Commonly Encountered Words/Phrases in Mathematics

S/N	WORD/ PHRASE	MEANING	REMARKS
1	Prove (an equation/ identity)	Start from one side of an equation and proceed in a number of steps to achieve the required expression or value on the other side of the equation.	It is insufficient to substitute values to demonstrate that L.H.S. = R.H.S.
2	Show		
3	Find	Obtain the expression or value and often involve some working steps.	
4	Determine		
5	Obtain		
6	Calculate		
7	Give the value of …		
8	Verify	We need to show that a given result fits a given equation; we are allowed to use the result itself in our working. Substitute a value into the equation and check LHS = RHS.	This is often confused with "Find" or "Determine" It is easier to verify than to find.
9	Evaluate	Find a numerical value of …	You cannot leave the answer in terms of a variable.
10	Exact	Answer must be in terms of fraction, surd form, ln, π, e.	It is not a rounded decimal.
11	Numerical	In decimal form.	Often imply use of GC.
12	Write down	No written working is required.	
13	State		
14	Estimate	A guess which can be justified mathematically; an approximate value of a calculation or measurement.	You either use a known approximation result or you only take the first few terms of an infinite series.
15	Approximate		
16	Solve	Find the value of all the unknowns in an equation.	
17	Define	Often apply to functions, must give the domain and rule.	
18	Comment	It is often enough to write a short sentence that goes to the point. Must be systematic.	Do not write an essay. The length depends on the number of marks allocated for the part. If it is a 2 mark question, give 2 different points.
19	Justify	To give more information about a particular piece of argument or proof, in order to make it convincing.	Do not give a description in words only.
20	Explain		
21	Describe	Self-explanatory but highlight the essential features or properties only.	Do not write an essay.
22	Suggest	A short explanation backed up by workings.	
23	Hence	Making use of previous result.	
24	Deduce		
25	Analytical	Working must be clearly shown.	
26	Algebraic	Graphical methods are not allowed.	
27	Conjecture	An educated guess.	
28	Simplifying your answer	To simplify an expression is to remove unnecessary numbers or terms, usually by cancelling or factorising. Give answer in the lowest form.	
29	Sketch a graph	Need not be to scale; key features need to be evident.	No graph paper needed.

Trigonometry

Identities and Formulae

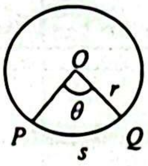
Basic Identities	$\sin^2\theta+\cos^2\theta=1$ $1+\tan^2\theta=\sec^2\theta$ $\cot^2\theta+1=\operatorname{cosec}^2\theta$
Double Angle Formulae **[in MF27]**	$\sin 2A = 2\sin A\cos A$ $\cos 2A=\cos^2 A-\sin^2 A$ $=1-2\sin^2 A$ $=2\cos^2 A-1$ $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$
Addition Formulae **[in MF27]**	$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$ $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$ $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$
R-formulae	$a\sin\theta+b\cos\theta=R\sin(\theta+\alpha)$ $a\sin\theta-b\cos\theta=R\sin(\theta-\alpha)$ $a\cos\theta+b\sin\theta=R\cos(\theta-\alpha)$ $a\cos\theta-b\sin\theta=R\cos(\theta+\alpha)$ where $R=\sqrt{a^2+b^2}$, $\alpha=\tan^{-1}\left(\dfrac{b}{a}\right)$

Circular Measure

Arc length, $s=r\theta$

Area of sector, $A=\dfrac{1}{2}r^2\theta$

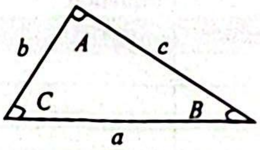
where θ **is in radians**



Sine and Cosine Rules

Sine Rule: $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$

Cosine Rule: $a^2=b^2+c^2-2bc\cos A$



Special Case

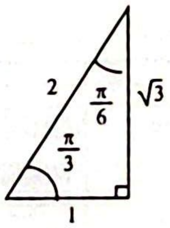
$\sin(-\theta)=-\sin\theta$

$\cos(-\theta)=\cos\theta$

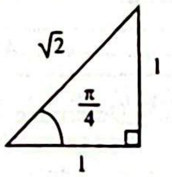
$\tan(-\theta)=-\tan\theta$

Special Angles

	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{3}$ or 60°
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$



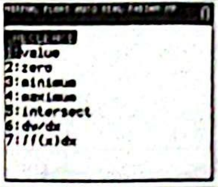
	$\frac{\pi}{4}$ or 45°
sin	$\frac{1}{\sqrt{2}}$
cos	$\frac{1}{\sqrt{2}}$
tan	1



Graphing Techniques

Procedure to Sketch a Curve

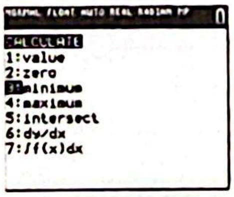
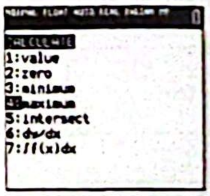
Step ①: Intercepts on Co-ordinate Axes

	x - intercepts	y - intercepts
Analytical Method (EXACT)	Let $y=0$ to find the corresponding value(s) of x.	Let $x=0$ to find the corresponding value(s) of y.
Use of G.C.		

Step ②: Asymptotes [must be drawn in dotted lines]

	Vertical	Horizontal	Oblique
Procedure	Let denominator of a rational function be zero and solve for the value(s) of x in the equation.	Find the value which y approaches as $x \to \pm\infty$.	Find the expression which y approaches as $x \to \pm\infty$. **Note:** A rational function has an oblique asymptote if the degree of numerator is 1 more than degree of denominator.
Example	$y=\dfrac{2x+1}{x+3}$	$y=\dfrac{2x+1}{x+3}$	$y=\dfrac{x^2-3x+6}{x-1}$
Solution	Let $x+3=0$ $\Rightarrow x=-3$ $\therefore x=-3$ is the vertical asymptote.	Using long division, $y=2-\dfrac{5}{x+3}$ As $x\to\pm\infty$, $y\to 2$. $\therefore y=2$ is the horizontal asymptote.	Using long division, $y=x-2+\dfrac{4}{x-1}$ As $x\to\pm\infty$, $y\to x-2$. $\therefore y=x-2$ is the oblique asymptote.

Step ③: Stationary Points

Analytical Method (EXACT)	Use of G.C.	
① Determine $\dfrac{dy}{dx}$. ② Put $\dfrac{dy}{dx}=0$ and find the corresponding x and y co-ordinates. ③ Use the first or second derivative test to check the nature of the stationary points.	Minimum point: 	Maximum point: 

Step ④: Use G.C. to obtain the Shape of the Graph

Functions

Definition

A function is defined by its rule and domain.

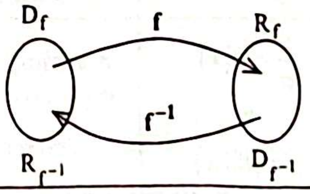
Graphical Test

Function	**Vertical Line Test** If any vertical line cuts the graph of $y = \mathrm{f}(x)$ at most once, then f is a function.
One-one Function	**Horizontal Line Test** If any horizontal line cuts the graph of $y = \mathrm{f}(x)$ at most once, then f is a one-one function.

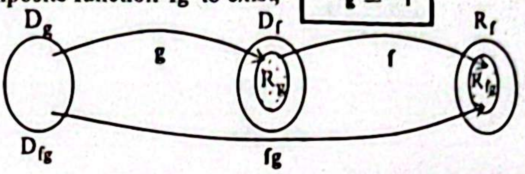
Restriction of a Function

A restriction of a function f, is a function
(i) that has the same rule as f and
(ii) whose domain is a subset of the domain of f.

Inverse Function

Test	If f is one-one, then f^{-1} exists.
Procedure	Find the expression for $\mathrm{f}^{-1}(x)$: • Let $y = \mathrm{f}(x)$ and then manipulate to make x the subject, i.e. express x in terms of y. • If the expression for x is not unique; choose the correct one by considering the domain of f, D_f. • Then $\mathrm{f}^{-1}(y) = x$ [expression in terms of y]. • Rewrite the expression by replacing y with x, to obtain $\mathrm{f}^{-1}(x)$.
Domain and Range	 $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_\mathrm{f}$ $\mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D}_\mathrm{f}$
Relationship between f and f^{-1}	The graphs of f and f^{-1} are reflections of each other in the line $y = x$. (When drawing, the scales of the x-axis and y-axis must be the same)

Composite Function

Test	For composite function fg to exist, $\boxed{R_g \subseteq D_f}$  **If fg exists:** • Write down the rule. • $D_{fg} = D_g$ • R_{fg} is determined by considering graph of fg or Substitute R_g into D_f and deduce the new range of function f.

Transformations

Translation, Scaling and Reflection

From	Replacement	To	Geometric Interpretation	From (x, y) to
$y = f(x)$	replace y with $(y-a)$	$y-a = f(x)$ $\Updownarrow$ $y = f(x)+a$	A translation of a units in the positive y-direction.	$(x, y+a)$
$y = f(x)$	replace y with $(y-(-a))$	$y-(-a) = f(x)$ $\Updownarrow$ $y = f(x)-a$	A translation of a units in the negative y-direction.	$(x, y-a)$
$y = f(x)$	replace x with $(x-a)$	$y = f(x-a)$	A translation of a units in the positive x-direction.	$(x+a, y)$
$y = f(x)$	replace x with $(x-(-a))$	$y = f(x-(-a))$	A translation of a units in the negative x-direction.	$(x-a, y)$
$y = f(x)$	replace y with $\frac{y}{a}$	$\frac{y}{a} = f(x)$ $\Updownarrow$ $y = af(x)$	A scaling parallel to the y-axis by a scale factor of a.	(x, ay)
$y = f(x)$	replace x with $\frac{x}{a}$	$y = f\left(\frac{x}{a}\right)$	A scaling parallel to the x-axis by a scale factor of a.	(ax, y)
$y = f(x)$	replace y with $-y$	$-y = f(x)$ $\Updownarrow$ $y = -f(x)$	A reflection in the x-axis.	$(x, -y)$
$y = f(x)$	replace x with $-x$	$y = f(-x)$	A reflection in the y-axis.	$(-x, y)$

Composite Transformations

$y = \mathrm{f}(x)$

↓ translate d units in the negative x-direction

$y = \mathrm{f}(x+d)$

↓ scale parallel to the x-axis by a scale factor $\frac{1}{c}$

$y = \mathrm{f}(cx+d)$

↓ scale parallel to the y-axis by a scale factor b

$y = b\mathrm{f}(cx+d)$

↓ translate a units in the positive y-direction

$y = a + b\mathrm{f}(cx+d)$

Graph of Modulus Functions

	Steps
$y = \lvert \mathrm{f}(x) \rvert$	Given the graph of $y = \mathrm{f}(x)$, ① Retain the portion above the $x-$axis (where $y \geq 0$) of $y = \mathrm{f}(x)$. ② Reflect the portion below the $x-$axis (where $y < 0$) in the x-axis.
$y = \mathrm{f}(\lvert x \rvert)$	Given the graph of $y = \mathrm{f}(x)$, ① Retain the portion of $y = \mathrm{f}(x)$ for $x \geq 0$. ② Reflect the retained portion about the y-axis.

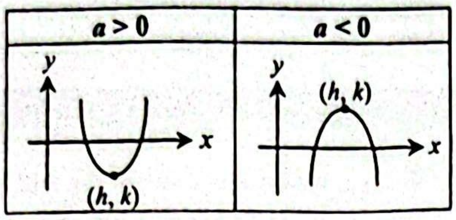
Graph of $y=\dfrac{1}{f(x)}$

	$y=f(x)$	$\longrightarrow$	$y=\dfrac{1}{f(x)}$
Markings	$(x,\ y)$	$\longrightarrow$	$\left(x,\dfrac{1}{y}\right),\ y\neq 0$
	y-intercept $(0,\ a)$	$\longrightarrow$	y-intercept (for $a\neq 0$) $\left(0,\ \dfrac{1}{a}\right)$
	$f(x)=0$ i.e. x-intercept, $(b,\ 0)$	$\longrightarrow$	$\dfrac{1}{f(x)}$ is not defined i.e. Vertical asymptote, $x=b$
	$f(x)\to\pm\infty$ i.e. Vertical asymptote, $x=c$ i.e. Oblique asymptote, $y=dx+e$	$\longrightarrow$	$\dfrac{1}{f(x)}\to 0^{\pm}$ i.e. x-intercept, $(c,\ 0)$ i.e. Horizontal asymptote, $y=0$
	Horizontal asymptote $y=f$	$\longrightarrow$	Horizontal asymptote (for $f\neq 0$) $y=\dfrac{1}{f}$
	• Maximum point $(g,\ h)$ • Minimum point $(i,\ j)$	$\longrightarrow$ $\longrightarrow$	Minimum point (for $h\neq 0$) $\left(g,\ \dfrac{1}{h}\right)$ Maximum point (for $j\neq 0$) $\left(i,\ \dfrac{1}{j}\right)$
Sketching	• $f(x)>0$ Graph lies above the x-axis • $f(x)<0$ Graph lies below the x-axis	$\longrightarrow$	$\dfrac{1}{f(x)}>0$ Graph lies above the x-axis $\dfrac{1}{f(x)}<0$ Graph lies below the x-axis
	$f(x)\uparrow$ $f(x)\downarrow$	$\longrightarrow$ $\longrightarrow$	$\dfrac{1}{f(x)}\downarrow$ $\dfrac{1}{f(x)}\uparrow$
	$f(x)=\pm 1$ The 2 graphs will intersect at the points where $f(x)=\pm 1$.	$\longrightarrow$	$\dfrac{1}{f(x)}=\pm 1$

Conics

Parabola

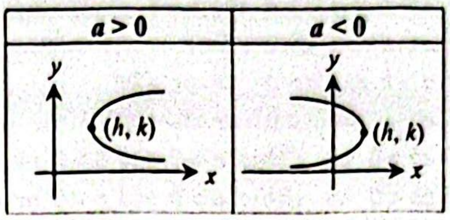
$$\boxed{y = a(x-h)^2 + k,\ a \neq 0}$$



Properties:

- Symmetrical about line $x = h$
- Turning point (h, k)

$$\boxed{(y-k)^2 = a(x-h),\ a \neq 0}$$

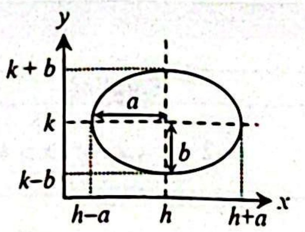


Properties:

- Symmetrical about line $y = k$
- Vertex (h, k)

Ellipse

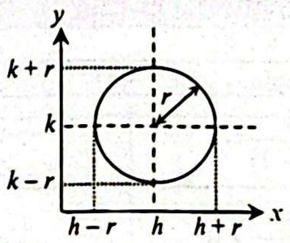
$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1} \qquad a > b > 0$$



Properties:

- Symmetrical about lines $x = h$ and $y = k$
- Centre (h, k)

Circle $\boxed{(x-h)^2 + (y-k)^2 = r^2}$

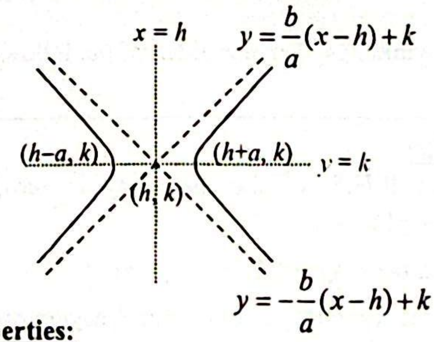


Properties:

- Symmetrical about any line that pass through (h,k)
- Centre (h,k)
- Radius r

Hyperbola

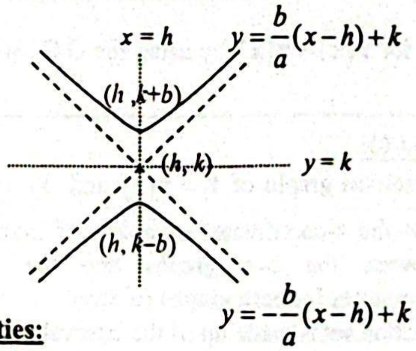
$$\boxed{\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1}$$



Properties:

- Symmetrical about lines $x = h$ and $y = k$
- Centre (h, k)
- Vertices are 'a' units from centre (h, k) in the direction parallel to x-axis
- Asymptotes at $y = \pm\frac{b}{a}(x-h)+k$

$$\boxed{\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1}$$



Properties:

- Symmetrical about lines $x = h$ and $y = k$
- Centre (h,k)
- Vertices are 'b' units from centre (h,k) in the direction parallel to y-axis
- Asymptotes at $y = \pm\frac{b}{a}(x-h)+k$

Inequalities

Basic Results

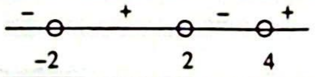
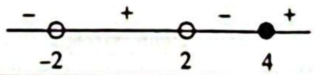
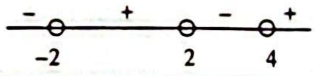
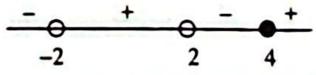
Given that $a, b, c \in \mathbb{R}$,

- If $a > b$ and $b > c \Rightarrow a > c$
- If $a > b \Rightarrow a + c > b + c$
- If $a > b$ and $c > 0 \Rightarrow ac > bc$
- If $a > b$ and $c < 0 \Rightarrow ac < bc$
- If $ab > 0 \Leftrightarrow$ either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$
- If $ab < 0 \Leftrightarrow$ either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$

Algebraic Approach

- Ensure R.H.S of the inequality is zero.
- Reduce L.H.S. to a single rational function.
- Factorise L.H.S. as far as possible. If a quadratic expression is non-factorisable, complete the square.
- Label critical values on a number line.
- Use a test-point method for each interval to find the solution.

Illustration of Test-Point Method

Example	Test-point	Solution
$\frac{(x-4)}{(x+2)(x-2)} > 0$		$-2 < x < 2$ or $x > 4$
$\frac{(x-4)}{(x+2)(x-2)} \geq 0$		$-2 < x < 2$ or $x \geq 4$
$\frac{(x-4)}{(x+2)(x-2)} < 0$		$x < -2$ or $2 < x < 4$
$\frac{(x-4)}{(x+2)(x-2)} \leq 0$		$x < -2$ or $2 < x \leq 4$

Graphical Approach

To solve for $f(x) > g(x)$ by using the G.C., we first identify the asymptotes, then use either of the following:

Method ①:	Method ②:
• Sketch the graphs of $Y_1 = f(x)$ and $Y_2 = g(x)$. • Find the x-coordinates of points of intersection between the two graphs and the vertical asymptotes for both graphs (if any). • Solution set is made up of the intervals where the graph of $Y_1 = f(x)$ is above the graph of $Y_2 = g(x)$.	• Ensure R.H.S. of the inequality is zero, i.e. $f(x) - g(x) > 0$. • Sketch the graph of $Y_1 = f(x) - g(x)$. • Find the x-intercepts and vertical asymptotes (if any). • Solution set is made up of the intervals where the graph of $Y_1 = f(x) - g(x)$ is above that of $y = 0$ (x-axis).

Definition of Modulus Function

$\lvert x\rvert = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$	For $x \in \mathbb{R}$, $a > 0$, $\lvert x\rvert < a \Leftrightarrow -a < x < a$ $\lvert x\rvert > a \Leftrightarrow x < -a$ or $x > a$

Inequalities involving Substitution

Given that $x < -a$ or $x > b$ where $a, b \in \mathbb{R}^+$

If replace x with e^x,	If replace x with $\ln x$,	If replace x with $\lvert x\rvert$,
$e^x < -a$ or $e^x > b$	$\ln x < -a$ or $\ln x > b$	$\lvert x\rvert < -a$ or $\lvert x\rvert > b$
(rej. $\because e^x > 0$) $x > \ln b$	$0 < x < e^{-a}$ $x > e^b$	(rej. $\because \lvert x\rvert > 0$) $x > b$ or $x < -b$

Given that $-a < x < b$ where $a, b \in \mathbb{R}^+$

If replace x with e^x, $-a < e^x < b$ $0 < e^x < b$ $x < \ln b$	If replace x with $\ln x$, $-a < \ln x < b$ $e^{-a} < x < e^b$	If replace x with $\lvert x\rvert$, $-a < \lvert x\rvert < b$ $\lvert x\rvert < b$ $-b < x < b$

System of Linear Equations

Procedure to Solve Problems Involving System of Linear Equations

Step ①: Define the **variables** to be used in the question. If necessary, draw a **diagram** labelling all the information to help you analyse the problem.

Step ②: Formulate the **equations** using the variables.

Step ③: Solve the equations using G.C. applications **PlySmlt2 (2: Simult Eqn Solver)**.

Step ④: Interpret the final solution based on the word problem.

Sigma Notation

Sequences

A sequence can be generated by a formula for the n^{th} term, $u_n = f(n)$, $n \in \mathbb{Z}^+$.

Series

$$\sum_{r=m}^{n} u_r = u_m + u_{m+1} + \cdots + u_{n-1} + u_n$$

Rules/ Useful Results

- $\sum_{r=m}^{n} u_r$ has $(n-m+1)$ terms
- $\sum_{r=m}^{n} cu_r = c\sum_{r=m}^{n} u_r$, where c is a constant.
- $\sum_{r=m}^{n} (u_r \pm v_r) = \sum_{r=m}^{n} u_r \pm \sum_{r=m}^{n} v_r$
- $\sum_{r=m}^{n} u_r = \sum_{r=k}^{n} u_r - \sum_{r=k}^{m-1} u_r$, where $k < m$.
- $\sum_{r=m}^{n} c = (n-m+1)c$, where c is a constant.
- $\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \frac{n}{2}(1+n)$ **(Arithmetic Series)**
- $\sum_{r=1}^{n} a^r = a + a^2 + a^3 + \ldots + a^n = \frac{a(a^n - 1)}{a-1}$ or $\frac{a(1-a^n)}{1-a}$ **(Geometric Series)**

Limit

The limit of a sequence u_n or a series $\sum_{r=m}^{n} u_r$ is the single value which it tends to as n tends to infinity. If such a limit exists then it is said to be convergent, else it is said to be divergent.

Recurrence Relation

It defines a term of a sequence using the previous term. It is a formula to work out the term from the previous one and the first term of the sequence must be given. For example, $u_n = u_{n-1} + 2$ where $n = 2, 3, 4, \ldots$ & $u_1 = 2$.

Relationship between Sequence and Series

$$u_n = \begin{cases} S_1 & , n = 1 \\ S_n - S_{n-1} & , n \in \mathbb{Z}^+, n \geq 2 \end{cases}$$

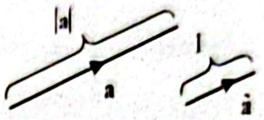
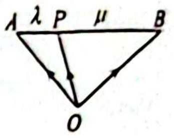
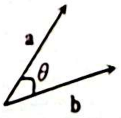
If $u_1 = S_1$, then $u_n = S_n - S_{n-1}$, $n \in \mathbb{Z}^+$

Arithmetic and Geometric Progression

Progression	**Arithmetic**	**Geometric**
Notation	First Term: a Common Difference: d	First Term: a Common Ratio: r
n^{th} Term	$u_n = a+(n-1)d$	$u_n = ar^{n-1}$
Proving	$u_{n+1} - u_n$ is a constant for all positive n	$\frac{u_{n+1}}{u_n}$ is a constant for all positive n
Sum of first n Term	$S_n = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}(a+u_n)$	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$
Sum to Infinity	- N.A. -	$S_\infty = \frac{a}{1-r},\ \lvert r\rvert < 1$
Mean	Given a sequence ..., x, y, z, ... such that x, y, z are three consecutive terms in an arithmetic sequence, then $y-x=z-y(=d)$ $y=\frac{1}{2}(x+z)$	Given a sequence ..., x, y, z, ... such that x, y, z are three consecutive terms in a geometric sequence, then $\frac{y}{x}=\frac{z}{y}(=r)$ $y^2 = xz$

Vectors

Properties

Magnitude	If $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, then $\lvert\mathbf{r}\rvert=\sqrt{x^2+y^2+z^2}$.
Unit Vectors	A unit vector is a vector whose magnitude is unity, i.e. 1. The unit vector in the direction of $\mathbf{a}$ is given by $\boxed{\hat{\mathbf{a}}=\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}}$. 
Parallel Vectors	Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **parallel** if and only if $\boxed{\mathbf{a}=k\mathbf{b}}$ for some $k\in\mathbb{R}\backslash\{0\}$.
Collinear Vectors	Three points *A*, *B* and *C* are **collinear**, i.e. lie on a straight line, if and only if $\boxed{\overrightarrow{AB}=k\overrightarrow{BC} \quad \text{for some } k\in\mathbb{R}\backslash\{0\} \text{ and } B \text{ is the common point}}$
Ratio Theorem	If *P* divides *AB* in the ratio $\lambda:\mu$, then $\overrightarrow{OP}=\boxed{\frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}}$ [in MF27].  In particular, if *P* is the mid-point of *AB*, i.e. $\lambda=1$ and $\mu=1$, $\overrightarrow{OP}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OB}\right)$
Scalar Product	$\boxed{\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta}$ **Results:** • $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$ (since $\mathbf{a}\ //\ \mathbf{a}\Rightarrow\theta=0°\Rightarrow\cos0°=1$) • $\mathbf{a}\cdot\mathbf{b}=0$ when $\mathbf{a}\perp\mathbf{b}$ (since $\theta=90°\Rightarrow\cos90°=0$) • $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$
Evaluation of Scalar Product	$\boxed{\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3}$
Angle	The angle, θ between 2 vectors $\mathbf{a}$ and $\mathbf{b}$ can be found using:  $\theta=\cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}\right)$ **Remark:** If cos θ is negative, θ will be obtuse. The acute angle between the two vectors will then be $\pi-\theta$.

<table>
<tr>
<td>Cross Product</td>
<td>

$\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both **a** and **b**.

$$\boxed{|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta}$$

Results:

- $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ (since $\mathbf{a}\,//\,\mathbf{a}\Rightarrow\theta=0^\circ\Rightarrow\sin 0^\circ=0$)
- $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|$ when $\mathbf{a}\perp\mathbf{b}$ (since $\theta=90^\circ\Rightarrow\sin 90^\circ=1$)
- $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$

</td>
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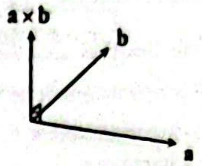
<tr>
<td>Evaluation of Cross Product</td>
<td>

$$\boxed{\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}}$$ [in MF27]



__Note:__
Vector product of **a** and **b** will give a vector which is ⊥ to both **a** and **b**.

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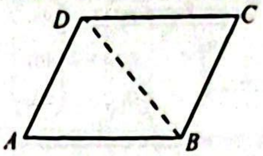
<tr>
<td>Area of Parallelogram / Triangle</td>
<td>

Area of parallelogram $ABCD=\boxed{\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|}$

Area of triangle $ABD=\boxed{\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|}$



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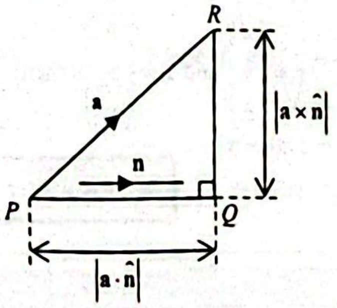
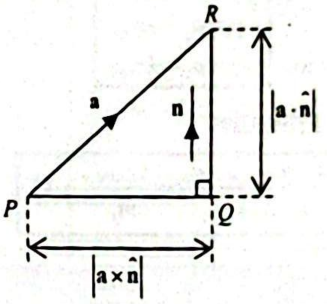
<tr>
<td>Further Applications of Dot and Cross Product</td>
<td>





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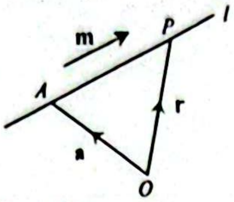
</table>

Scalar Product (Dot Product)	Vector Product (Cross Product)
$\mathbf{a}\cdot\mathbf{a}=\|\mathbf{a}\|^2$	$\mathbf{a}\times\mathbf{a}=0$
$\|\mathbf{a}\cdot\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|$ if and only if **a** and **b** are parallel to each other.	$\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|$ if and only if **a** is perpendicular to **b**
$\mathbf{a}\cdot\mathbf{b}=0$ if and only if **a** and **b** are perpendicular to each other.	$\mathbf{a}\times\mathbf{b}=\mathbf{0}$ if and only if **a** and **b** are parallel
$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ (Commutative Law)	$\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$ (Anti-commutative property)
$\mathbf{a}\cdot(k\mathbf{b})=k(\mathbf{a}\cdot\mathbf{b})=(k\mathbf{a})\cdot\mathbf{b}$ (Associative Law)	$k(\mathbf{a}\times\mathbf{b})=(k\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(k\mathbf{b})$ (Associative Law)
$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ $(\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}$ (Distributive Law)	$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ $(\mathbf{b}+\mathbf{c})\times\mathbf{a}=\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}$ (Distributive Law)

Equations of Line and Plane

Line



① **Vector Equation**

$$\boxed{\mathbf{r}=\mathbf{a}+\lambda\mathbf{m},\ \lambda\in\mathbf{R}}$$

$\mathbf{a}$: position vector of a point on l

$\mathbf{m}$: direction vector of line l

λ is a parameter

② **Parametric Equations**

$$\mathbf{r}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}m_1\\m_2\\m_3\end{pmatrix}$$

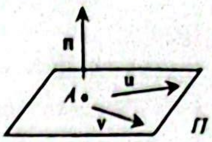
then the parametric equations of l are:

$$\boxed{\begin{aligned}x&=a_1+\lambda m_1\\y&=a_2+\lambda m_2\\z&=a_3+\lambda m_3\end{aligned}}$$

③ **Cartesian Equation**

$$\boxed{\frac{x-a_1}{m_1}=\frac{y-a_2}{m_2}=\frac{z-a_3}{m_3}(=\lambda)}$$

Plane



① **Parametric Form**

Π contains the point A with position vector $\mathbf{a}$ and is parallel to $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}$ is not // to $\mathbf{v}$

$$\boxed{\mathbf{r}=\mathbf{a}+\lambda\mathbf{u}+\mu\mathbf{v},\lambda,\ \mu\in\mathbf{R}}$$

② **Scalar Product Form**

To convert parametric form into scalar product form, $\mathbf{n}=\mathbf{u}\times\mathbf{v}$

$$\boxed{\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}=d}$$

③ **Cartesian Form**

$\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\mathbf{n}=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$, then

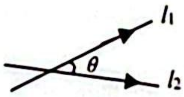
$$\boxed{n_1x+n_2y+n_3z=d}$$

Angles

2 Lines

Given 2 non-parallel lines,

$$l_1:\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{m}_1,\ \lambda\in\mathbb{R}$$
$$l_2:\mathbf{r}=\mathbf{a}_2+\mu\mathbf{m}_2,\mu\in\mathbb{R}$$



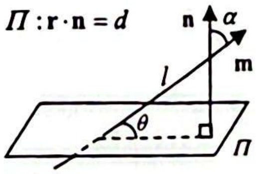
$$\theta=\cos^{-1}\left(\frac{\mathbf{m}_1\cdot\mathbf{m}_2}{|\mathbf{m}_1||\mathbf{m}_2|}\right)$$

Acute angle between l_1 and l_2 can be directly found using

$$\theta=\cos^{-1}\left|\frac{\mathbf{m}_1\cdot\mathbf{m}_2}{|\mathbf{m}_1||\mathbf{m}_2|}\right|.$$

Line and Plane

$$l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{m},\ \lambda\in\mathbb{R}$$
$$\Pi:\mathbf{r}\cdot\mathbf{n}=d$$



Method ①:

First, obtain the angle between the normal and the line, i.e.

$$\alpha=\cos^{-1}\left(\frac{\mathbf{m}\cdot\mathbf{n}}{|\mathbf{m}||\mathbf{n}|}\right)$$

① If α obtained is acute,

$$\theta=\frac{\pi}{2}-\alpha.$$

② If α obtained is obtuse,

$$\theta=\alpha-\frac{\pi}{2}.$$

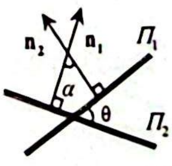
Method ②:

$$\alpha=\sin^{-1}\left(\frac{\mathbf{m}\cdot\mathbf{n}}{|\mathbf{m}||\mathbf{n}|}\right)$$

2 Planes

Given equations of two non-parallel planes as

$$\Pi_1:\mathbf{r}\cdot\mathbf{n}_1=d_1$$
$$\Pi_2:\mathbf{r}\cdot\mathbf{n}_2=d_2$$



Angle between the normals, i.e.

$$\alpha=\cos^{-1}\left(\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

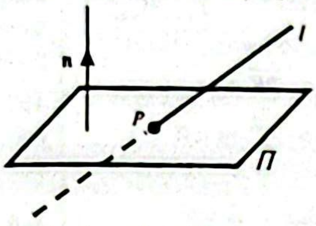
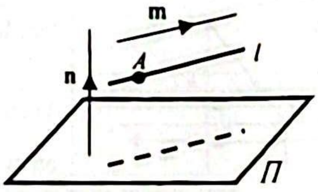
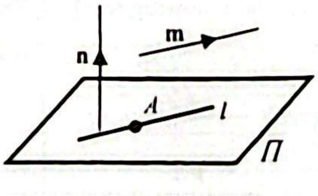
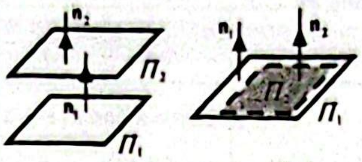
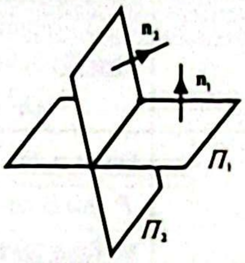
① If α obtained is acute,

$$\theta=\alpha.$$

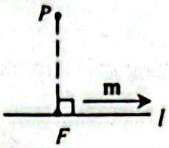
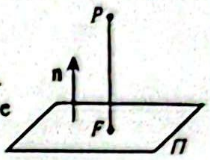
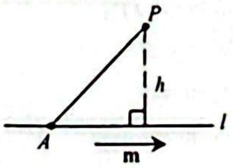
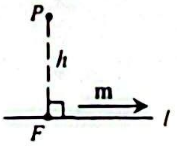
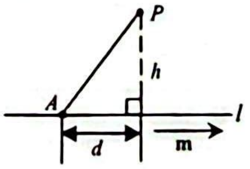
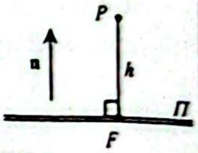
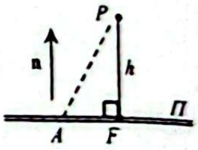
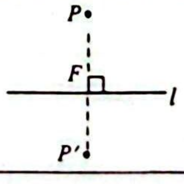
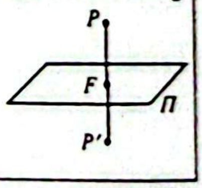
② If α obtained is obtuse,

$$\theta=\pi-\alpha.$$

Relationship

2 Lines	Line and Plane	2 Planes
Given 2 lines $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{m}_1,\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{m}_2, \mu \in \mathbb{R}$ **Case ①:** **Lines are parallel.** $\mathbf{m}_1 = k\mathbf{m}_2$ **Case ②:** **Lines intersect at a point.** $\mathbf{m}_1 \neq k\mathbf{m}_2$ for any k, the 2 lines are not parallel. $\mathbf{a}_1 + \lambda\mathbf{m}_1 = \mathbf{a}_2 + \mu\mathbf{m}_2$ Unique solution for λ and μ is obtained. **Case ③:** **Lines are skew lines.** $\mathbf{m}_1 \neq k\mathbf{m}_2$ for any k, the 2 lines are not parallel. $\mathbf{a}_1 + \lambda\mathbf{m}_1 = \mathbf{a}_2 + \mu\mathbf{m}_2$ No solution found for λ and μ.	Given a line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{m}, \lambda \in \mathbb{R}$ and a plane $\Pi : \mathbf{r}\cdot\mathbf{n} = d$ **Case ①:** **Line intersects the plane at one point.** $\mathbf{m}\cdot\mathbf{n} \neq 0$ $\because$ line is not parallel to the plane.  Let P be the point of intersection. Since P lies on l, $\overrightarrow{OP} = \mathbf{a} + \lambda\mathbf{m}$ for a λ value. Since P also lies on Π, $\overrightarrow{OP}\cdot\mathbf{n} = d$ $(\mathbf{a} + \lambda\mathbf{m})\cdot\mathbf{n} = d$ $\mathbf{a}\cdot\mathbf{n} + \lambda\mathbf{m}\cdot\mathbf{n} = d$ $\lambda = \dfrac{d - \mathbf{a}\cdot\mathbf{n}}{\mathbf{m}\cdot\mathbf{n}}$ Hence $\overrightarrow{OP} = \mathbf{a} + \left(\dfrac{d - \mathbf{a}\cdot\mathbf{n}}{\mathbf{m}\cdot\mathbf{n}}\right)\mathbf{m}$ **Case ②:** **Line does not intersect the plane.**  $\mathbf{m}\cdot\mathbf{n} = 0$ and $\mathbf{a}\cdot\mathbf{n} \neq d$. **Case ③:** **Line lies on the plane.**  $\mathbf{m}\cdot\mathbf{n} = 0$ and $\mathbf{a}\cdot\mathbf{n} = d$.	**Case ①:** **Two non-intersecting parallel planes or coincident planes.**  $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. $\mathbf{n}_1 = k\mathbf{n}_2$ for some $k \in \mathbb{R}$. **Case ②:** **Two non-parallel planes intersect at a line.**  $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. $\mathbf{n}_1 \neq k\mathbf{n}_2$ for any $k \in \mathbb{R}$. Given equations of two planes as $\Pi_1 : \mathbf{r}\cdot\mathbf{n}_1 = d_1$ $\Pi_2 : \mathbf{r}\cdot\mathbf{n}_2 = d_2$ **Method ①: Using G.C.** Rewrite the equations of both planes in Cartesian form: $a_1x + a_2y + a_3z = d_1$ $b_1x + b_2y + b_3z = d_2$ Use the G.C. application PlySmlt2 to solve the above equations for two variables in terms of the third. This will give the equation of line of intersection in parametric form. **Method ②: A point P with position vector $\mathbf{p}$ that is common to both planes is known.** Line of intersection is $l : \mathbf{r} = \mathbf{p} + \lambda\mathbf{m}, \lambda \in \mathbb{R}$ where $\mathbf{m} = \mathbf{n}_1 \times \mathbf{n}_2$.

Relationship between a Point and a Line/Plane

	Line	Plane
Is point lying on ...?	To check whether a point P lies on a line $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{m},\ \lambda\in\mathbb{R}$, equate the x, y and z components of P to the equation of the line, and solve for λ. There will be a unique value of λ if P lies on the line.	To check whether a point P lies in a plane $\Pi:\mathbf{r}\cdot\mathbf{n}=d$, substitute the position vector of P into the equation of the plane. If L.H.S. = R.H.S., P lies on the plane.
Foot of ⊥ from a point to	Given a line $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{m},\ \lambda\in\mathbb{R}$.  **Step ①**: Since F lies on l, $\overrightarrow{OF}=\mathbf{a}+\lambda\mathbf{m}$ for a λ value. **Step ②**: Find $\overrightarrow{PF}$. **Step ③**: Since $\overrightarrow{PF}\perp l \Rightarrow \overrightarrow{PF}\perp\mathbf{m}$ $\Rightarrow \overrightarrow{PF}\cdot\mathbf{m}=0$ Solve for λ. **Step ④**: Substitute the value of λ into $\overrightarrow{OF}=\mathbf{a}+\lambda\mathbf{m}$.	Given a point P with position vector $\mathbf{p}$ and a plane $\Pi:\mathbf{r}\cdot\mathbf{n}=d$.  Let F be the foot of ⊥ from the point P to the plane. The vector equation of the line that is perpendicular to the plane and passing through P is $l_{PF}:\mathbf{r}=\mathbf{p}+\lambda\mathbf{n},\lambda\in\mathbb{R}$. F is the point of intersection between the line and the plane.
...⊥ distance between point and ...	Given a line $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{m},\ \lambda\in\mathbb{R}$ and a point P that is not on the line. **Method ①: Using Cross Product** $h=\lvert\overrightarrow{AP}\times\widehat{\mathbf{m}}\rvert$  **Method ②: Finding Foot of Perpendicular** Find the foot of the perpendicular F from P to l. Then $h=\lvert\overrightarrow{PF}\rvert$.  **Method ③: Finding Length of Projection** **Step ①**: Find length of projection of $\overrightarrow{AP}$ on $\mathbf{m}$, $d=\lvert\overrightarrow{AP}\cdot\widehat{\mathbf{m}}\rvert$.  **Step ②**: Using Pythagoras Theorem, $h=\sqrt{AP^2-d^2}=\sqrt{\lvert\overrightarrow{AP}\rvert^2-\lvert\overrightarrow{AP}\cdot\widehat{\mathbf{m}}\rvert^2}$.	Given a plane $\Pi:\mathbf{r}\cdot\mathbf{n}=d$ and a point P that does not lie on Π. **Method ①:** **Finding Foot of Perpendicular** Find the foot of the perpendicular F from P to Π. Then the perpendicular distance from P to Π is $h=\lvert\overrightarrow{PF}\rvert$.  **Method ②: Using Length of Projection** If we know a point A with position vector $\mathbf{a}$ that lies on Π.  Then h is the length of projection of $\overrightarrow{AP}$ on $\mathbf{n}$, i.e. $h=\lvert\overrightarrow{AP}\cdot\widehat{\mathbf{n}}\rvert$.
Image by reflection of point in ...	Find $\overrightarrow{OF}$ first and then use the following result to find $\overrightarrow{OP'}$. $\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OP'}\right)$ 	Find $\overrightarrow{OF}$ first and then use the following result to find $\overrightarrow{OP'}$. $\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OP'}\right)$ 

Techniques of Differentiation

First Principles

The differentiation of a function $f(x)$ with respect to x is defined as $\boxed{f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}}$.

Strictly Increasing/Decreasing Functions and Concavity

Let $y = f(x)$ be a function that can be differentiated in the interval I .

(i) If $f'(x) > 0$, then f is strictly increasing in the given interval I .

(ii) If $f'(x) < 0$, then f is strictly decreasing in the given interval I .

(iii) If $f''(x) > 0$, then the graph of $y = f(x)$ is concave upwards in the given interval I .

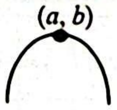
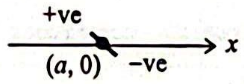
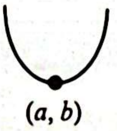
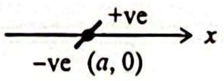
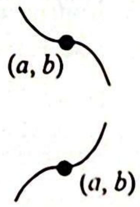
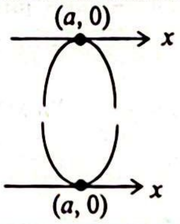
(iv) If $f''(x) < 0$, then the graph of $y = f(x)$ is concave downwards in the given interval I .

Sketch the Graph of $y = f'(x)$

Step ①: Analyse the asymptotes on $y = f(x)$

$y = f(x)$	To	$y = f'(x)$
Vertical Asymptote: $x = a$		**Vertical Asymptote:** $x = a$
Horizontal Asymptote: $y = a$		**Horizontal Asymptote:** $y = 0$
Oblique Asymptote: $y = ax + b$		**Horizontal Asymptote:** $y = a$

Step ②: Analyse the stationary points on $y = f(x)$

$y = f(x)$	To	$y = f'(x)$
Maximum Point: 		***x*-intercept:** 
Minimum Point: 		***x*-intercept:** 
Stationary Point of Inflexion: 		**Minimum/Maximum Point:** 

Step ③: Analyse the portions of the graph where...

$y = f(x)$	To	$y = f'(x)$
Gradient is negative		Graph is below x**-axis**
Gradient is positive		Graph is above x**-axis**

Step ④: Using the information obtained from steps ① – ③, sketch the graph of $y = f'(x)$.

Rules of Differentiation

Product Rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	**Quotient Rule**	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
Chain Rule	$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$		
Logarithmic Differentiation	This method is used for differentiating • expression of the form u^v, where u, v are functions of x, e.g. x^x • complicated products and quotients		
Implicit Differentiation	$\frac{d}{dx}g(y)=\left[\frac{d}{dy}g(y)\right]\frac{dy}{dx}$		

Formulae

	Basic	General
Powers of x	$\frac{d}{dx}x^n=nx^{n-1}$	$\frac{d}{dx}\left[f(x)\right]^n=n\left[f(x)\right]^{n-1}\cdot\frac{df(x)}{dx}$
Trigonometric Functions	$\frac{d}{dx}\sin x=\cos x$ $\frac{d}{dx}\cos x=-\sin x$ $\frac{d}{dx}\tan x=\sec^2 x$ $\frac{d}{dx}\sec x=\sec x\tan x$ [in MF27] $\frac{d}{dx}\operatorname{cosec} x=-\operatorname{cosec} x\cot x$ [in MF27] $\frac{d}{dx}\cot x=-\operatorname{cosec}^2 x$	$\frac{d}{dx}\sin f(x)=\cos f(x)\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\cos f(x)=-\sin f(x)\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\tan f(x)=\sec^2 f(x)\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\sec f(x)=\sec f(x)\tan f(x)\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\operatorname{cosec} f(x)=-\operatorname{cosec} f(x)\cot f(x)\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\cot f(x)=-\operatorname{cosec}^2 f(x)\cdot\frac{df(x)}{dx}$
Inverse Trigonometric Functions	$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$ [in MF27] $\frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{1-x^2}}$ [in MF27] $\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$ [in MF27]	$\frac{d}{dx}\sin^{-1}f(x)=\frac{1}{\sqrt{1-f(x)^2}}\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\cos^{-1}f(x)=-\frac{1}{\sqrt{1-f(x)^2}}\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\tan^{-1}f(x)=\frac{1}{1+f(x)^2}\cdot\frac{df(x)}{dx}$
Logarithmic Functions	$\frac{d}{dx}\ln x=\frac{1}{x}$ $\frac{d}{dx}\log_a x=\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)=\frac{1}{x\ln a}$	$\frac{d}{dx}\ln f(x)=\frac{1}{f(x)}\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}\log_a f(x)=\frac{d}{dx}\left[\frac{\ln f(x)}{\ln a}\right]=\frac{1}{f(x)\ln a}\cdot\frac{df(x)}{dx}$
Exponential Functions	$\frac{d}{dx}e^x=e^x$ $\frac{d}{dx}a^x=a^x\ln a$	$\frac{d}{dx}e^{f(x)}=e^{f(x)}\cdot\frac{df(x)}{dx}$ $\frac{d}{dx}a^{f(x)}=a^{f(x)}\ln a\cdot\frac{df(x)}{dx}$

Applications of Differentiation

Equations of Tangent and Normal

To find equation of tangent to the curve at P, we need:

- the coordinates of point $P(x_1, y_1)$,
- the gradient of the tangent at P, $f'(x_1)$.

The equation of tangent at $P(x_1, y_1)$ is

$$y - y_1 = f'(x_1)(x - x_1)$$

To find equation of normal to the curve at P, we need:

- the coordinates of point $P(x_1, y_1)$,
- the gradient of the normal at P, $-\dfrac{1}{f'(x_1)}$.

The equation of normal at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$$

Determine the Nature of the Stationary Points

Method ①: First Derivative Test

x	a^-	a	a^+	a^-	a	a^+	a^-	a	a^+
Sign of $\dfrac{dy}{dx}$	>0	0	<0	<0	0	>0	+ ve − ve	0 or 0	+ ve − ve
Tangent to curve	/	—	\	\	—	/	/—/	or	\—\
Nature of Stationary points	Maximum Point			Minimum Point			Point of Inflexion		

Method ②: Second Derivative Test

Sign of $\dfrac{d^2y}{dx^2}$ at $x = a$	<0	>0	0
Nature of Stationary Points	Maximum Point	Minimum Point	Inconclusive. Use first derivative test.

Solve Questions Involving Rate of Change

Step ①: Denote each changing quantity by a variable.
Step ②: Find equations relating the variables.
Step ③: Use chain rule to link up the derivatives.
Step ④: Write down the values of the variables and the given rates of change.
Step ⑤: Solve for the unknown rate.

Solve Maximization and Minimization Problems

Step ①: **Identify variables** (if necessary, draw a diagram).
Step ②: **Form the equations** involving these variables
Step ③: Express the dependent variable (whose max. and min. are to be determined) **in terms of a single variable** by using all the information given.
Step ④: **Differentiate** to find the stationary values.
Step ⑤: **Test the nature of the stationary value** using either first or second derivative test and answer the question.

Maclaurin's Series

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + b^n$$

where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$. **[in MF27]**

Maclaurin Expansion

In general, if a function $f(x)$ can be expressed as an infinite series or power series in ascending powers of x, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

[in MF27]

Standard Series Expansion

$$\begin{array}{ll}
f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \ldots + \dfrac{x^n}{n!}f^{(n)}(0) + \ldots & \\
(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots & (|x|<1) \\
e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^r}{r!} + \ldots & (\text{all } x)
\end{array}$$

$$\left.\begin{array}{ll}
\bigstar\ \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + \dfrac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots & (\text{all } x) \\
\text{✦}\ \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots + \dfrac{(-1)^r x^{2r}}{(2r)!} + \ldots & (\text{all } x)
\end{array}\right\} x \text{ is in radians.}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

[In MF27]

If x (in radians) is sufficiently small for x^3 and higher powers of x to be neglected, then we have

$$\begin{array}{l}
\bigstar\ \sin x \approx x \\
\text{✦}\ \cos x \approx 1 - \dfrac{x^2}{2!} \\
\tan x \approx x
\end{array}$$

Techniques of Integration

Formulae

Integration	**Basic**	**General**
form x^n	$\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\int \mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n \mathrm{d}x = \frac{\left[\mathrm{f}(x)\right]^{n+1}}{n+1} + C,\ n \neq -1$
form x^{-1}	$\int \frac{1}{x} \,\mathrm{d}x = \ln\lvert x\rvert + C$	$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} \,\mathrm{d}x = \ln\lvert \mathrm{f}(x)\rvert + C$
Exponential Functions	$\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + C$ $\int a^x \,\mathrm{d}x = \frac{a^x}{\ln a} + C$	$\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)} \,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + C$ $\int \mathrm{f}'(x) a^{\mathrm{f}(x)} \,\mathrm{d}x = \frac{a^{\mathrm{f}(x)}}{\ln a} + C$
Trigonometric Functions	$\int \cos x \,\mathrm{d}x = \sin x + C$ $\int \sin x \,\mathrm{d}x = -\cos x + C$ $\int \sec^2 x \,\mathrm{d}x = \tan x + C$ $\int \operatorname{cosec}^2 x \,\mathrm{d}x = -\cot x + C$ $\int \sec x \tan x \,\mathrm{d}x = \sec x + C$ $\int \operatorname{cosec} x \cot x \,\mathrm{d}x = -\operatorname{cosec} x + C$	$\int \mathrm{f}'(x)\cos\left[\mathrm{f}(x)\right] \mathrm{d}x = \sin\left[\mathrm{f}(x)\right] + C$ $\int \mathrm{f}'(x)\sin\left[\mathrm{f}(x)\right] \mathrm{d}x = -\cos\left[\mathrm{f}(x)\right] + C$ $\int \mathrm{f}'(x)\sec^2\left[\mathrm{f}(x)\right] \mathrm{d}x = \tan\left[\mathrm{f}(x)\right] + C$ $\int \mathrm{f}'(x)\operatorname{cosec}^2\left[\mathrm{f}(x)\right] \mathrm{d}x = -\cot\left[\mathrm{f}(x)\right] + C$ $\int \mathrm{f}'(x)\sec\left[\mathrm{f}(x)\right]\tan\left[\mathrm{f}(x)\right] \mathrm{d}x = \sec\left[\mathrm{f}(x)\right] + C$ $\int \mathrm{f}'(x)\operatorname{cosec}\left[\mathrm{f}(x)\right]\cot\left[\mathrm{f}(x)\right] \mathrm{d}x = -\operatorname{cosec}\left[\mathrm{f}(x)\right] + C$
Other Trigonometric Functions **[Basic in MF27]**	★ $\int \tan x \,\mathrm{d}x = \ln\lvert \sec x\rvert + C$ ★ $\int \cot x \,\mathrm{d}x = \ln\lvert \sin x\rvert + C$ ★ $\int \operatorname{cosec} x \,\mathrm{d}x = -\ln\lvert \operatorname{cosec} x + \cot x\rvert + C$ ★ $\int \sec x \,\mathrm{d}x = \ln\lvert \sec x + \tan x\rvert + C$	$\int \mathrm{f}'(x)\tan\left[\mathrm{f}(x)\right] \mathrm{d}x = \ln\left\lvert \sec\left[\mathrm{f}(x)\right]\right\rvert + C$ $\int \mathrm{f}'(x)\cot\left[\mathrm{f}(x)\right] \mathrm{d}x = \ln\left\lvert \sin\left[\mathrm{f}(x)\right]\right\rvert + C$ $\int \mathrm{f}'(x)\operatorname{cosec}\left[\mathrm{f}(x)\right] \mathrm{d}x = -\ln\left\lvert \operatorname{cosec}\left[\mathrm{f}(x)\right] + \cot\left[\mathrm{f}(x)\right]\right\rvert + C$ $\int \mathrm{f}'(x)\sec\left[\mathrm{f}(x)\right] \mathrm{d}x = \ln\left\lvert \sec\left[\mathrm{f}(x)\right] + \tan\left[\mathrm{f}(x)\right]\right\rvert + C$
Others **[Basic in MF27]**	$\int \frac{1}{x^2+a^2} \,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2-x^2}} \,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + C$ ★ $\int \frac{1}{x^2-a^2} \,\mathrm{d}x = \frac{1}{2a}\ln\left\lvert \frac{x-a}{x+a}\right\rvert + C$ ★ $\int \frac{1}{a^2-x^2} \,\mathrm{d}x = \frac{1}{2a}\ln\left\lvert \frac{a+x}{a-x}\right\rvert + C$	$\int \frac{\mathrm{f}'(x)}{\left[\mathrm{f}(x)\right]^2+a^2} \,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left[\frac{\mathrm{f}(x)}{a}\right] + C$ $\int \frac{\mathrm{f}'(x)}{\sqrt{a^2-\left[\mathrm{f}(x)\right]^2}} \,\mathrm{d}x = \sin^{-1}\left[\frac{\mathrm{f}(x)}{a}\right] + C$ $\int \frac{\mathrm{f}'(x)}{\left[\mathrm{f}(x)\right]^2-a^2} \,\mathrm{d}x = \frac{1}{2a}\ln\left\lvert \frac{\mathrm{f}(x)-a}{\mathrm{f}(x)+a}\right\rvert + C$ $\int \frac{\mathrm{f}'(x)}{a^2-\left[\mathrm{f}(x)\right]^2} \,\mathrm{d}x = \frac{1}{2a}\ln\left\lvert \frac{a+\mathrm{f}(x)}{a-\mathrm{f}(x)}\right\rvert + C$

★ Presented differently in MF27

Techniques

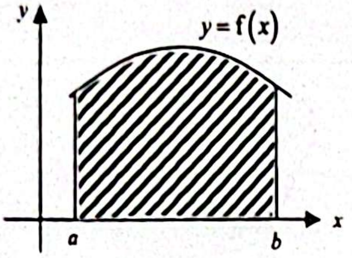
Integration	Suggested Approach
By Standard Form	• Powers of x — $\left[f(x)\right]^n$ where $n \neq -1$ • Logarithmic functions — $\frac{f'(x)}{f(x)}$ • Exponential functions — $e^{f(x)}$ • Trigonometric functions • Inverse trigonometric functions — $\int \frac{1}{ax^2+bx+c}\,dx$
Using Trigonometric Formulae	For integrals of the form $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$ where n is even, apply **Double Angle Formula**
Involving Long Division	For integrals containing **improper rational functions** such as $\int \frac{3x}{1-x}\,dx$ and $\int \frac{2x^2}{x+4}\,dx$.
Involving Partial Fractions	Used for **proper rational functions,** $\frac{p(x)}{q(x)}$, where the denominator is completely factorized.
Involving 'Splitting the Numerator'	For integrals that are **unable to be expressed in the form** $\int \frac{f'(x)}{f(x)}\,dx$ **directly**.
By Substitution	Used to introduce a suitable substitution so that the original integral can be transformed to one that fits into one of the standard formulae. **Step ①:** Differentiate the given substitution. **Step ②:** Ensure all the terms in x have been substituted with the new variable, say θ. **Step ③:** Integrate the expression with respect to θ. **Step ④:** Remember to express the final answer in terms of the original variable, x.
By Parts	Useful for integrals that contain **inverse trigonometric functions, logarithmic functions and mixed functions.** $\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$ **Guide on Choice of Function u: (term to differentiate)** Logarithmic, Inverse trigonometry, Algebraic, Trigonometry, Exponential (LIATE)

Definite Integrals

Area

... x-axis

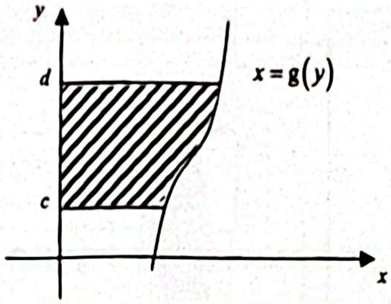
Area of region bounded by the curve $y = f(x)$, the vertical lines $x = a$ and $x = b$ and the x-axis:



$$A = \int_a^b y \, dx$$

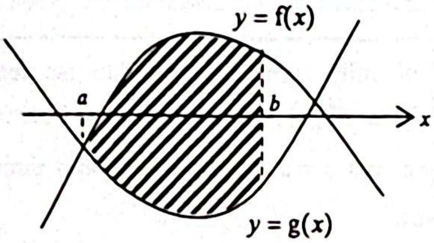
... y-axis

Area of region bounded by the curve $x = g(y)$, the horizontal lines $y = c$ and $y = d$ and the y-axis:



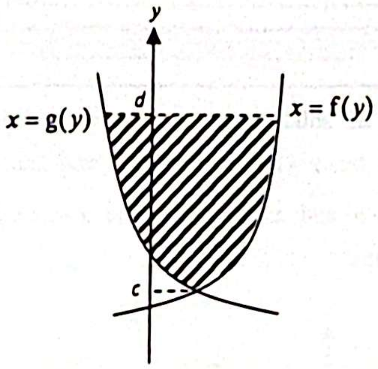
$$A = \int_c^d x \, dy$$

Area of region bounded between the curves $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$:



$$A = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$

Area of region bounded between the curves $x = f(y)$ and $x = g(y)$ and the horizontal lines $y = c$ and $y = d$:

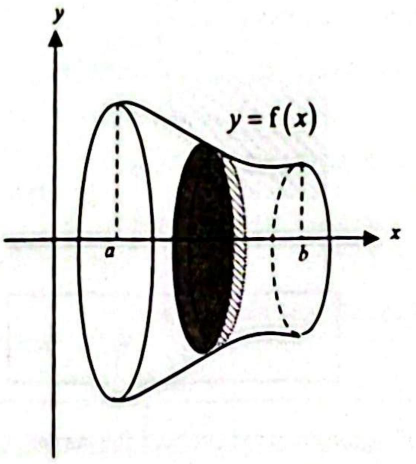


$$A = \int_c^d f(y) \, dy - \int_c^d g(y) \, dy$$

Volume

<table>
<tr><th>... x-axis</th><th>... y-axis</th></tr>
<tr>
<td>

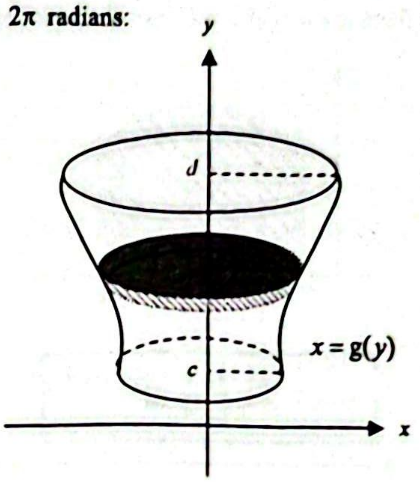
Volume of solid formed by rotating the region bounded by the curve $y = f(x)$, the vertical lines $x = a$ and $x = b$ and the x-axis, about the x-axis through 2π radians:



$$V = \pi\int_a^b y^2\,dx$$

</td>
<td>

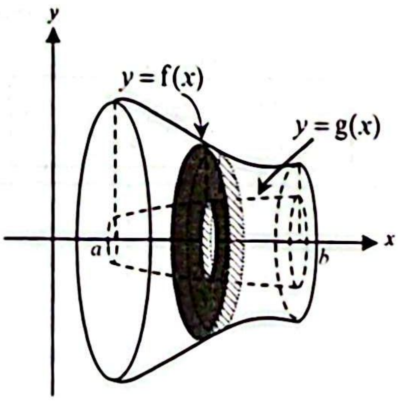
Volume of solid formed by rotating the region bounded by the curve $x = g(y)$, the horizontal lines $y = c$ and $y = d$ and the y-axis, about the y-axis through 2π radians:



$$V = \pi\int_c^d x^2\,dy$$

</td>
</tr>
<tr>
<td>

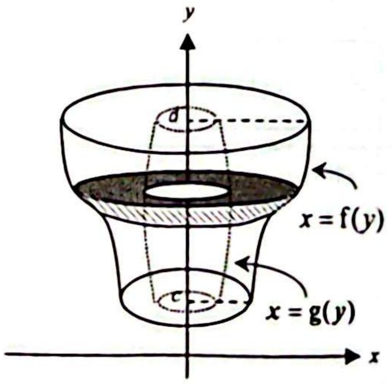
Volume of solid formed by rotating the region bounded by $y = f(x)$, $y = g(x)$ and the vertical lines $x = a$ and $x = b$, about the x-axis through 2π radians:



$$V = \pi\int_a^b [f(x)]^2\,dx - \pi\int_a^b [g(x)]^2\,dx$$

</td>
<td>

Volume of solid formed by rotating the region bounded by $x = f(y)$, $x = g(y)$ and the horizontal lines $y = c$ and $y = d$, about the y-axis through 2π radians:



$$V = \pi\int_c^d [f(y)]^2\,dy - \pi\int_c^d [g(y)]^2\,dy$$

</td>
</tr>
</table>

Parametric Equations

Parametric Equation

It is any set of equations that describes the curve using a real variable, t, called the parameter.

Elimination of parameter →

Cartesian Equation

It is any equation that describes the curve using the x- and y- coordinates in the plane.

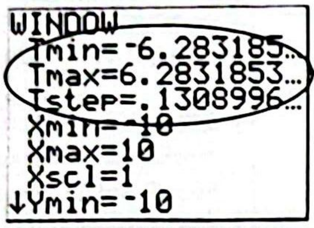
Use of G.C.

(i)

```
NORMAL  SCI  ENG
FLOAT  0123456789
RADIAN  DEGREE
FUNC  PAR  POL  SEQ
CONNECTED  DOT
SEQUENTIAL  SIMUL
REAL  a+bi  re^θi
FULL  HORIZ  G-T
↓NEXT↓
```

Ensure that G.C. is in parametric (PAR) mode

(ii)



```
WINDOW
 Tmin=-6.283185…
 Tmax=6.2831853…
 Tstep=.1308996…
 Xmin=-10
 Xmax=10
 Xscl=1
↓Ymin=-10
```

$T_{min} = -2\pi$

$T_{max} = 2\pi$

Use this range for sketching of parametric graphs, unless specified in question

Parametric Differentiation

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Differential Equations

Differential Equations

The order of a differential equation is the highest derivative present in the equation.

Solve using

Direct Integration

Equation of the form $\frac{dy}{dx} = f(x)$

$$\frac{dy}{dx} = f(x)$$

Integrating with respect to x,

$$\int \frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\int 1\,dy = \int f(x)\,dx$$

$$y = \int f(x)\,dx$$

Method of Separation of Variables

$$\frac{dy}{dx} = f(x)g(y)$$

Rewriting, $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$

Integrating with respect to x,

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Method of Substitution

Step ①:

Differentiate the given substitution with respect to x.

Step ②:

Replace the $\frac{dy}{dx}$ term in the differential equation by the expression obtained in Step ① and the terms in y are also replaced accordingly by the new variable.

Step ③:

The new simplified differential equation can be solved by **direct integration or method of separation of variables.**

Step ④:

Replace the new variable with y in the general solution.

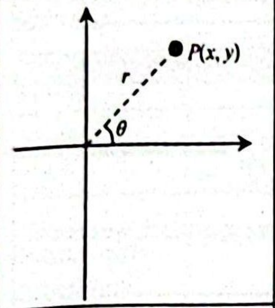
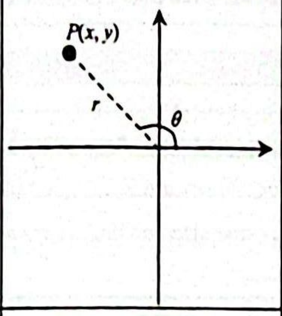
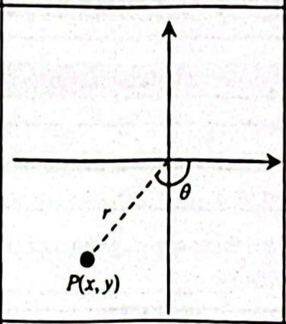
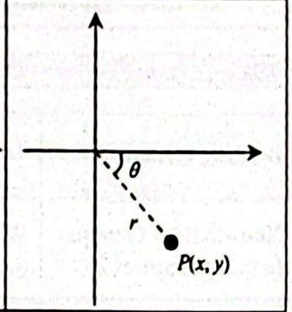
Complex Numbers

$\boxed{i=\sqrt{-1}}$ $\boxed{i^2=-1}$

Conjuagte of a Complex Number

If $\boxed{z=x+iy}$ where $x, y \in \mathbb{R}$, then the conjugate of z, denoted by z^* is defined by $\boxed{z^*=x-iy}$.

Modulus and Argument of a Complex Number

1st Quadrant	2nd Quadrant	3rd Quadrant	4th Quadrant
$x>0$ and $y>0$	$x<0$ and $y>0$	$x<0$ and $y<0$	$x>0$ and $y<0$
			
$r=\sqrt{x^2+y^2}$	$r=\sqrt{x^2+y^2}$	$r=\sqrt{x^2+y^2}$	$r=\sqrt{x^2+y^2}$
$\theta=\tan^{-1}\left\|\frac{y}{x}\right\|$	$\theta=\pi-\tan^{-1}\left\|\frac{y}{x}\right\|$	$\theta=-\left(\pi-\tan^{-1}\left\|\frac{y}{x}\right\|\right)$	$\theta=-\tan^{-1}\left\|\frac{y}{x}\right\|$

Geometrical Effects

	On the Argand diagram, suppose *P* is the point which represents the complex number z, then the geometric effect of ...
Conjugation	conjugating a complex number is reflecting *P* about the Re-axis.
Negation (multiplying by −1)	negating a complex number is rotating *OP* 180° about the origin.
Multiplying by i	multiplying a complex number by i is rotating *OP* 90° anti-clockwise.about the origin.

	On the Argand diagram, the geometric effect of ...
Addition/ Subtraction	adding/subtracting one complex number to another complex number is a form of translation/is translating one point to a different point.

Conjugate Roots Theorem

A polynomial equation of degree n,

$$a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n=0$$

has exactly n roots (considering repeated roots and complex roots).

Now consider a polynomial equation with **real coefficients**,

$$a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n=0 \text{ where } a_0, a_1, a_2, a_3, ..., a_n \in \mathbb{R}.$$

If z is a complex root of the equation, then **z^* is also a complex root** of the equation. Hence, there is always an even number of complex roots, and the rest are real roots.

Permutations and Combinations

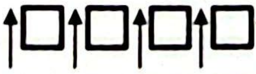
General Principles

Addition Principle	Suppose there are n_1 ways for event E_1 to occur, n_2 ways for event E_2 to occur, … , n_k ways for event E_k to occur, and these ways are **mutually exclusive**, then the number of ways for at least one of the events $E_1, E_2, \dots, E_k$ to occur is $n_1 + n_2 + \cdots + n_k$.
Multiplication Principle	Suppose that an event E can be spilt into k events, $E_1, E_2, \dots, E_k$, in ordered stages. If there are n_1 ways for E_1 to occur, n_2 ways for E_2 to occur, …, and n_k ways for E_k to occur, and these ways are **independent**, then the number of ways for event E to occur is given by $n_1 n_2 \cdots n_k$.

Selection of Objects

Distinct Groups	To select r objects out of n distinct objects, ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ for $0 \le r \le n$.
Non-distinct Groups (e.g. 3 groups of 2)	When there are r groups of the same size and the groups are not distinct, we need to divide by $r!$.
Selection from Non-distinct Objects	List down cases.

Arrangement of Objects

Distinct Objects	In general, the number of ways to arrange r objects from n distinct objects ($r \le n$) in a row could be done in two stages: Stage ①: Number of ways to select r objects out of n $= {}^nC_r$ Stage ②: Number of ways to arrange the r objects $= r!$ (${}^rP_r = r!$) Total no. of ways $= {}^nC_r \times r!$
Non-distinct Objects	In general, the number of ways to arrange n objects where n_1 objects are indistinguishable of type 1, n_2 objects are indistinguishable of type 2, … and n_k objects are indistinguishable of type k, is $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$
With Objects Combined	Stage ①: Combine the objects Stage ②: Arrange the non-combined objects and the combined objects Stage ③: Arrange the combined objects
With Objects Separated	Stage ①: Arrange the objects $(n-r)$ excluding those (r) that need to be separated Stage ②: Insert r objects into "spaces" (denoted by arrows) and arrange them. 
In a Circle with Non-distinct Labels	The number of permutations of n distinct objects in a circle is $(n-1)!$
In a Circle with Distinct Labels	The number of permutations of n distinct objects in a circle of n labelled position. $(n-1)! \times n$

Probability

Probability	If the finite sample space S consists of equally likely outcomes, then the probability of an event E, written as $P(E)$ is defined as $\boxed{P(E)=\frac{n(E)}{n(S)}}$
Results	Let E denote some event. • $0 \le P(E) \le 1$ • $P(E')=1-P(E)$ Let A and B denote some events. • $P(A\cup B)=P(B\cup A)$ • $P(A\cap B)=P(B\cap A)$ • $P(A\cup B)=P(A)+P(B)-P(A\cap B)$
Mutually Exclusive Events	Events A and B are said to be mutually exclusive if they **cannot occur simultaneously.** To show whether two events A and B are mutually exclusive, we need to determine either $P(A\cap B)=0$ or $P(A\cup B)=P(A)+P(B)$
Conditional Probability	Given that A and B are events such that $P(A)\neq 0$ and $P(B)\neq 0$, the probability of A occurring given that B has already occurred is written as $P(A\|B)$. $\boxed{P(A\|B)=\frac{P(A\cap B)}{P(B)}}$
Independent Events	Two processes are independent if knowing the outcome of one provides no useful information about the outcome of the other. To show whether two events A and B are independent, we need to show either $P(A\|B)=P(A)$ or $P(B\|A)=P(B)$ or $P(A\cap B)=P(A)\cdot P(B)$
Tools	• Table of Outcomes • Venn Diagram • Permutation and Combination • Tree Diagram

Tree Diagram

$P(A)=a$

$P(A')=1-a$

$P(B)=ab+(1-a)c$

$P(B')=a(1-b)+(1-a)(1-c)$

$P(A\cap B)=ab$

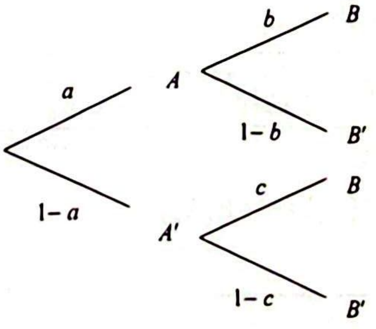
$P(A'\cap B')=(1-a)(1-c)$

$P(B|A)=b$

$P(B'|A)=1-b$

$P(B|A')=c$

$P(B'|A')=1-c$



Discrete Random Variables

A random variable is a function that associates a real number with each element in the sample space.

Definition	**A random variable is called a discrete random variable if its set of possible outcomes is countable.**
Probability Distribution Function	- $P(X=x)=f(x)$ - $0\le f(x)\le 1$ - $\sum_{\text{all } x} f(x)=1$ **Probability Distribution Table** $\begin{array}{|c|c|c|c|c|} \hline x & a & b & c & d \\ \hline P(X=x) & P(X=a) & P(X=b) & P(X=c) & P(X=d) \\ \hline \end{array}$
Mode	Value of x such that $P(X=x)$ is the largest.
Median, m	$P(X\le m)\ge 0.5$ and $P(X\ge m)\ge 0.5$ **Note:** If $P(X\le m_1)\ge 0.5$ and $P(X\ge m_2)\ge 0.5$, then $m=\frac{m_1+m_2}{2}$
Mean, $\mu=E(X)$	$E(X)=\sum_{\text{all } x}[xf(x)]$
$E[g(x)]$	$\sum_{\text{all } x}[g(x)f(x)]$
Variance, $\sigma^2=\text{Var}(X)$	$\text{Var}(X)=E\left[(X-\mu)^2\right]=\sum_{\text{all } x}\left[(x-\mu)^2 f(x)\right]$ $\text{Var}(X)=E(X^2)-[E(X)]^2$ **Note:** $\sigma=\sqrt{\text{Var}(X)}$ is called the standard deviation of X.

Mean and Variance of Linear Combinations of Random Variables

Let X and Y be two independent random variable and a and b are constants.

Mean	**Variance**
$E(a)=a$	$\text{Var}(a)=0$
$E(aX)=aE(X)$	$\text{Var}(aX)=a^2\text{Var}(X)$
$E(aX\pm b)=aE(X)\pm b$	$\text{Var}(aX\pm b)=a^2\,\text{Var}(X)$
$E(aX\pm bY)=aE(X)\pm bE(Y)$	$\text{Var}(aX\pm bY)=a^2\,\text{Var}(X)+b^2\,\text{Var}(Y)$
$E(X_1+X_2+\ldots+X_n)$ $=E(X_1)+E(X_2)+E(X_3)+\cdots+E(X_n)$ $=nE(X)$	$\text{Var}(X_1+X_2+\ldots+X_n)$ $=\text{Var}(X_1)+\text{Var}(X_2)+\cdots+\text{Var}(X_n)$ $=n\,\text{Var}(X)$

Note:

$$2X\neq X_1+X_2$$

The random variable $2X$ means twice the random variable X, while X_1+X_2 is the sum of two independent observations of X.

Binomial Distribution

<table>
<tr><td>Conditions*</td><td>1. There is a fixed number of trials.
2. Each trial is independent of any other trial.
3. Each trial has 2 mutually exclusive possible outcomes i.e. 'success' or 'failure' outcomes.
4. The probability of a success, denoted by p, is constant in each trial.</td></tr>
<tr><td>Definition</td><td>Let X be the random variable denoting number of ... out of ...
$$X \sim \mathrm{B}(n, p)$$
where n is the number of trials
p is the probability of success for each trial.</td></tr>
<tr><td>Probability Distribution</td><td><u>Method ①:</u>
$$\mathrm{P}(X = x) = \binom{n}{x}(p)^x (1-p)^{n-x}, \text{for } x = 0,1,2,\ldots,n$$
where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ [in MF27]
<u>Method ②:</u>
G.C. allows us to find $\mathrm{P}(X = x)$ using binompdf.</td></tr>
<tr><td>$\mathrm{P}(X \le x)$</td><td>G.C. only allows us to find $\mathrm{P}(X \le x)$ using binomcdf. To find probabilities involving other inequality signs, we need to rewrite the expressions as follows:
<table>
<tr><th>To find</th><th>Calculate</th></tr>
<tr><td>$\mathrm{P}(X < k)$</td><td>$\mathrm{P}(X \le k-1)$</td></tr>
<tr><td>$\mathrm{P}(X > k)$</td><td>$1-\mathrm{P}(X \le k)$</td></tr>
<tr><td>$\mathrm{P}(X \ge k)$</td><td>$1-\mathrm{P}(X \le k-1)$</td></tr>
</table>
where $k \in \mathbb{Z}$.</td></tr>
<tr><td>Mean</td><td>np [in MF27]</td></tr>
<tr><td>Variance</td><td>$np(1-p)$ [in MF27]</td></tr>
<tr><td>Mode (Most Likely Value)</td><td>G.C. allows us to find the mode by entering binompdf in $Y_1 =$ and observing the TABLE for the highest probability.</td></tr>
</table>

* In examination questions, the fixed number of trials and two mutually exclusive possible outcomes are usually stated in the question. As such, one should answer with regards to independence and constant probability of success in context of the question (examples shown on next page).

Example ①: [2009/II/11(i)]

A fixed number, n, of cars is observed and the number of those cars that are red is denoted by R. State, in context, two assumptions needed for R to be well modelled by a binomial distribution.

Solution:

1. Whether each car is red or not is independent of any other cars.
2. Probability of a car being red is a constant throughout the sample.

Example ②: [2011/II/7(i)]

When I try to contact (by telephone) any of my friends in the evening, I know that **on average** the probability that I succeed is 0.7. On one evening I attempt to contact a fixed, n, of different friends. If I do not succeed with a particular friend, I do not attempt to contact that friend again that evening. The number of friends whom I succeed in contacting is the random variable R. State, in the context of this question, two assumptions needed to model R by a binomial distribution.

Solution:

1. Whether I succeed in contacting a friend or not is independent of any other attempts.
2. Probability of successfully contacting a friend is a constant at 0.7.

Example ③: [2012/II/9(i)]

In an opinion poll before an election, a sample of 30 voters is obtained. The number of voters in the sample who support Alliance Party is denoted by A. State, in context, what must be assumed for A to be well-modelled by a binomial distribution.

Solution:

1. Whether a voter who supports Alliance Party or not is independent of any other voters.
2. Probability of a voter voting for Alliance party is a constant.

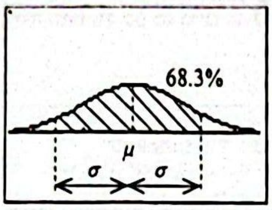
Example ④: [2013/II/7(i)]

On average one in 20 packets of a breakfast cereal contains a free gift. Jack buys n packets from a supermarket. The number of these packets containing a free gift is the random variable F. State, in context, two assumptions needed for F to be well modelled by a binomial distribution.

Solution:

1. Whether a packet contains a free gift or not is independent of any other packets.
2. Probability of getting a free gift in a cereal packet is a constant at $\frac{1}{20}$.

Normal Distribution

Definition	Let X be the random variable denoting … of a … $X \sim \mathrm{N}(\mu,\sigma^2)$
Mean	μ
Variance	σ^2
Properties	 Note: $\mathrm{P}(X=a)=0$ $\mathrm{P}(\lvert X-\mu\rvert \le \sigma)=0.683$ $\mathrm{P}(\lvert X-\mu\rvert \le 2\sigma)=0.954$ $\mathrm{P}(\lvert X-\mu\rvert \le 3\sigma)=0.997$ • Normal distribution curve is bell shaped and symmetrical about the line $x=\mu$. • Total area under a normal curve is 1. • Area under a normal curve between $x=a$ and $x=b$ gives the value of the probability $\mathrm{P}(a \le X \le b)$. • Shape and location of a normal curve are determined by the values of μ and σ.
Standardization	If $\mu=0$ and $\sigma=1$, then $Z \sim \mathrm{N}(0,1)$ is called the **standard normal distribution.** If μ and/or σ are unknown, there is a need to standardize $X \sim \mathrm{N}(\mu,\sigma^2)$ so as to find the unknown(s) before other probabilities can be computed. If $X \sim \mathrm{N}(\mu,\sigma^2)$ and $Z=\frac{X-\mu}{\sigma}$, then $Z \sim \mathrm{N}(0,1)$.
Sum of Independent Normal Variables	If X and Y are two independent normal variables such that $X \sim \mathrm{N}(\mu_x,\sigma_x^2)$ and $Y \sim \mathrm{N}(\mu_y,\sigma_y^2)$, then $\boxed{\begin{gathered} X+Y \sim \mathrm{N}(\mu_x+\mu_y,\, \sigma_x^2+\sigma_y^2) \\ X-Y \sim \mathrm{N}(\mu_x-\mu_y,\, \sigma_x^2+\sigma_y^2) \end{gathered}}$ $\boxed{\begin{gathered} aX+bY \sim \mathrm{N}(a\mu_x+b\mu_y,\, a^2\sigma_x^2+b^2\sigma_y^2) \\ aX-bY \sim \mathrm{N}(a\mu_x-b\mu_y,\, a^2\sigma_x^2+b^2\sigma_y^2) \end{gathered}}$
Sum of Independent and Identically Distributed Normal Variables	If $X \sim \mathrm{N}(\mu,\sigma^2)$ and $X_1, X_2, X_3, \ldots, X_n$ are n independent random observations of X, then $X_1+X_2+X_3+\ldots+X_n$ will follow a normal distribution with $\boxed{X_1+X_2+X_3+\ldots+X_n \sim \mathrm{N}(n\mu,\, n\sigma^2)}$ If $X \sim \mathrm{N}(\mu,\sigma^2)$ and nX is n times of one random observation of X, then nX will follow a normal distribution with $\boxed{nX \sim \mathrm{N}(n\mu,\, n^2\sigma^2)}$

Sampling

Definition	• A **population** consists of the totality of the observations with which we are concerned. • A **sample** is a collection of observations or measurements taken from a population, i.e a sample is a subset of a population. • A **simple random sample** is a sample in which all elements in the population are selected **independently** from one another and each element has an **equal chance** of being selected. • A statistic is **unbiased** if the expectation of the statistic is equal to the population parameter, i.e. a statistic $\hat{\Theta}$ is said to be an unbiased estimator of the parameter θ if $\mu_{\hat{\Theta}} = \mathrm{E}\left(\hat{\Theta}\right) = \theta$.
Sampling Distribution	**Given that X has Mean μ and Variance σ^2,** **<u>Case ①</u>: If X is Normally Distributed** $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ **<u>Case ②</u>: If X is <u>NOT</u> given to be Normally Distributed** Since n is sufficiently large, i.e. $n \geq 30$ $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ approx. by CLT **or** $X_1 + X_2 + X_3 + \ldots + X_n \sim \mathrm{N}\left(n\mu, n\sigma^2\right)$ approx. by CLT
Estimation of Population Parameters	Unbiased estimate of population mean, $\hat{\mu} = \bar{x} = \frac{\sum x}{n} = \frac{\sum(x-a)}{n} + a$ Unbiased estimate of population variance, $s^2 = \frac{n}{n-1}\left(\frac{\sum(x-\bar{x})^2}{n}\right)$ [In MF27] $= \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$ [In MF27] $= \frac{1}{n-1}\left(\sum(x-a)^2 - \frac{\left(\sum(x-a)\right)^2}{n}\right)$ $= \frac{n}{n-1}(\text{sample variance})$

Hypothesis Testing

Definition

- *p*-value is the probability of obtaining the observed sample data or more extreme than it is, given that the population mean value stated in the null hypothesis is true.
- $\alpha\%$ level of significance means that there is $\alpha\%$ chance of rejecting H_0 when H_0 is true, i.e. $\mathrm{P}(\text{rejecting } \mathrm{H}_0 \mid \mathrm{H}_0 \text{ is true}) = \alpha\%$
- **Relationship:** *p*-value is the smallest level of significance at which the null hypothesis H_0 will be rejected.

Partial Question: Given $\mathrm{H}_0 : \mu = 30$

$\mathrm{H}_1 : \mu > 30$, where μ is the population mean length of an adult fish, in cm.

with $\bar{x} = 30.15$, $n = 100$ and *p*-value $= 0.0473$

(i) Explain what is meant by the phrase 'a 5% significance level' in this context.
(ii) Explain what is meant by '*p*-value' in this context.

Possible solution:

(i) 5% significance level means that there is a 0.05 probability that we claim that the mean length of an adult fish is more than 30 cm when it is actually 30 cm.

(ii) *p*-value means that there is a probability of 0.0473 to obtain a sample of 100 adult fish whose mean length is 30.15 cm or more when the population mean length is 30 cm.

Procedure for Conducting a Hypothesis Test

Calculate the unbiased estimate of population mean, $\hat{\mu}$ if $\bar{x}$ if not given in question.

Unbiased estimate of population mean, $\hat{\mu} = \bar{x} = \dfrac{\sum x}{n}$ or $\dfrac{\sum(x-a)}{n} + a$

Calculate the unbiased estimate of the population variance, s^2 if σ^2 is not given in question.

Unbiased estimate of population variance, s^2

$$= \frac{n}{n-1}\left(\frac{\sum(x-\bar{x})^2}{n}\right) \text{ or } \frac{1}{n-1}\left(\sum x^2 - \frac{(\sum x)^2}{n}\right) \text{ or } \frac{1}{n-1}\left(\sum(x-a)^2 - \frac{(\sum(x-a))^2}{n}\right) \text{ or } \frac{n}{n-1}(\text{sample variance})$$

[In MF27] [In MF27]

Step ①: **Define Random Variable if it is not defined in the question.**

Step ②: **State Hypotheses.**

Null hypothesis $\mathrm{H}_0 : \mu = \mu_0$

Alternative hypothesis

Left-tailed Test	Two-tailed Test	Right-tailed Test
$\mathrm{H}_1 : \mu < \mu_0$	$\mathrm{H}_1 : \mu \neq \mu_0$	$\mathrm{H}_1 : \mu > \mu_0$

where μ is the population mean <in context of question>.

Step ④: State the Distribution of $\bar{X}$ and the Test Statistic.

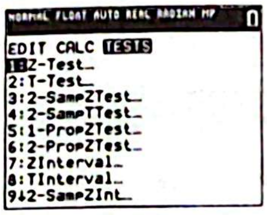
If X is Normally Distributed	If X is NOT given to be Normally Distributed
Under H_0, $\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$ $Z = \frac{\bar{X}-\mu_0}{\sqrt{\sigma^2/n}} \sim N(0,1)$	Under H_0, Since n is large, i.e. $n \geq 30$, $\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$ approx. by CLT $Z = \frac{\bar{X}-\mu_0}{\sqrt{\sigma^2/n}} \sim N(0,1)$

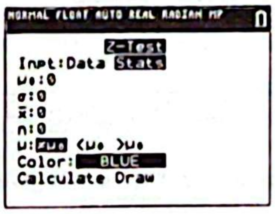
Note: The value of σ^2 will be replaced by s^2 calculated earlier if σ^2 is not given in question.

Step ⑤: Decide on doing *p*-value or critical value.

No unknowns:	use either *p*-value or critical value method
Unknown μ_0, $\bar{x}$, n, σ^2 or s^2:	use critical value method
Unknown α :	use *p*-value method

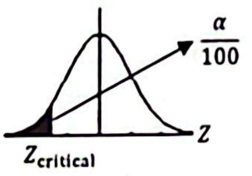
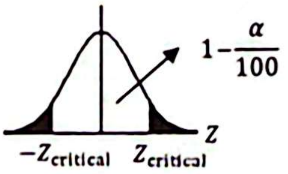
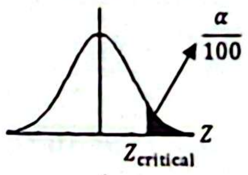
Method ①: *p*-value


NORMAL FLOAT AUTO REAL RADIAN MP
EDIT CALC TESTS
1:Z-Test…
2:T-Test…
3:2-SampZTest…
4:2-SampTTest…
5:1-PropZTest…
6:2-PropZTest…
7:ZInterval…
8:TInterval…
9↓2-SampZInt…


NORMAL FLOAT AUTO REAL RADIAN MP
Z-Test
Inpt:Data Stats
μ₀:0
σ:0
x̄:0
n:0
μ:≠μ₀ <μ₀ >μ₀
Color: BLUE
Calculate Draw

To reject H_0: $p\text{-value} \leq \frac{\alpha}{100}$

Method ②: Critical Value

$H_1: \mu < \mu_0$	$H_1: \mu \neq \mu_0$	$H_1: \mu > \mu_0$
		
$z_{critical} = \text{invNorm}\left(\frac{\alpha}{100}, 0, 1, \text{left}\right)$	$z_{critical} = \text{invNorm}\left(1-\frac{\alpha}{100}, 0, 1, \text{centre}\right)$	$z_{critical} = \text{invNorm}\left(\frac{\alpha}{100}, 0, 1, \text{right}\right)$
To reject H_0: $z_{test} \leq z_{critical}$	To reject H_0: $\lvert z_{test} \rvert \geq z_{critical}$	To reject H_0: $z_{test} \geq z_{critical}$

Step ⑥: Conclusion

p-value

Since $p\text{-value} \leq \frac{\alpha}{100}$, we reject H_0 and conclude that there is sufficient evidence at $\alpha\%$ level of significance to claim that <H_1 in context of the question>.

OR

Since $p\text{-value} > \frac{\alpha}{100}$, we do not reject H_0 and conclude that there is insufficient evidence at $\alpha\%$ level of significance to claim that <H_1 in context of the question>.

Critical value

Since <test statistics lies in rejection region>, we reject H_0 and conclude that we have sufficient evidence at $\alpha\%$ level of significance to claim that <H_1 in context of the question>.

OR

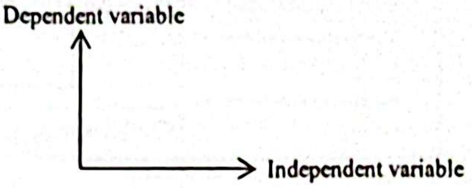
Since <test statistics does not lies in rejection region>, we do not reject H_0 and conclude that there is insufficient evidence at $\alpha\%$ level of significance to claim that <H_1 in context of the question>.

Correlation and Linear Regression

Scatter Diagram

- **A scatter diagram** can be used to represent a set of bivariate data.
- **A bivariate data** is a set that consists of 2 variables (independent and dependent) in ordered pairs.
- **Independent variable** provides the basis for estimation. It can be manipulated or controlled by the researcher.
- Dependent variable is to be estimated or predicted. It is observed for changes in order to assess the effect of manipulating the independent variable.
- **General pattern of a scatter diagram**

Relationship	Description
Linear	① Positive/ Negative ② Strong/ Weak
Non-linear	
No clear relationship	



- Scatter diagram gives a visual evidence of possible extreme data points known as **outliers**.
- The inclusion or exclusion of **outliers** affect the value of correlation coefficient.

Product Moment Correlation Coefficient, r

$$r=\frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\left\{\sum(x-\bar{x})^2\right\}\left\{\sum(y-\bar{y})^2\right\}}}=\frac{\sum xy-\frac{\sum x\sum y}{n}}{\sqrt{\left(\sum x^2-\frac{\left(\sum x\right)^2}{n}\right)\left(\sum y^2-\frac{\left(\sum y\right)^2}{n}\right)}}$$

[In MF27]

- r is a numerical measure of the linear relationship between two variable, x and y.
- r can only take values between -1 and 1 (inclusive), i.e. $-1 \le r \le 1$.

r	Implications	Remarks
-1	Perfect negative linear correlation between y and x.	All data points lie on a straight line with a negative gradient.
<0	A negative linear relationship between y and x.	y decreases as x increases.
0	No linear relationship between y and x.	Non-linear relationship or no clear relationship.
>0	A positive linear relationship between y and x.	y increases as x increases.
$+1$	Perfect positive linear correlation between y and x.	All data points lie on a straight line with a positive gradient.

- **A strong linear correlation between two variables does not necessarily imply that one variable directly causes the other.**
- Correlation should not be used to explain that the independent variable affects the dependent variable, i.e. **there is no causal relationship between the two variables.**

Linear Regression

- Linear regression attempts to **model the relationship between two variables by fitting a linear equation to a set of observed data.**
- One variable is considered to be under control and it is called the independent variable while the other is considered a dependent variable.

Regression line of ...	***y* on *x***	***x* on *y***
Equation	$\boxed{y=a+bx}$ where x is the independent variable and y is the dependent variable	$\boxed{x=c+dy}$ where y is the independent variable and x is the dependent variable
	$\boxed{y-\bar{y}=b(x-\bar{x})}$	$\boxed{x-\bar{x}=d(y-\bar{y})}$
	Both regression lines will pass through $(\bar{x}, \bar{y})$.	
Regression coefficients	$a=\bar{y}-b\bar{x}$ $b=\frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$ [In MF27]	$c=\bar{x}-d\bar{y}$ $d=\frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2}$

- **Using appropriate regression lines**

Cases	**Estimate value of y for a given value of x**	**Estimate value of x for a given value of y**
x is taken to be the independent variable	Use regression line y on x	
y is taken to be the independent variable	Use regression line x on y	
Neither x nor y can be identified as the independent variable	Use regression line y on x	Use regression line x on y

- **Interpolation** is making a prediction **within the range of values** of the predictor in the sample used to generate the model. It is **generally reliable**.
- **Extrapolation** is making a prediction **outside the range of values** of the predictor in the sample used to generate the model. It is **not reliable**.

Question ①: Is the linear model appropriate?

Possible solution:

- Context of question – A linear model is not appropriate since time cannot be negative.
- Scatter diagram – As x increases, y increases at a decreasing rate, therefore a linear model is not appropriate.
- r – A linear model is appropriate for the given data range since $|r|$ is close to 1.

Question ②: Is the estimation reliable?

Possible solution:

- Reliable – Since r is close to 1 (or -1) and the estimation was carried out using interpolation, the estimate is reliable.
- Not reliable – Since r is close to 0, the estimate is not reliable.
  – Model is valid only in the given data range, extrapolation cannot be done.

Question ③: Is the claim valid?

Possible solution: