H2 Further Mathematics 9649

Pure Mathematics

Zhuo Zhuzhen

Hwa Chong Institution

Preface

One of my most vivid memory from JC is how I was contemplating my life decisions after finding out that my friend who dropped FM for computing was getting easy As while I was stuck with my Ds no matter how hard I tried. Indeed, I don't think many can deny the fact that FM is a difficult subject, and I would argue its one of the most difficult A level subject there is. Perhaps you are feeling the same struggle - and maybe even a bit of helplessness - as I once did.

Pure mathematics can feel really demoralising at times, especially given how creative examiners can be. However, just remember that the very fact that you are reading this means that you have the ability to do well in this subject (since you passed the selection test). The power of spamming papers and questions should not be understated - remember to never give up no matter how difficult it seems.

I created this set of notes as a convenient reference for myself for everything that I needed to know about FM and as a tool for memorising the dozens of formulas. Now that it has served its purpose, I hope it will be as useful to you in achieving your desired goals, just as I was able to achieve mine.

All the best for all your future exams!

Acknowledgements

This entire set of notes is an adapted and summarised version of the Further Mathematics notes created by HCI Math Department. Most figures are taken directly from the source. Special thanks must also be given to my FM teachers, Mr Ng Say Tiong and Miss Ho Jia Yuan, for working so hard and being so selfless.

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CHAPTER 1

Recurrence Relations and Mathematical Induction

1.1 Convergence of Infinite Sequence

When given a recurrence relation, if $u_n \to L$ when $n \to \infty$, then $u_{n+1} \to L$. Hence, a recurrence relation can be reduced to an expression with one algebraic term to find the limit of the equation.

1.2 Proving

To prove that $u_{n+1} > u_n$ or vice versa, there are two ways to prove:

1. Show that $u_{n+1} - u_n > 0$.
2. Let $y_1 = u_{n+1}$ and $y_2 = u_n$ and compare graphically.

1.3 Solutions to RR

1.3.1 First Order

If we consider the equation $u_n = au_{n-1} + b$, to solve:

Method 1 - Iteration

$$\begin{aligned} u_n &= au_{n-1} + b \\ &= a(au_{n-2} + b) + b \\ &= a^2 u_{n-2} + ab + b \\ &\vdots \\ &= a^{n-1}u_1 + a^{n-2} + \cdots + ab + b \\ &= a^{n-1}u_1 + \frac{b\left[a^{n-1} - 1\right]}{a-1} \end{aligned}$$

Method 2 - Formula

$$u_n = \left(u_1 - \frac{b}{1-a}\right)a^{n-1} + \frac{b}{1-a}, \quad n \geq 1.$$

1.3.2 Second Order

Consider the equation $u_n = pu_{n-1} + q$. To solve, we solve the characteristic equation $m^2 - pm - q = 0$, where the solution of the RR is:

$$u_n = \begin{cases} Am_1^n + Bm_2^n, & m_1 \neq m_2 \\ (A + Bn)m_1^n, & m_1 = m_2 \\ \left(\sqrt{\alpha^2 + \beta^2}\right)^n [A\cos n\theta + B\sin n\theta], & m_1 = \alpha + i\beta, m_2 = \alpha - i\beta \end{cases}$$

1.4 Mathematical Induction

Proof by mathematical induction consisits of two steps:

1. The basis (or base case) - showing that the statement is true for the smallest value of n that is given in the statement.
2. The induction step - showing that the statement is true for some $n = k$, then its true for some $n = k + 1$.

Example 1. Using Mathematical Induction, prove that Q(n) = T(n) for all $n \in \mathbb{Z}^+$.

1 Write down the statement provided, denoting it as $P(n)$ for convenience.

Let $P(n)$ be the statement $Q(n) = T(n)$ for all $n \in \mathbb{Z}^+$.

2 Show that the base case $P(\alpha)$ is true.

When $n = 1$, LHS = Q(1) = γ
RHS = $T(1) = \gamma$
Since LHS = RHS, $P(1)$ is true.

3 Important assumption to prove $P(k + 1)$ to be true.

Assume $P(k)$ is true for *some* $k \in \mathbb{Z}^+$, i.e. $Q(k) = T(k)$.

4 Write down $P(k + 1)$ explicitly (not super important).

To prove $P(k + 1)$ is true, i.e. $Q(k + 1) = T(k + 1)$,

5 Using the assumption, prove that $P(k + 1)$ is true.

$$\begin{aligned} \text{LHS} = Q(k+1) &= \cdots \\ &= \vdots \\ &= T(k+1) = \text{RHS} \end{aligned}$$

6 State the conclusion

Since $P(1)$ is true and $P(k)$ is true $\implies P(k + 1)$ is true for all $k \in \mathbb{Z}^+$, by MI, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

CHAPTER 2

Numerical Methods

A number α is known as the *root* of the equation $f(x) = 0$ if $f(\alpha) = 0$.

2.1 Determining the Existence of Roots

2.1.1 Graphical Method

Method 1: Sketch the graph of $f(x) = 0$. The x values of the points of intersection of the graph with $y = 0$ are the roots.

Method 2: Rewrite the equation $f(x) = 0$ into the form $f_1(x) = f_2(x)$. The x value of the intersections are the roots.

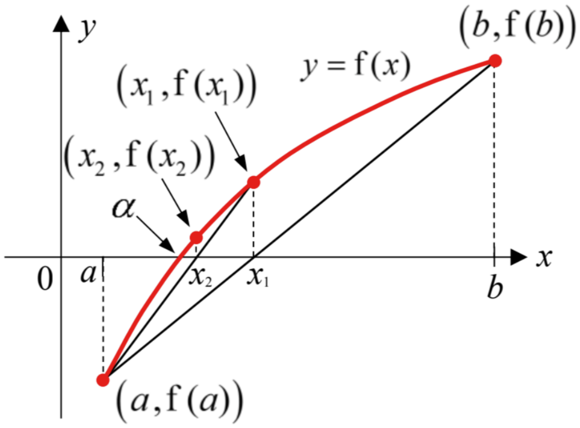
2.1.2 Analytical Method

The equation $f(x) = 0$ has **at least one real root** in the interval (a, b) if 1 f is continuous in the interval $[a, b]$ and 2 $f(a)f(b) < 0$.

To show that there is **only one root**, one needs to show that $f(x)$ is *always* increasing or decreasing in the interval [a,b].

2.2 Approximate Roots

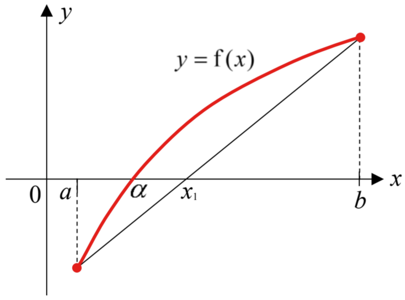
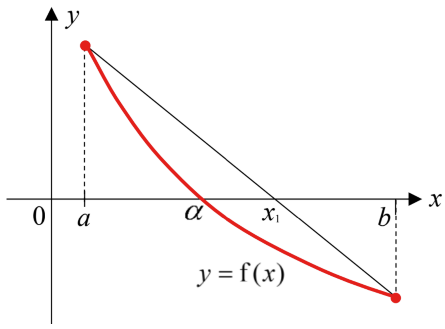
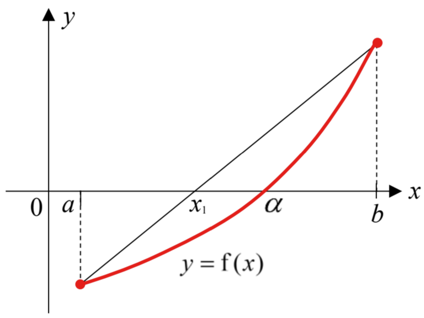
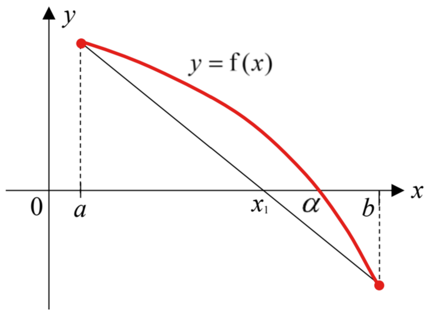
2.2.1 Linear Interpolation Method



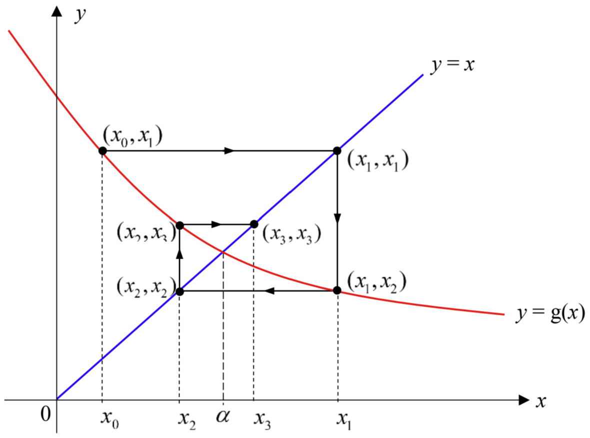
$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

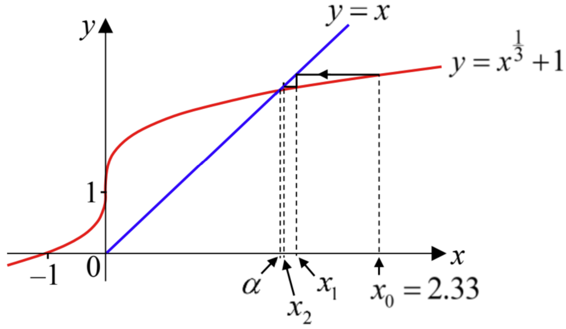
Ensure that α is always bracketed by x_n and a or b. This is done by ensuring that if $f(x_1) > 0$ and $f(b) > 0$, discard $(b, f(b))$ and replace with $(x_1, f(x_1))$. If $f(x_2) < 0$ and $f(b) > 0$, discard $(a, f(a))$.

Error in Linear Interpolation

Error	$f'(x)$	$f''(x)$	Diagram	Shape
Overestimation	> 0	< 0		Increasing, Concave Upwards
	< 0	> 0		Decreasing, Concave Downwards
Underestimation	> 0	> 0		Increasing, Concave Upwards
	< 0	< 0		Decreasing, Concave Downwards

2.2.2 Fixed Point Iteration





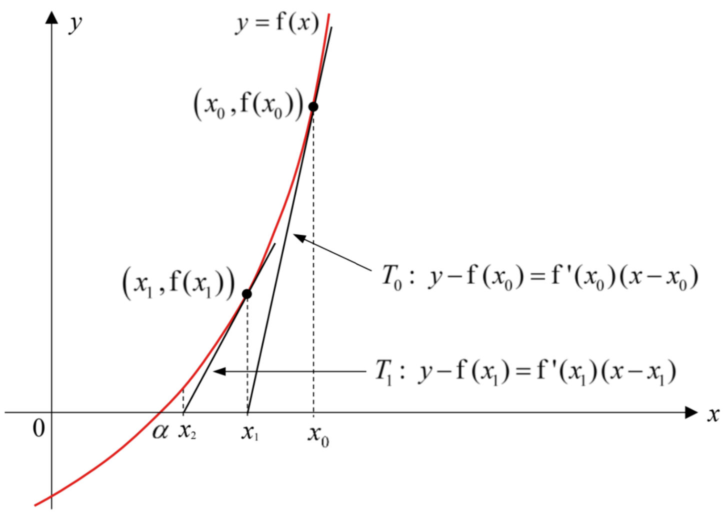
Steps to use fixed point iteration:

1. Rewrite $f(x)$ into the form $x = g(x)$.
2. Form the iteration by changing it to $x_{n+1} = g(x_n)$.
3. Substitute x_0, x_1 etc into the iteration formula until $x_{n+1} = x_n$ is found to the required degree of accuracy.
4. Use the fact that f is continuous in the interval $[a, b]$ and $f(a)f(b) < 0$ to confirm that x_n is the required root.

Convergence of Iteration. Consider the equation $x = g(x)$. If $g(x)$ and $g'(x)$ are continuous and $|g'(x)| < 1$ for all $x \in [a, b]$, then for any number x_0 in $[a, b]$, the sequence defined by $x_{n+1} = g(x_n)$ converges to a fixed point α in $[a, b]$.

Failure of Iteration. If $|g'(x)| > 1$, x_n *may not* converge.

2.2.3 Newton-Raphson Method



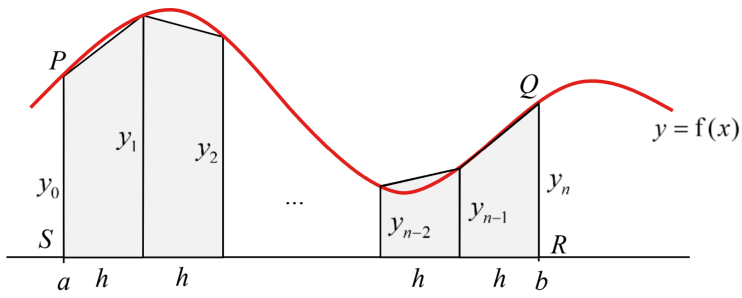
Steps to use Newton-Raphson Method

1. Use x_0 in the iteration formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (formula given in MF26) to obtain x_1.
2. Repeat until one has attained $x_{n+1} = x_n$ to the required degree of accuracy of α.
3. Use the fact that f is continuous in the interval $[a, b]$ and that $f(a)f(b) < 0$ to confirm that x_n is the required root.

Divergence/ Failure of Newton-Raphson. One needs to ensure that 1 The initial value x_0 is close to α and away from the turning points of the curve, and 2 ensure that $|f'(x)|$ will not be very small.

2.3 Approximate Definite Integrals

2.3.1 Trapezium Rule



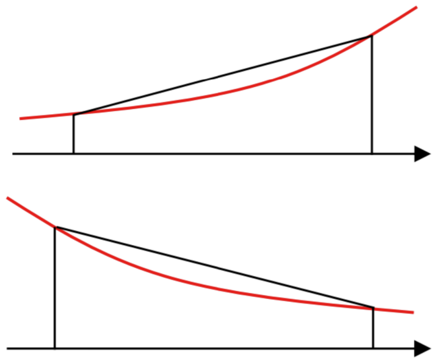
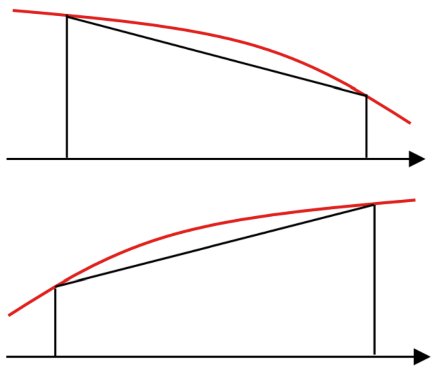
The formula for trapezium rule (somewhat given in MF26) is:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

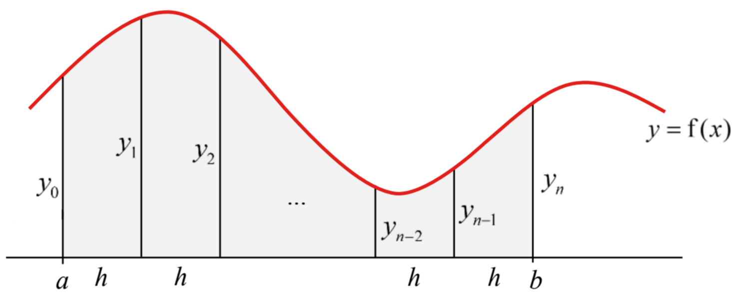
Note:

1. The values $y_0, y_1, \cdots y_n$ are known as *ordinates*. There are exactly $(n+1)$ ordinates in n trapeziums.
2. The number of *intervals* is equal to the number of trapeziums.
3. Larger n lead to better approximations.

Error in Trapezium Rule

Error	Diagram	Shape
Overestimation		Concave Upwards
Underestimation		Concave Downwards

2.3.2 Simpson's Rule



Unlike trapezium rule which uses trapeziums to estimate the area, Simpson's rule a quadratic equation to estimate the area. Note that when using Simpson's rule, the number of intervals n **must be an *even* number**. It has the formula:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}[y_0 + 4(y_{\text{odd indices}}) + 2(y_{\text{even indices}}) + y_n]$$

CHAPTER 3

Polar Curves and Conic Sections

3.1 Polar Equations

Cartesian	Polar
x	$r\cos\theta$
y	$r\sin\theta$
$\sqrt{x^2+y^2}$	r
(x, y)	Base angle: $\theta = \tan^{-1}(\frac{y}{x})$

Note:

1. The origin is called a *pole*.
2. Positive x-axis is known as the initial line, $\theta = 0$, or polar axis.
3. θ is measured from the initial line and spans the range $(-\pi, \pi]$.
4. $P(r, \theta) = P(r, \theta + 2k\pi)$.
5. $(-a, \theta) = (a, \theta + \pi)$.

The max and min points can be found via differentiation, i.e. via:

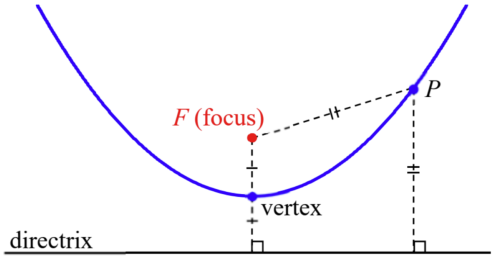
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} = 0$$

3.1.1 Symmetry

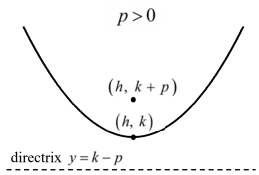
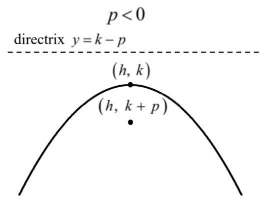
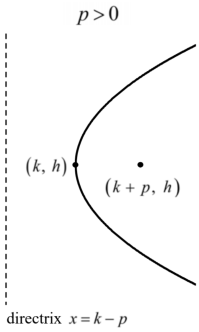
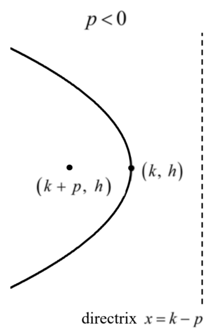
Type of Symmetry	Polar Equation	Can be expressed as an equation in:
About polar (x-axis)	$r = f(\theta) = f(-\theta)$	$\cos\theta$
About vertical (y-axis)	$r = f(\theta) = f(\pi - \theta)$	$\sin\theta$
About pole	$r = f(\theta) = f(\pi + \theta)$	-

3.2 Conic Sections

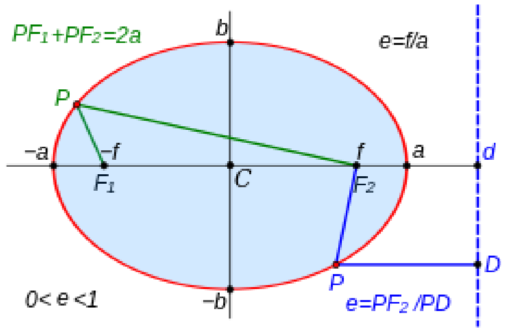
3.2.1 Parabola



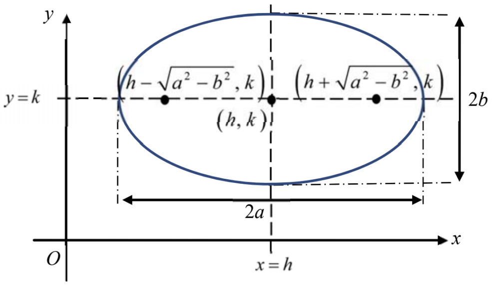
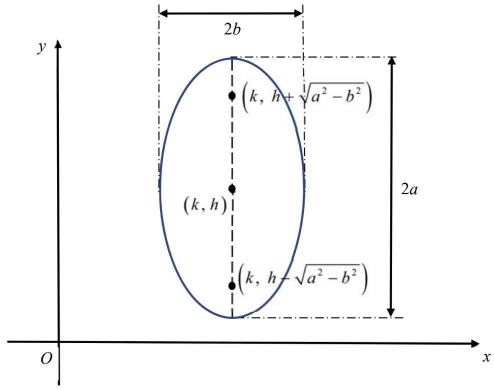
> Parabola is the set of points that are *equidistant* from the focus and the directrix.

Equation	Directrix	Focus	Shape	
$(x-h)^2 = 4p(y-k)$	$y = k - p$	$(h, k+p)$		
$(y-h)^2 = 4p(x-k)$	$x = k - p$	$(k+p, h)$		

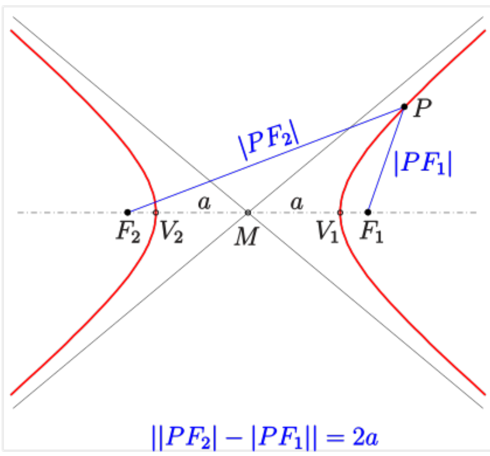
3.2.2 Ellipse



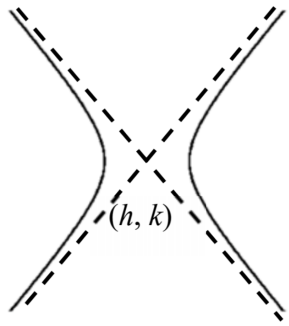
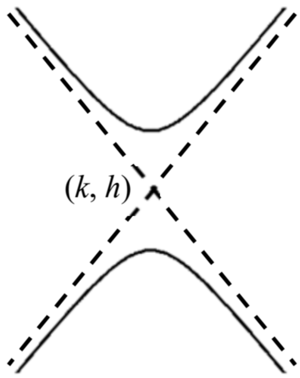
> Parabola is the set of points such that the *sum* of the distance to the 2 focus F_1 and F_2 is *constant* for every point on the curve.

Equation	Directrix	Focus	Shape
$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1,\ a>b$	$(h \pm \dfrac{a}{e}, k)$	$(h \pm c, k)$	
$\dfrac{(y-h)^2}{a^2}+\dfrac{(x-k)^2}{b^2}=1,\ a>b$	$(k, h \pm \dfrac{a}{e})$	$(k, h \pm c)$	

3.2.3 Hyperbola



> Hyperbola is the set of points such that the *difference* of the distance to the 2 focus F_1 and F_2 is *constant* for every point on the curve.

Equation	Directrix	Focus	Shape
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,\ a > b$	$(h \pm \frac{a}{e}, k)$	$(h \pm c, k)$	
$\frac{(y-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,\ a > b$	$(k, h \pm \frac{a}{e})$	$(k, h \pm c)$	

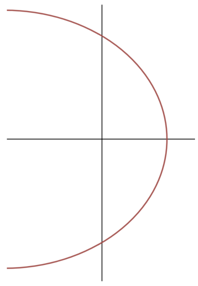
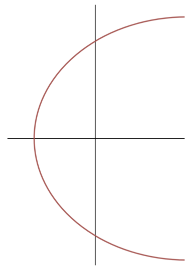
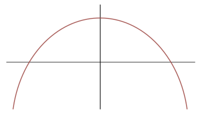
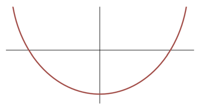
3.2.4 Directrix-Eccentricity-Focus Definitions

Eccentricity is the ratio of the distance of any point on a conic to the focus to that of the directrix.

Conic	Eccentricity	c
Ellipse	$e = \frac{c}{a}$ $= \sqrt{1 - \frac{b^2}{a^2}}$ $0 < e < 1$	$c^2 = a^2 - b^2$
Parabola	$e = 1$	$c = p$
Hyperbola	$e = \frac{c}{a}$ $= \sqrt{1 + \frac{b^2}{a^2}}$ $e > 1$	$c^2 = a^2 + b^2$

3.2.5 Polar Equations of Conic Sections

Note that one of the focus is always on the pole. If e is the eccentricity and d is the distance from the centre of the conic to the directrix,

Equation	Directrix	Diagram
$r = \dfrac{ed}{1 + e\cos\theta}$	$x = d$	
$r = \dfrac{ed}{1 - e\cos\theta}$	$x = -d$	
$r = \dfrac{ed}{1 + e\sin\theta}$	$y = d$	
$r = \dfrac{ed}{1 - e\sin\theta}$	$y = -d$	

Note that $-r = \dfrac{ed}{1 + \cos\theta}$ is the same as $r = \dfrac{ed}{1 + \cos(\theta + \pi)} = \dfrac{ed}{1 - \cos\theta}$ and that for hyperbolas, one of the lines follows the diagram.

CHAPTER 4

Application of Definite Integrals

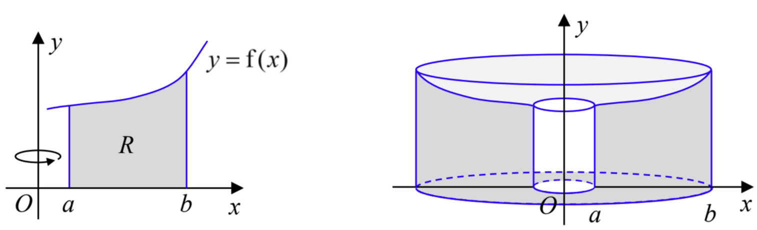
4.1 Area of Curve in Polar

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

4.2 Arc Length of Curve

Coordinate System	Formula
Cartesian	$l_x = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, dx$
	$l_y = \int_{f(a)}^{f(b)} \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\, dy$
Parametric	$l = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, dt$
Polar	$l = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2}\, d\theta$

4.3 Volume of Revolution of Curve



For **shell method**, the axis of rotation is parallel to the lines that are drawn/ area under the graph, with the formula:

$$V_y = 2\pi \int_a^b xy\, dx$$

$$V_x = 2\pi \int_a^b xy\, dy$$

To use solve in parametric form, simply use the fact that:

$$dx = \frac{\mathrm{d}x}{\mathrm{d}t}\, dt$$

4.4 Surface Area of Revolution of Curve

Coordinate System	Axis	Formula
Cartesian	x	$S_x = 2\pi \int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,dx$
		$S_x = 2\pi \int_{f(a)}^{f(b)} y\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,dy$
	y	$S_y = 2\pi \int_a^b x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,dx$
		$S_y = 2\pi \int_{f(a)}^{f(b)} x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,dy$
Parametric	x	$S_x = 2\pi \int_{t_1}^{t_2} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,dt$
	y	$S_y = 2\pi \int_{t_1}^{t_2} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,dt$

CHAPTER 5

Matrices

5.1 Terminologies and Notations

A matrix is defined as a rectangular array denoted by:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. Each item in a matrix is known as an element or entry.
2. If A has m rows and n columns, we say that A is a matrix of order $m \times n$(read as m by n), or A is a $m \times n$ matrix.
3. The element in the i^{th} row and j^{th} column of the matrix A is denoted by a_{ij}. Hence, a $m \times n$ matrix can be represented by $(a_{ij})_{m\times n}$.
4. If $m = 1$, then A is a row matrix/ row vector.
5. If $n = 1$, then A is a column matrix/ column vector.
6. If $m = n$, then A is a square matrix of order n/m.
   (a) A square matrix is diagonal if $a_{ij} = 0$ whenever $i \neq j$.
   (b) A diagonal matrix of order n whose diagonal entries are all 1 is called an identity matrix denoted as I_n or simply I.
   (c) A square matrix is called an upper triangular matrix if all entries below the diagonal entries are 0.
   (d) A square matrix is called a lower triangular matrix if all entries above the diagonal entries are 0.
7. A matrix where all entries = 0 is called a zero matrix or null matrix.
8. Two elements are equal if and only if they are of the same order and their corresponding elements are equal.

5.2 Properties of Matrix Addition and Scalar Multiplication

Let A, B, C be $m \times n$ matrices, and O be the 0 matrix, and $a, b \in \mathbb{R}$. Then,

1. $A + B = B + A$.
2. $a(A + B) = aA + aB$.
3. $(a + b)A = aA + bA$.

4. $a(bA) = (ab)A$.
5. $(A + B) + C = A + (B + C)$.
6. $A + O = A$.
7. $OA = O$.

5.3 Properties of Matrix Multiplication

Let $k \in \mathbb{R}$, and A, B, and C be square matrices of the same order,

1. $AB \neq BA$.
2. $AB = 0$ does not imply that A or B is a zero matrix.
3. $(AB)C = A(BC)$.
4. $A(B + C) = AB + AC$.
5. $k(AB) = (kA)B = A(kB)$.
6. $A^n = A \times A \times \cdots \times A$ (n times).

5.4 Transpose of a Matrix

If A is a $m \times n$ matrix, then the transpose of A, denoted by A^T, is obtained from A by interchanging the rows and columns of A, i.e. if $B = A$, then $b_{ij} = a_{ji}$. For e.g.:

$$A^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}^T = \begin{pmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix}$$

5.4.1 Properties of Transpose Matrices

If A and B are $m \times n$ matrices, C be a $n \times r$ matrix, $k \in \mathbb{R}$, then:

1. $(A^T)^T = A$.
2. $(A + B)^T = A^T + B^T$.
3. $(kA)^T = k(A^T)$.
4. $(AC)^T = C^T A^T$

5.5 Augmented Matrix

The matrix is called an augmented matrix of the linear system when each row represents the coefficients of the variables, and the constant on the other side of each equation.

For instance, a system of linear equations:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= \alpha \\ a_{21}x + a_{22}y + a_{23}z &= \beta \\ a_{31}x + a_{32}y + a_{33}z &= \gamma \end{aligned}$$

Can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

The first matrix is known as the coefficient matrix of the system of linear equations. The augmented matrix, X of the system is $(A : b)$, i.e.

$$X = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \alpha \\ a_{21} & a_{22} & a_{23} & \beta \\ a_{31} & a_{32} & a_{33} & \gamma \end{pmatrix}$$

5.6 Elementary Row Operations

The 3 types of elementary row operations are:

1. Multiplying each element in a row by a non-zero scalar.

   $A \xrightarrow{R_x \to kR_x} B$

2. Multiply each row by a non-zero scalar and add the result to another row.

   $A \xrightarrow{R_y \to kR_x + R_y} B$

3. Interchange 2 rows

   $A \xrightarrow{R_x \leftrightarrow R_y} B$

Performing elementary row operations on the augmented matrix does not change the set of solutions of the original linear system, a concept called *row equivalence.*

5.7 Echelon Form

A matrix is in ecehlon form is it is:

1. All non-zero rows are above any rows of all zeros.
2. Leading term of each row is 1.
3. In two consecutive non-zero rows, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Meanwhile, a matrix in reduced row echelon form (rref) if:

1. It is in ref.
2. All entries in the column above a leading 1 are zero.

Note:

1. Both ref and rref can be used to solve a system of linear equations.
2. Rref provides us with the solution directly, while ref requires solving by substitution.
3. The method to reduce the augmented matrix to ref in order to solve a system of linear equations is known as the Gaussian elimination (or row reduction) method.

4. The method to reduce the augmented matrix to rref in order to solve a system of linear equations is known as the Gauss-Jordan elimination method.

5. The ref of an augmented matrix is not unique.

6. The rref of an augmented matrix is unique.

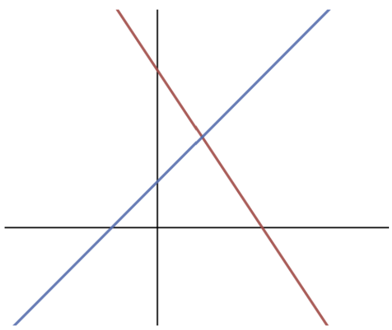
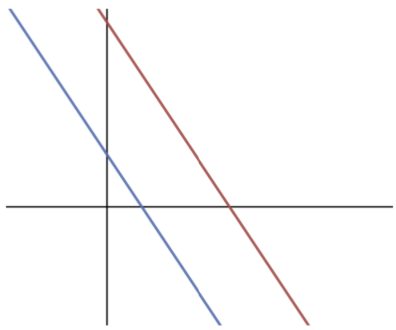
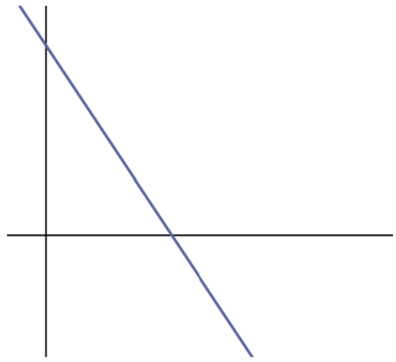
5.8 Geometrical Interpretation of Solution to System of Linear Equations.

If a system of equations does not have a solution, the system is inconsistent.
If a system of equations has a solution, the system is consistent.

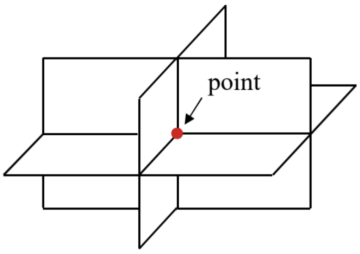
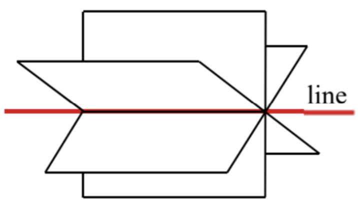
Each system of linear equations has exactly one of the three outcomes:

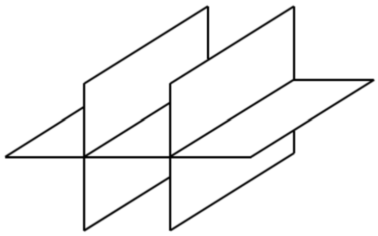
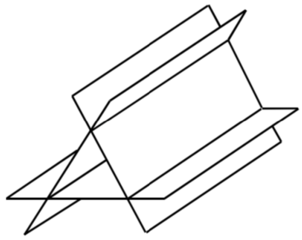
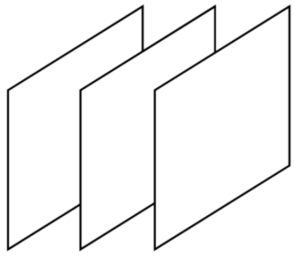
1. No solution (empty)
2. Exactly one solution (unique)
3. Infinitely many solutions

5.8.1 2 Variables

Unique Solution	No Solutions	Infinitely Many Solutions
		
2 lines intersect at 1 point	2 lines are parallel	2 lines overlap (same line)

5.8.2 3 Variables

Unique Solution	No Solutions
	
3 planes intersect at a point	3 planes intersect at a line

No Solutions		
		
2 planes are parallel	All 3 planes are not parallel	All 3 planes are parallel

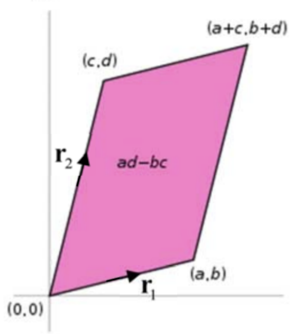
5.9 Determinant of Matrix

5.9.1 2 × 2 Matrix

The determinant is given by the formula:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{1}$$

Geometrically, the absolute value of the determinant of a 2 × 2 matrix is the area of a parallelogram with sides represented by the row vectors of the matrix.

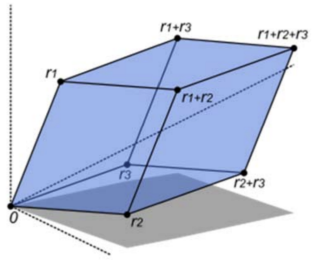


5.9.2 3 × 3 Matrix

Meanwhile, the determinant of a 3 × 3 matrix is given by:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Geometrically, the absolute value of the determinant of a 3 × 3 matrix is the volume of a parallelepiped with sides represented by the row vectors of the matrix.



Note:

1. Determinant is only defined for square matrices only.
   (a) Only 2×2 and 3×3 matrix is in syllabus.
2. It is important to note the '+' and '-' assigned to each element in the computation of the determinant, and is given by:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

3. The determinant can be computed using different columns and rows, under the method of *Cofactor Expansion*

5.9.3 Properties of Determinants

If A, B, C are $n \times n$ matrices,

1. If A has a row/ column of 0, then $|A| = 0$.
2. If A has two identical or equal rows or columns, or is a scalar multiple, then $|A| = 0$.
3. $|A| = |A^T|$.
4. If A is an upper or lower triangular matrix, then $|A| = a_{11}a_{22}a_{33}\cdots a_{nn}$.
   (a) In particular, the determinant of $I = 1$.
5. $|AB| = |A||B|$.
6. In general, $|A| + |B| \neq |A + B|$
7. If A, B, and C differs only in a single row or column, the r^{th} column (or row), with the corresponding entries in all other rows (or columns) equal to each other, and suppose that the r^{th} row (or column) of C is the sum of the corresponding entries in the r^{th} rows (or columns) of A and B, then $|C| = |A| + |B|$

   E.g., $\begin{vmatrix} a_1 & b_1 & c_1 + d_1 \\ a_2 & b_2 & c_2 + d_2 \\ a_3 & b_3 & c_3 + d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$
8. $|k^n| = k^n|A|$.
   (a) Follows the fact that if B is the matrix obtained when a row (or column) of A is multiplied by k, then $|B| = k|A|$.

   E.g., $\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix}$
9. If B is the matrix obtained when two rows (or columns) of A are interchanged, then $|B| = -|A|$.

E.g., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$

10. If B is the matrix obtained when a multiple of one row (or column) of A is added to another row (or column), then $|A| = |B|$.

E.g., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_3 & b_1 + kb_3 & c_1 + kc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note:

1. Properties 9, 10, 11 above are very useful in evaluating matrices using Gaussian elimination with the objective to:
   - Reduce the matrix to an upper triangular matrix whose determinant is the product of the diagonal entries (property 5).
   - Create as many 0s as possible to make computing determinants easier.

5.10 Non-singular Matrices and its Inverse

A matrix A is non-singular/ invertible/ non degenerate if there exists a matrix of the same order B such that $AB = BA = I$, where $B = A^{-1}$. If A has no inverse, it is non-invertible/ singular/ degenerate.

Properties of Non-Singular Matrices:

1. A^{-1} is unique.
2. $(A^{-1})^{-1} = A$.
3. $(kA)^{-1} = \frac{1}{k}A^{-1}$.
4. $(AB)^{-1} = B^{-1}A^{-1}$.
   (a) $(A^n)^{-1} = (A^{n-1}A)^{-1} = \cdots = A^{-1}A^{-1} \ldots A^{-1} = (A^{-1})^n$.
5. $(A^T)^{-1} = (A^{-1})^T$.

5.10.1 Computing Inverse

For a 2×2 matrix, the formula is:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Meanwhile, to find the inverse of a 3×3 matrix, we form the matrix $\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$ and reduce it to the form $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right)$, and the inverse is simply:

$$A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Do remember that the following can be found on the GC:

1. Determinant
2. Inverse
3. Multiplication

CHAPTER 6

Vector Space

Let V be a non-empty set, denoted by $(V, +, \cdot)$, on which vector addition and scalar multiplication are defined. Then, $(V, +, \cdot)$ is a vector space if the following axioms are true:

A1: $\boldsymbol{v}_1 + \boldsymbol{v}_2 \in V$ (Closed under addition)

A2: $k\boldsymbol{v}_1 \in V$ (Closed under Scalar multiplication)

A3: $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1$ (Commutativity of vector addition)

A4: $(\boldsymbol{v}_1 + \boldsymbol{v}_2) + \boldsymbol{v}_3 = \boldsymbol{v}_1 + (\boldsymbol{v}_2 + \boldsymbol{v}_3)$ (Associativity of vector addition)

A5: $\{\mathbf{0} \in V \mid \boldsymbol{v} + \mathbf{0} = \mathbf{0} + \boldsymbol{v} = \boldsymbol{v}\}$ exists (Additive identity – 0 is the zero vector)

A6: $\{-\boldsymbol{v} \in V \mid \boldsymbol{v} + (-\boldsymbol{v}) = 0\}$ exists ($(-\boldsymbol{v})$ is the additive inverse)

A7: $k(\boldsymbol{v}_1 + \boldsymbol{v}_2) = k\boldsymbol{v}_1 + k\boldsymbol{v}_2$ (Distributivity of vector addition)

A8: $(k + m)\boldsymbol{v} = k\boldsymbol{v} + m\boldsymbol{v}$ (Distributivity of scalar multiplication)

A9: $(km)\boldsymbol{v} = k(m\boldsymbol{v})$ (Associativity of scalar multiplication)

A10: $\{1 \in \mathbb{R} \mid 1(\boldsymbol{v}) = \boldsymbol{v}\ \}$ (Scalar multiplication identity)

6.1 Subspace

To prove that $(W, +, \cdot)$ is a subspace of $(V, +, \cdot)$:

1. $(W, +, \cdot)$ is a non-empty set.
2. For any $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$, $\boldsymbol{w}_1 + \boldsymbol{w}_2 \in W$ (Closed under addition)
3. For any $\boldsymbol{w} \in W$, $k \in \mathbb{R}$, $k\boldsymbol{w} \in W$ (Closed under scalar multiplication)
4. $\mathbf{0} \in W$

OR

1. Every element in W is also in V.
   (a) Every element in the basis of W can be written as a linear combination of the elements in the basis of V.

Note: any line or plane in $\mathbb{R}^n$ that passes through the origin is a subspace of $\mathbb{R}^n$

Definition 6.1 (Linear Combination)**.** A vector $\boldsymbol{v}$ is a linear combination of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ if there are scalars $c_1, c_2, \ldots, c_n$ such that $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$.

Definition 6.2 (Spanning Set)**.** V $=$ span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ if every vector in V is a *linear combination* of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$.

Theorem 1. Let $(V, +, \cdot)$ be a vector space and let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in V$. The set $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is linearly independent if and only if the vector equation

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \cdots + \alpha_k \boldsymbol{v}_k = \mathbf{0} \qquad \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$$

has only one solution, namely the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

Definition 6.3 (Basis)**.** Set S is a basis if 1 it is *linearly independent* and 2 it *spans* V.

Note:

1. Bases for vector space is *not* unique.
2. Bases for the same vector space has the *same* number of spaces.
3. Zero vector space $\{0\}$ is defined as a vector space with basis $\varnothing$.

Theorem 2 (Uniqueness of Basis Representation)**.** If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_3\}$ is a basis for a vector space V, then every vector v in V can be written as a linear combination of vectors in S in one and only one way.

Theorem 3. Let V be a vector space, and $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a basis of V. Let $U = \{u_1, u_2, \ldots, u_k\}$ be a set of k vectors in V.

1. If $k > n$, then U is linearly dependent.
2. If $k < n$, then U does not span V.

Theorem 4. All bases of a vector space have the same number of vectors.

Definition 6.4 (Dimension)**.** The number of dimensions, denoted by $\dim(V)$, is the number of vectors in any basis of V.

Note that the dimension of a polynomial of degree n is $n + 1$ as degree 1 is the constants. Moreover, $\dim(\{0\}) = 0$.

Theorem 5. Let V be a vector space such that $\dim(V) = n$. Then, if $S = \{v_1, v_2, \ldots, v_n\}$, and S is linearly independent/ spans V, then S is a basis for V.

For the following definitions, let A be a $m \times n$ matrix,

Definition 6.5 (Row Vectors)**.** Row vectors are the row matrices in vector space $\mathbb{R}^n$, and spans row space.

Definition 6.6 (Column Vectors)**.** Column vectors are the column of the matrix in vector space $\mathbb{R}^m$, and spans column space.

Definition 6.7 (Null Space)**.** Null space of matrix A, also known as kernal of A, is defined to be the set of all solutions of $Ax = 0$, and is a *subspace* of $\mathbb{R}^n$

Theorem 6. If A and B are row equivalent matrices, then,

1. A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
2. A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

6.1.1 Find Subspace of $\mathbb{R}^n$

M1:

1. Apply Gaussian elimination to matrix A where the vectors are row vectors to obtain matrix B, the ref form of A.
2. The non-zero rows of B form a basis for V.

M2:

1. Apply Gaussian elimination to the matrix A where the vectors are column vectors to obtain B, the ref form of A.
2. The column vectors of A that correspond to the column vectors containing the leading 1's in B form a basis for A.

6.2 Linear Transformation

If T: $V \to W$, then T is a linear transformation if:

(a) $\mathrm{T}(\boldsymbol{u}+\boldsymbol{v}) = \mathrm{T}(\boldsymbol{u}) + \mathrm{T}(\boldsymbol{v})$ [Additive Property]

(b) $\mathrm{T}(k\boldsymbol{u}) = k\mathrm{T}(\boldsymbol{u})$ [Homogeneity Property]

Theorem 7. Let A be a $m \times n$ matrix. The function T: $\mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathrm{T}(x) = Ax$ for all $x \in \mathbb{R}^n$ is a linear transformation. Conversely, every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by a $m \times n$ matrix, with matrix A called the *transformation matrix* of A.

6.2.1 Null Space and Range Space

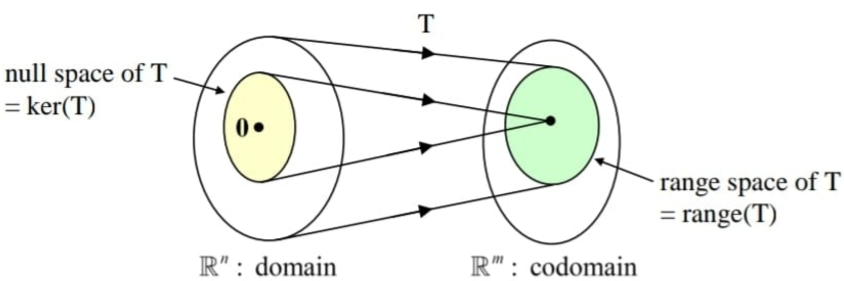
Let T: $\mathbb{R}^n \to \mathbb{R}^m$, then:

1. The null space (or kernel) of T, denoted by ker(T), is the set of all vectors x in $\mathbb{R}^n$ that satisfies $\mathrm{T}(x) = 0$.

   i.e. $\ker(\mathrm{T}) = \{x \in \mathbb{R}^n \mid \mathrm{T}(x) = 0\}$

2. The range space (or range) of T, denoted by range(T), the set of all vectors $\mathrm{T}(x)$ where $x \in \mathbb{R}^n$

   i.e. $\mathrm{range}(\mathrm{T}) = \{\mathrm{T}(x) : x \in \mathbb{R}^n\}$



Note that:

$\ker(\mathrm{T}) = \ker(A)$

$\mathrm{Range}(\mathrm{T}) =$ column space of A

6.2.2 Rank and Nullity

Definition 6.8 (Rank)**.** A common dimension of the row space and column space of matrix A is called the rank of A, denoted as $\text{rank}(A)$.

Definition 6.9 (Nullity)**.** The dimension of the null space of a matrix A is called the nullity of A, denoted as $\text{nullity}(A)$

Let T: $\mathbb{R}^n \to \mathbb{R}^m$, then:

1. $\text{nullity}(A) = \text{nullity}(\text{T})$
2. $\text{rank}(A) = \text{rank}(\text{T})$
3. $\text{rank}(A) + \text{nullity}(A) = n$

Theorem 8. If A is a $m \times n$ matrix, then the row space and the column space of A have the same dimension.

6.3 Eigenvalues and Eigenvectors

Definition 6.10. A scalar λ is called an **eigenvalue** of a $n \times n$ matrix A if there exists a non-zero vector x such that $Ax = \lambda x$. Every non-zero vector x satisfying $Ax = \lambda x$ is called an **eigenvector** of A corresponding to the eigenvalue λ.

Note:

1. Eigenvectors corresponding to distinct eigenvalues of A are linearly independent
2. Eigenvector cannot be 0, but eigenvalue can be 0
3. If $Ax = 0$ for some non-zero vector $x \in \mathbb{R}^n$, then x is an eigenvector of A and 0 is an eigenvalue of A, since $Ax = 0 = 0x$. As x is non-zero, A is non-invertible. Hence, 0 is an eigenvalue of A if and only if A is non-invertible.
4. Each scalar multiple of an eigenvector x is an eigenvector, as $A(kx) = k(Ax) = k(\lambda x) = \lambda(kx)$
5. If A is the transformation matrix of T: $\mathbb{R}^n \to \mathbb{R}^n$, then $\text{T}(x) = Ax = \lambda$, the eigenvector x points in the direction that is stretched by the transformation and λ is the factor by which it is stretched. If λ is negative, the direction is reversed. If $\lambda = 0$, the transformation maps x to 0.

6.3.1 Finding Eigenvalues

To find the eigenvalue, simply make use of the characteristic equation:

$$\det(\lambda I - A) = 0$$

When it is expanded, the resulting polynomial of degree n is known as the characteristic polynomial of A, with the roots being the eigenvalues of A (n distinct values).

Theorem 9. If A is a $n \times n$ matrix, the following statements are equivalent:

1. λ is an eigenvalue of A.
2. The system of equations $(\lambda I - A)x = 0$ has non-trivial solutions.

3. There is a non-zero vector $x \in \mathbb{R}^n$, known as an eigenvalue, such that $Ax = \lambda x$.
4. λ is a solution of, or satisfies, $\det(\lambda I - A) = 0$.

Theorem 10. If A is a $m \times m$ triangular matrix, then the eigenvalues of A are its diagonal values.

Theorem 11. Let x be an eigenvector of matrix A corresponding to eigenvalue λ.

1. If $B = A + kI$, $k \in \mathbb{R}$, then x is also an eigenvector of B with eigenvalue $\lambda + k$.
2. If $B = kA$, then x is also an eigenvector of B with eigenvalue $k\lambda$.
3. If $B = A^{-1}$, then x is also an eigenvector of B with eigenvalue $\frac{1}{\lambda}$.
4. If $B = PAP^(-1)$ for some non-singular matrix P, then $P^{-1}x$ is an eigenvector of B with eigenvalue λ.
5. If x is an eigenvector for matrix B corresponding to eigenvalue μ, then x is also an eigenvector for matrix $A + B$ with eigenvalue $\lambda + \mu$.
6. If x is an eigenvector for a matrix B corresponding to eigenvalue μ, then x is also an eigenvector for matrix AB with eigenvalue $\lambda\mu$.

6.4 Diagonalisation of Matrices

An $n \times n$ matrix A is diagonlaisable if it can be expressed in the form PDP^{-1}, i.e.

$$A = PDP^{-1}$$

Where $P = [\boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \boldsymbol{v}_3 \mid \cdots]$ and $D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

Theorem 12. An $n \times n$ matrix is diagonalisable if and only if A has n linearly independent eigenvectors.

Theorem 13. If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalisable.

Note that since $\det(A) = \det(D)$,

$$A^n = PD^nP^{-1}$$

CHAPTER 7

Differentiation Equation

7.1 Solving D.E. Analytically

7.1.1 First Order

Variable Separable

Used to solve D.E. with the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$.

1. Rearrange into $\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$.
2. Integrate both sides to get $\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$

Integrating Factor

Used to solve D.E. with the form $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$.

1. Find the integrating factor $u(x) = e^{\int p(x)\, dx}$.
2. Solve $u(x) \cdot y = \int u(x)q(x)\, dx$

7.1.2 Second Order

Homogeneous

With the form $a\frac{d^2x}{dy^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$.

1. Solve characteristic equation $ar^2 + br + c = 0$.
2. General solution is given by:

$$y = \begin{cases} Ae^{r_1x} + Be^{r_2x}, & r_1 \neq r_2 \\ (A + Bx)e^{r_1x}, & r_1 = r_2 \\ e^{\alpha x}(A\cos\beta x + B\sin\beta x), & r_1 = \alpha + i\beta, r_2 = \alpha - i\beta, a, b \in \mathbb{R} \end{cases}$$

Non-homogeneous

With the form $a\frac{d^2x}{dy^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x)$, where $f(x) \neq 0$.

1. Solve $f(x) = 0$ to obtain the complementary function y_c.

2. Guess the particular integral y_p by looking at $f(x)$.
3. Sub y_p into the given D.E. and determine unknowns by computing coefficients.
4. General solution is given by: $y = y_p + y_c$.

$f(x)$	Particular Integral	Remarks
$a_0 + a_1 x + a_2 x^2 + \cdots$	$y = \lambda_0 + \lambda_1 x + \lambda_2 x + \cdots$	Use polynomial of the same degree (usually)
pe^{kx}	$y = \lambda e^{kx}$ Or $y = \lambda x e^{kx}$ Or $y = \lambda x^2 e^{kx}$	Introduce an x if $e^k x$ is in the complementary function, or an x^2 if both e^{kx} and xe^{kx} are in complementary function.
$p \sin kx + q \cos kx$	$y = \lambda \sin kx + \mu \cos kx$	p or q may be 0.

Good to know: If y_p is a particular solution of a non-homogeneous linear second-order D.E., and y_1 and y_2 are linearly independent solutions, the general solution is $y = c_1 y_1(x) + c_2 y_2(x) + y_p$, where c_1, c_2 are constants.

7.2 Approximation to Solutions of First Order D.E.

The Formula (found in MF26) for Euler's method is given by:

$$y_{i+1} = y_i + hf(x_i, y_i), \quad i = 0, 1, \cdots, N-1$$

i	x_i	y_i	$f(x_i, y_i) = \cdots$
0	initial x	initial y	Find f by substituting
1	$x_0 + h$	Use formula	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Meanwhile, the improved Euler's Method (Heun's Formula) is:

$$\text{Predictor: } y^*_{i+1} = y_i + hf(x_i, y_i)$$
$$\text{Corrector: } y_{i+1} = y_i + \left[\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}\right]$$

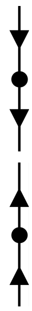
i	x_i	y^*_i	$f(x_i, y^*_i) = \cdots$	y_i	$f(x_i, y_i) = \cdots$
0	Initial x	-	-	Initial y	Sub in
1	$x_0 + h$	Formula	Sub in	Sub in	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

7.3 Equilibria and Stability

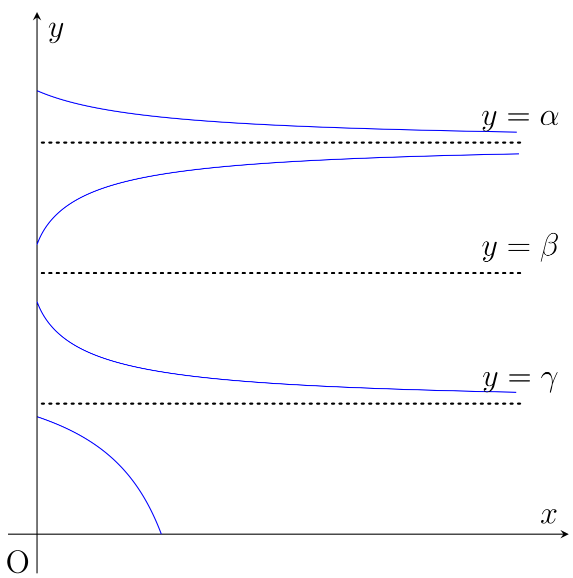
The equilibrium points of the D.E. are given by the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t) = 0$$

The equilibrium solutions can be classified into the following:

Type	Behaviour	Phase line
Unstable/ Source	Nearby points diverge from the equilibrium as $t \to \infty$	
Stable/ Sink	Nearby solutions converge to the equilibrium as $t \to \infty$	
Node	Neither a sink nor source	

Questions may ask one to sketch a graph, showing the behaviour of different initial conditions. An example is as shown:



Where $y = \alpha$ is stable, $y = \beta$ is unstable, and $y = \gamma$ is a node.

Note:

1. Questions often require the y-axis to be a "thing" and the x-axis to be time. Remember that for such cases, show clearly the *origin* and do not sketch for $y < 0$ and $x < 0$.
2. If $y = 0$ or any other value is not an equilibrium solution, make sure to show it clearly.

7.4 Mathematical Modelling

Exponential Growth Model

The rate of change of population is given by:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP,\ P(0) = P_0$$

Where k is the (net per-capita) growth rate, which is given by birth rate − death rate.

Logistic Growth Model

The rate of change of population is given by:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{N}\right)$$

Where r is the intrinsic growth rate and N is the carrying capacity.

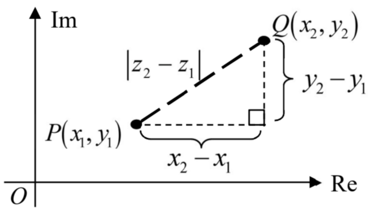
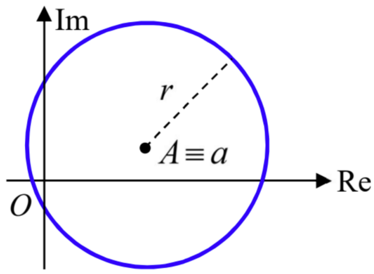
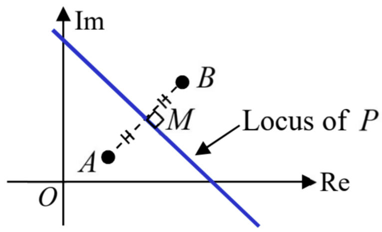
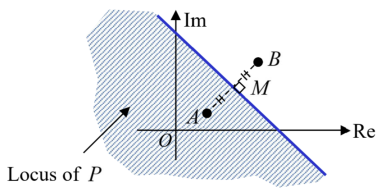
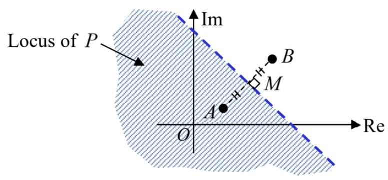
Sometimes, a harvesting term is introduced, where the population is *continually* being harvested at a rate of $H(P)$ per unit time. In such cases, the equation is modified to be:

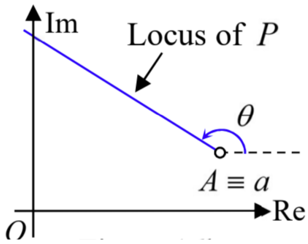
$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{N}\right) - H(P)$$

CHAPTER 8

Complex Numbers

8.1 Geometric Interpretation

Equation	Description	Figure
$z_2 - z_1$	Distance between the points z_2 and z_1	
$\mid z - z_1 \mid = r$	Circle of radius r centre z_1	
$\mid z - a \mid = \mid z - b \mid$	Perpendicular bisector of points A and B	
Inequality	If $\leq$ or $\geq$, draw a solid line. Shade region needed	
Note: one easy way to easily check which region is required is to substitute values in.	If $<$ or $>$, draw a dotted line. Shade region needed	

Equation	Description	Figure
$\arg(z-a)=\theta$	Half-line with end-point A (*excluding* A) and making an angle of θ measured in radians from the positive real axis	

Note:

1. When asked to sketch, always try to express the expression in one of the forms above.
2. As a last resort, one can sub in $z = x + yi$ and sketch y against x.

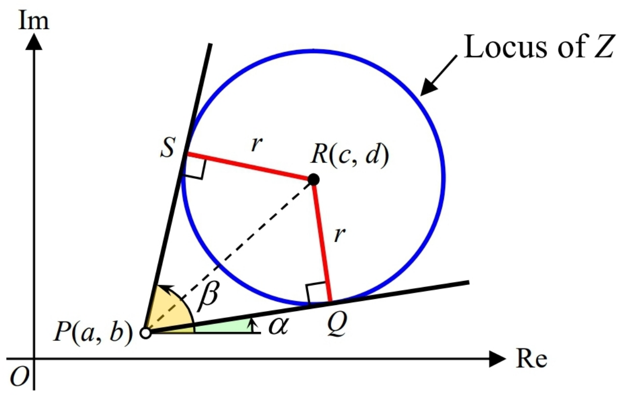
8.1.1 Problem Solving

Maximum and Minimum Distance

Steps:

1. Draw Argand Diagram.
2. Identify correct maximum and/or minimum distances in Argand diagram.
3. Use Pythagoras Theorem, simple trigonometrical ratios or coordinate geometry as appropriate to find the value of the distances.
4. Common question types usually involves:
   (a) Making use of radius of circles.
   (b) Perpendicular bisectors.

Maximum and Minimum Arguments



Steps:

1. Draw Argand Diagram.
2. Identify correct maximum and/or minimum distances in Argand diagram.
3. Use Pythagoras Theorem, sine/ cosine rules, simple trigonometrical ratios or coordinate geometry as appropriate to find the value of the distances.

8.2 De Moivre's Theorem and its Application

For any complex number $z = r\cos\theta + i\sin\theta$ and any integer n,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

8.2.1 nth root of a Complex Number

Given that $w = r(\cos\theta + i\sin\theta)$, then, to find the roots of $z^n = w$, where $n \in \mathbb{Z}^+$ is simply given by (usually write in exponential form):

$$z = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \qquad k = 0, 1, 2, \ldots, n-1$$

The roots of $z^n = 1$, $n \in \mathbb{Z}^+$ is known as n^{th} root of nullity.

8.2.2 Deriving Trigonometric Identities

Identities in $\sin n\theta$, $\cos n\theta$, and $\tan n\theta$ can be expressed in terms of $\sin\theta$, $\cos\theta$, and $\tan\theta$. For instance,

$$\begin{aligned}\tan n\theta &= \frac{\sin n\theta}{\cos n\theta}\\ &= \frac{\mathrm{Im}(\cos n\theta + i\sin n\theta)}{\mathrm{Re}(\cos n\theta + i\sin n\theta)}\\ &= \frac{\mathrm{Im}(\cos\theta + i\sin\theta)^n}{\mathrm{Re}(\cos\theta + i\sin\theta)^n}\\ &= \vdots\end{aligned}$$

One can often obtain the desired result by expanding and simplifying the expression. Do remember to define $c = \cos(\theta)$, $s = \sin\theta$, and $t = \tan\theta$ for convenience and neatness.

When questions include $\sum$ and sequence and series, try to simplify them (Maclaurin series is useful sometimes).