begin{pmatrix}-4\\0\\-4\end{pmatrix}.$ $\left\|\overrightarrow{CN}\right\|=\sqrt{4^2+4^2}=\sqrt{32}=4\sqrt{2}$	Some students converted l_2: $x-3=-z+2,\ y=1$ wrongly and resulted in using a wrong position vector of a point on l_2. Line l_2 should be $\boldsymbol{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\beta\begin{pmatrix}1\\0\\-1\end{pmatrix}, \beta\in\mathbb{R}.$ Some students did not proceed to find $\overrightarrow{CN}$ and wrongly applied $\overrightarrow{ON}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}=0$

10 (d)	Line l: $\mathbf{r}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-1\end{pmatrix}, \mu\in\mathbb{R}.$  Since point C lies on line l, $\overrightarrow{OC}=\begin{pmatrix}-2+\mu\\1\\-1-\mu\end{pmatrix}$ ---- (1) $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}-2+\mu\\1\\-1-\mu\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}-3+\mu\\0\\-1-\mu\end{pmatrix}$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}-2\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\0\\1\end{pmatrix}$ Since $\angle ABC=90^\circ$, $\therefore \overrightarrow{AB}\bullet\overrightarrow{BC}=0$. $\begin{pmatrix}3\\0\\1\end{pmatrix}\bullet\begin{pmatrix}-3+\mu\\0\\-1-\mu\end{pmatrix}=0$ $-9+3\mu+0-1-\mu=0$ $\therefore \mu=5$ Sub $\mu=5$ into (1): $\overrightarrow{OC}=\begin{pmatrix}-2+5\\1\\-1-5\end{pmatrix}=\begin{pmatrix}3\\1\\-6\end{pmatrix}$.	Angle ABC is 90° Hence $\overrightarrow{BA}\bullet\overrightarrow{BC}=0$ not $\overrightarrow{AB}\bullet\overrightarrow{AC}$ Some students mistook $\begin{pmatrix}-2+\mu\\1\\-1-\mu\end{pmatrix}$ to be $\overrightarrow{BC}$
10 (e)	**Method 1:** Let point N be the foot of perpendicular from point A to p_2. 	Some students who tried to formulate line AN uses vector $\overrightarrow{AN}$ as the line which is incorrect.

	Line AN: $\mathbf{r}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}, \alpha\in\mathbb{R}.$ ------(1) Plane p_2: $\mathbf{r}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=2$ ------(2) Sub. (1) into (2): $\begin{pmatrix}-2+\alpha\\1+\alpha\\-1+\alpha\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=2$ $-2+\alpha+1+\alpha-1+\alpha=2$ $\alpha=\frac{4}{3}$ Sub. $\alpha=\frac{4}{3}$ into (1), $\overrightarrow{ON}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}+\left(\frac{4}{3}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}-\frac{2}{3}\\\frac{7}{3}\\\frac{1}{3}\end{pmatrix}$	
10 (e)	**Method 2:** Let point N be the foot of perpendicular from point A to p_2. By observation, (2,0,0) lies on p_2.  	

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$$

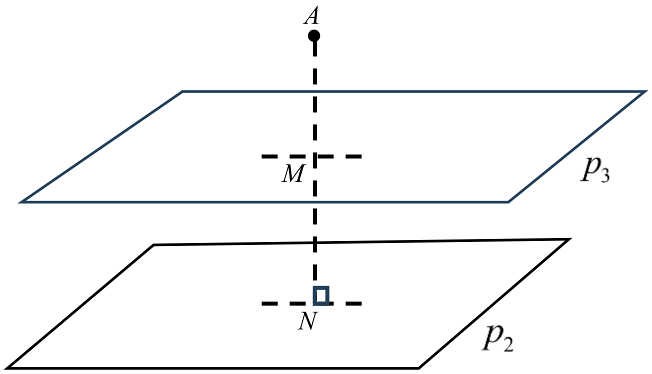
$$\overrightarrow{AN} = \left(\overrightarrow{AD} \bullet \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|} \right) \left(\frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|} \right)$$

$$= \left(\frac{\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{3}} \right) \left(\frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{3}} \right) = \frac{4-1+1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA}$$

$$\frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \overrightarrow{ON} - \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{ON} = \frac{4}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{7}{3} \\ \frac{1}{3} \end{pmatrix}$$

<table>
<tr>
<td>**10 (f)**</td>
<td>


Point M is the midpoint of point A and point N.

$\overrightarrow{AM} = \overrightarrow{MN}$

$\overrightarrow{OM} - \overrightarrow{OA} = \overrightarrow{ON} - \overrightarrow{OM}$

$2\overrightarrow{OM} = \overrightarrow{ON} + \overrightarrow{OA}$

$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{ON} + \overrightarrow{OA}\right)$

$\overrightarrow{OM} = \frac{1}{2}\left(\begin{pmatrix} -\frac{2}{3} \\ \frac{7}{3} \\ \frac{1}{3} \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} -\frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{pmatrix}$

Plane p_3: $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ \frac{5}{3} \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$

Plane p_3: $x + y + z = 0$.
</td>
<td>
Some students **wrongly** use $\frac{1}{2}\overrightarrow{AN}$ as the midpoint's position vector.

Some students managed to reach the step $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$, but did not proceed to express the equation in Cartesian form $x + y + z = 0$.
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ACJC Mathematics Department
2025 H2 Mathematics Promotional Examinations Solutions

<table>
<tr><th>Qn</th><th>Solutions</th><th>Remarks</th></tr>
<tr><td>11 (a)</td><td>$a=10, d=2$
$u_{10}=a+9d=10+9(2)=28$
He would have completed 28 floors on his 10th training session.</td><td>Some students misunderstood the question and found S_{10} instead of u_{10}.</td></tr>
<tr><td>11 (b)</td><td>$u_n=a+(n-1)d=62$
$10+(n-1)(2)=62$
$n=27$
He must train for 9 weeks to be ready for the vertical marathon.</td><td>Error commonly seen:
$10+(n-1)(2)=62$
$n-1=26$
$n=25$

Many students misread the question and did not realise that a week does not imply 7 days in this training program.</td></tr>
<tr><td>11 (c)</td><td>$S_{27}=\frac{27}{2}[10+62]=972$
$S_{27}+62(11\times 3)=972+2046=3018$

He would have completed a total of 3018 floors after training for 20 weeks.</td><td>Note:
Do not simply find S_{60}.</td></tr>
<tr><td>11 (d)</td><td>$a=180, r=0.9$
$u_n<82$
$ar^{n-1}<82$
$180(0.9)^{n-1}<82$
$(0.9)^{n-1}<\frac{41}{90}$
$(n-1)\ln(0.9)<\ln\left(\frac{41}{90}\right)$
$n-1>7.46$
$n>8.46$
Least $n=9$
At the 9th minute, Caleb's heart rate will be below 82 bpm

Alternatively, using GC table
<table><tr><th>n</th><th>$180(0.9)^{n-1}$</th></tr><tr><td>8</td><td>86.1</td></tr><tr><td>9</td><td>77.5</td></tr><tr><td>10</td><td>69.7</td></tr></table></td><td>Students did not change the inequality when dividing ln0.9 on both sides. Note that ln0.9 is a negative number.</td></tr>
</table>

11 (e)	$H_{n+1} = 0.85H_n + 10.5$, $H_1 = 160$ $H_{10} = 93.2$, $H_{11} = 89.7$ Least $n = 11$ At the **11**th minute, Alex's heart rate will be below 90 bpm.	
11 (f)	At rest, Alex's heart rate stabilises, $H = 0.85H + 10.5$ $H = 70$ Alex's heart rate after he has rested long enough is 70 bpm.	$H_{n+1} = 0.85H_n + 10.5$ is not a GP thus students who used $\frac{a}{1-r}$ will not get the full credit. Students who used the formula to find H_n has to present as $H_n = 170(0.85)^n + \frac{10.5}{1-0.85}$ in order to obtain full credit.