POST PRELIM PRACTICE – A LEVEL SOLUTIONS (STANDARD)

1. Graphing Techniques

(Q1) [Cambridge/9794/2007]

The equation of a circle is $x^2 + y^2 - 8y = 9$. Find the coordinates of the centre of the circle, and the radius of the circle. [3]

A straight line has equation $x = 3y + k$, where k is a constant. Show that the y-coordinates of the points of intersection of the line and the circle are given by

$$10y^2 + (6k - 8)y + (k^2 - 9) = 0. \quad [2]$$

Hence determine the exact values of k for which the line is a tangent to the circle. [4]

$x^2+y^2-8y=9$ — ①

$(x-0)^2+(y^2-8y+16)=16+9=25$

$(x-0)^2+(y-4)^2=5^2$

Coord. of centre of circle : $(0,4)$

Radius of circle : 5 units

$x=3y+k$ — ②

Subst. ② in ①: $(3y+k)^2+y^2-8y=9$

$9y^2+6yk+k^2+y^2-8y-9=0$

$10y^2+(6k-8)y+(k^2-9)=0$ — ③

For the line to be a tangent to the circle,

Discriminant for ③, $D=0$

$(6k-8)^2-4(10)(k^2-9)=0$

$36k^2-96k+64-40k^2+360=0$

$-4k^2-96k+424=0$

$k^2+24k-106=0$

$k=\dfrac{-24\pm\sqrt{24^2+4(106)}}{2}$

$=\dfrac{-24\pm\sqrt{1000}}{2}$

$=-12\pm5\sqrt{10}$

(Q2) [Cambridge/9231/2007]

The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + 4},$$

where a, b and c are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Given also that C has a turning point at $x = -1$, find the value of c. [3]

(iii) Find the set of values of y for which there are no points on C. [4]

(iv) Draw a sketch of the curve with equation

$$y = \frac{2(x-7)^2 + 3(x-7) - 2}{x - 3}.$$ [3]

[You should state the equations of the asymptotes and the coordinates of the turning points.]

(i) $y = 2x - 5 + \frac{A}{x+4}$

$= \frac{(2x-5)(x+4) + A}{x+4}$

$= \frac{2x^2 + 3x + (A-20)}{x+4}$

$\therefore\ a = 2,\ b = 3,\ c = A - 20$

(ii) $y = \frac{2x^2+3x+c}{x+4}$

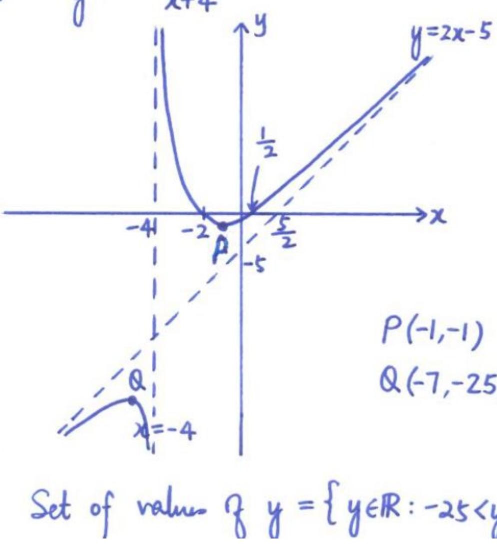
$\frac{dy}{dx} = \frac{(x+4)(4x+3) - (2x^2+3x+c)}{(x+4)^2}$

At $x=-1$, $\frac{dy}{dx} = 0$

$\frac{(3)(-1) - (2-3+c)}{3^2} = 0$

$c = -2$

(iii) $y = \frac{2x^2+3x-2}{x+4}$



$P(-1,-1)$

$Q(-7,-25)$

Set of values of $y = \{y \in \mathbb{R} : -25 < y < -1\}$

(iii) Analytical method (for reference)

$y = \frac{2x^2+3x-2}{x+4}$

$xy + 4y = 2x^2 + 3x - 2$

$2x^2 + (3-y)x - 2 - 4y = 0$

For real x, $(3-y)^2 - 4(2)(-2-4y) \geq 0$

$9 - 6y + y^2 + 16 + 32y \geq 0$

$y^2 + 26y + 25 \geq 0$

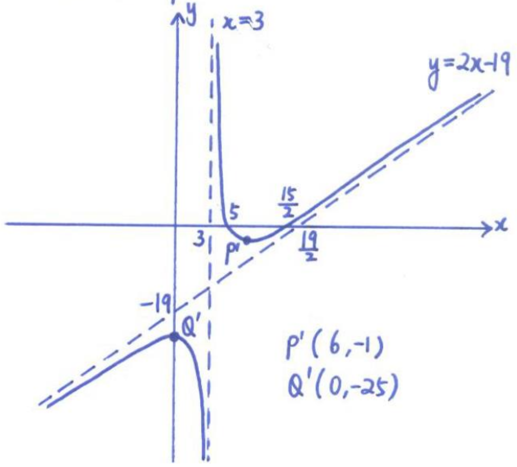
$(y+25)(y+1) \geq 0$

$y \geq -1$ OR $y \leq -25$

$\therefore$ Set of values of y for which there are no pts on C

$= \{y \in \mathbb{R} : -25 < y < -1\}$

(iv) Translate $y=\frac{2x^2+3x-2}{x+4}$ 7 units in the positive x-direction.



(Q3) [Cambridge/H2 Maths 2007]

Show that the equation $y=\dfrac{2x+7}{x+2}$ can be written as $y=A+\dfrac{B}{x+2}$, where A and B are constants to be found. Hence state a sequence of transformations which transform the graph of $y=\dfrac{1}{x}$ to the graph of $y=\dfrac{2x+7}{x+2}$. [4]

Sketch the graph of $y=\dfrac{2x+7}{x+2}$, giving the equations of any asymptotes and the coordinates of any points of intersection with the x- and y-axes. [3]

$$y=\frac{2x+7}{x+2}$$
$$=\frac{2(x+2)+3}{x+2}$$
$$=2+\frac{3}{x+2}$$

$\therefore$ $A=2$, $B=3$

$y=\frac{1}{x}$

$\downarrow$ Translation by 2 units in the negative x-direction

$y=\frac{1}{x+2}$

$\downarrow$ Scaling // y-axis by a scale factor 3

$\frac{y}{3}=\frac{1}{x+2}$

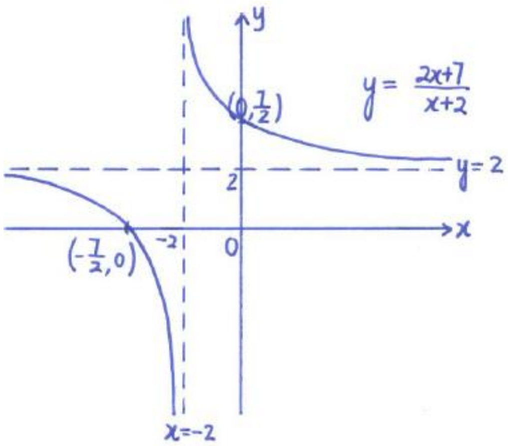
$y=\frac{3}{x+2}$

$\downarrow$ Translation by 2 units in the positive y-direction

$y-2=\frac{3}{x+2}$

$y=2+\frac{3}{x+2}$

$=\frac{2x+7}{x+2}$



(Q4)

A curve whose equation is $y = \dfrac{1}{x+1}$, undergoes in succession the following transformations:

A: A reflection in the x-axis,
B: A translation of magnitude 1 unit in the y-direction,
C: A scaling parallel to the y-axis by a factor 2.

Give the equation of the resulting curve.

A curve undergoes in succession the transformations **A**, **B** and **C** as above and the equation of the resulting curve is $y = 2 - |2x+6|$. Determine the equation of the curve before the transformations were effected.

$y = \frac{1}{x+1}$

$\downarrow$ Replace y by $-y$

$-y = \frac{1}{x+1}$

$y = -\frac{1}{x+1}$

$\downarrow$ Replace y by $y-1$

$y-1 = -\frac{1}{x+1}$

$y = 1 - \frac{1}{x+1}$

$\downarrow$ Replace y by $\frac{y}{2}$

$\frac{y}{2} = 1 - \frac{1}{x+1}$

$y = 2\left(1 - \frac{1}{x+1}\right)$

$= \frac{2x}{x+1}$

$y = 2 - |2x+6|$

$\downarrow$ Replace y by $\frac{y}{\left(\frac{1}{2}\right)}$

$\frac{y}{\left(\frac{1}{2}\right)} = 2 - 2|x+3|$

$y = 1 - |x+3|$

$\downarrow$ Replace y by $y+1$

$y+1 = 1 - |x+3|$

$y = -|x+3|$

$\downarrow$ Replace y by $-y$

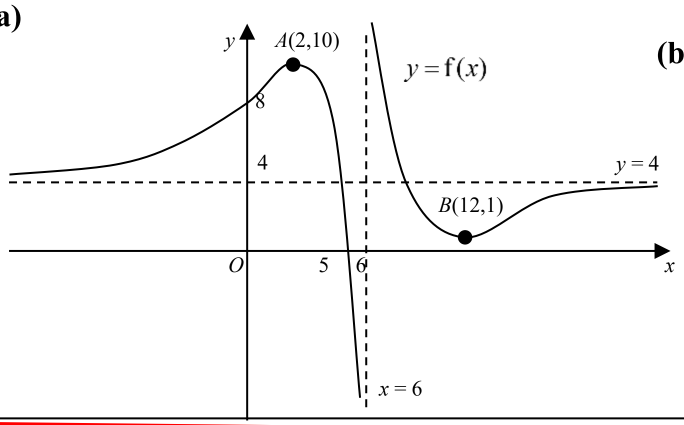
$-y = -|x+3|$

$y = |x+3|$

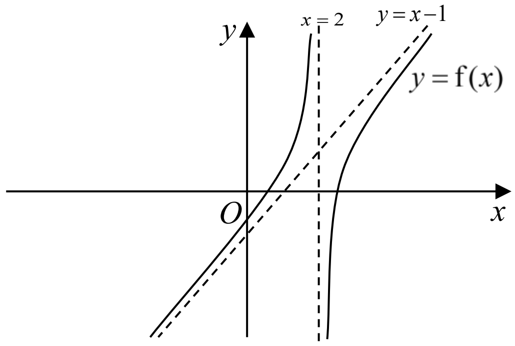
(Q5)

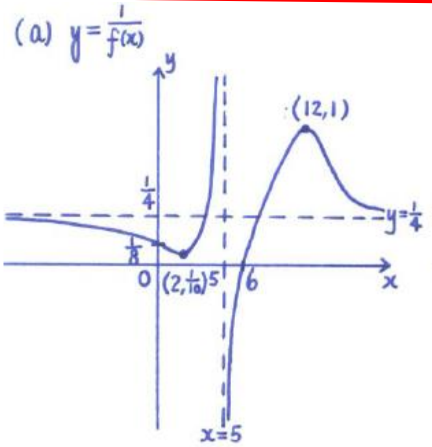
Sketch the graphs of $y=\dfrac{1}{\mathrm{f}(x)}$ (for (a) only), $y=\mathrm{f}'(x)$, $y=|\mathrm{f}(x)|$, $y=\mathrm{f}(|x|)$..

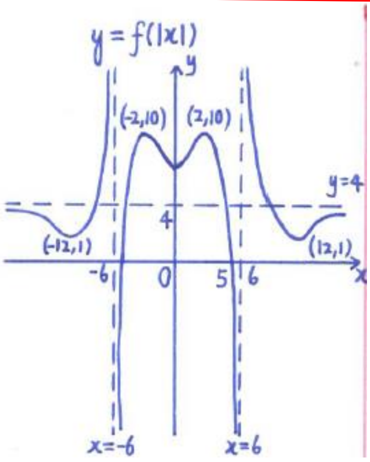
(a)

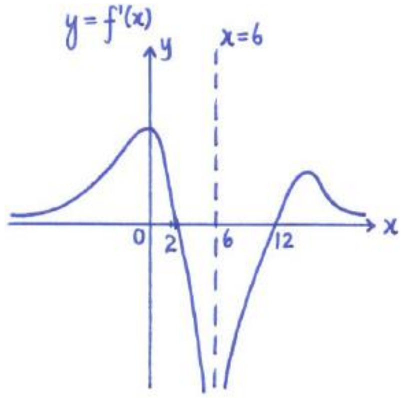


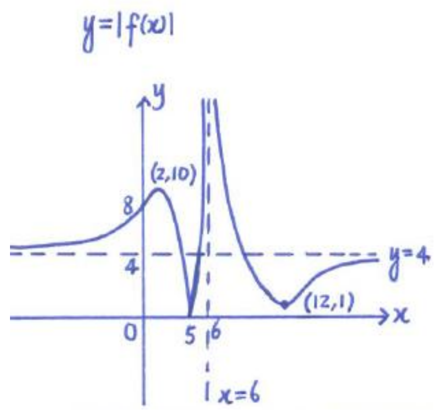
(b)

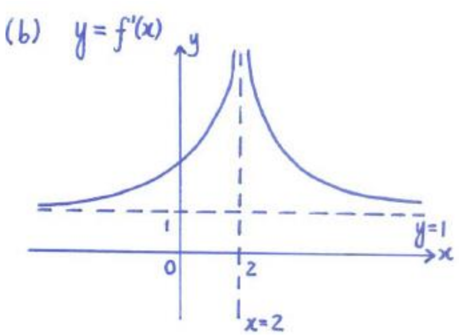


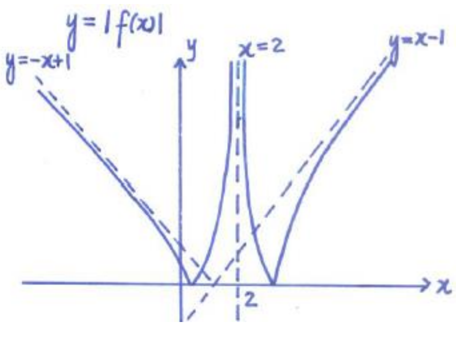


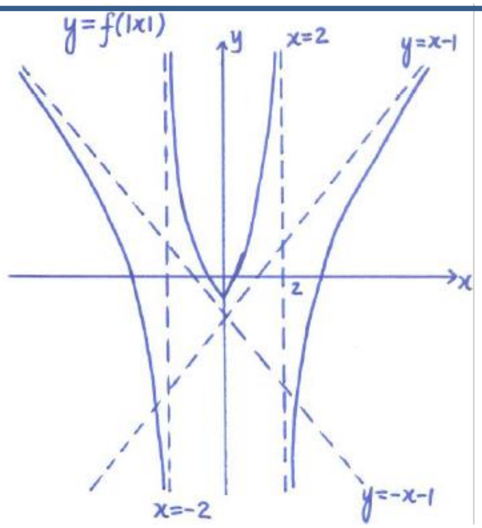












2. Functions

(Q1) [Cambridge/9709/2007]

The function f is defined by $f : x \mapsto 2x^2 - 8x + 11$ for $x \in \mathbb{R}$.

(i) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

(ii) State the range of f. [1]

(iii) Explain why f does not have an inverse. [1]

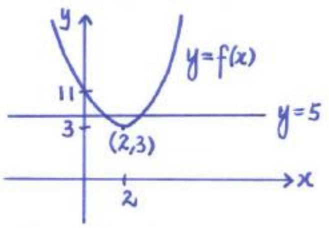
The function g is defined by $g : x \mapsto 2x^2 - 8x + 11$ for $x \leqslant A$, where A is a constant.

(iv) State the largest value of A for which g has an inverse. [1]

(v) When A has this value, obtain an expression, in terms of x, for $g^{-1}(x)$ and state the range of g^{-1}. [4]

(i)
$$f(x) = 2x^2 - 8x + 11$$
$$= 2(x^2 - 4x + 4 - 4) + 11$$
$$= 2(x-2)^2 + 3$$

(ii)



$R_f = [3, \infty)$

(iii) The horizontal line $y=5$ cuts the graph of f at 2 distinct pts. Thus f is not one-one and f^{-1} does not exist.

(iv) Largest value of $A = 2$

(v) $g : x \mapsto 2x^2 - 8x + 11, \ x \leqslant 2$

Let $y = 2(x-2)^2 + 3$

$$(x-2)^2 = \frac{y-3}{2}$$
$$x - 2 = \pm\sqrt{\frac{y-3}{2}}$$
$$x = 2 \pm \sqrt{\frac{y-3}{2}}$$

Since $x \leqslant 2$, $x = 2 - \sqrt{\frac{y-3}{2}}$

$$\therefore \ g^{-1}(x) = 2 - \sqrt{\frac{x-3}{2}}$$

$R_{g^{-1}} = D_g = (-\infty, 2]$

(Q2) [Cambridge/9794/Specimen Paper for 2010]

The function f is defined by

$$\mathrm{f} : x \mapsto x^3 - 1, \quad x \in \mathbb{R}.$$

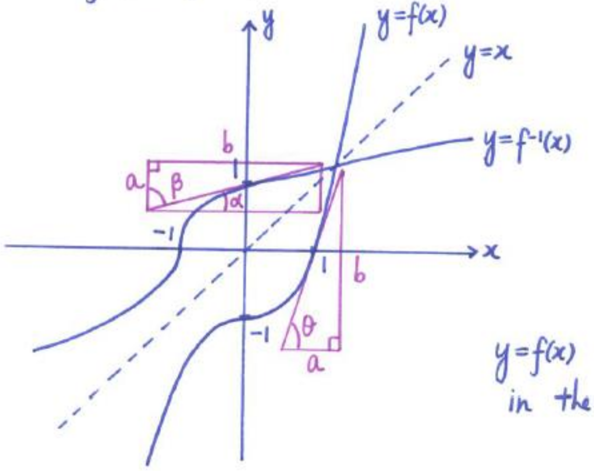
(i) Find an expression for $\mathrm{f}^{-1}(x)$. [2]

(ii) Sketch on a single diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, making clear the relationship between these graphs. [2]

(iii) Write down $\mathrm{f}'(x)$ and hence determine the gradient of $y = \mathrm{f}^{-1}(x)$ at the point where it crosses the y-axis. [3]

(i) Let $y = f(x)$
$= x^3 - 1$
$x^3 = y + 1$
$x = (y+1)^{\frac{1}{3}}$
$\therefore f^{-1}(x) = (x+1)^{\frac{1}{3}}$

(ii)



$y=f(x)$ is a reflection of $y=f^{-1}(x)$ in the line $y=x$.

(iii) $f'(x) = 3x^2$

Where the $y=f^{-1}(x)$ graph crosses the y-axis, $x=0$, $y=1$

Reflected pt of $(0,1)$ in $y=x$ is $(1,0)$ on $y=f(x)$.

Gradient of f at $(1,0) = f'(1) = 3 = \tan\theta = \frac{b}{a}$

By symmetry, $\tan\beta = \tan\theta = \frac{b}{a} = 3$

Gradient of $y=f^{-1}(x)$ at $(0,1) = \tan\alpha = \frac{a}{b} = \frac{1}{3}$

(Q3) [Cambridge/H2 Maths 2007]

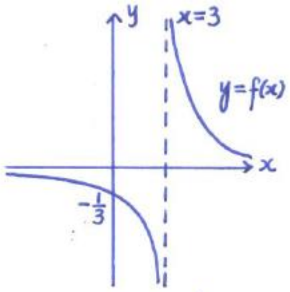
Functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{1}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3,$$
$$\mathrm{g} : x \mapsto x^2 \quad \text{for } x \in \mathbb{R}.$$

(i) Only one of the composite functions fg and gf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist. [3]

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(i)



$R_f = \mathbb{R} \setminus \{0\}$

$R_g = [0, \infty)$

Since $R_f \subseteq D_g$, gf exists.

Since $R_g \nsubseteq D_f$, fg does not exist.

$gf : x \mapsto \frac{1}{(x-3)^2}, \quad x \in \mathbb{R}, x \neq 3$

(ii) Let $y = f(x)$

$= \frac{1}{x-3}$

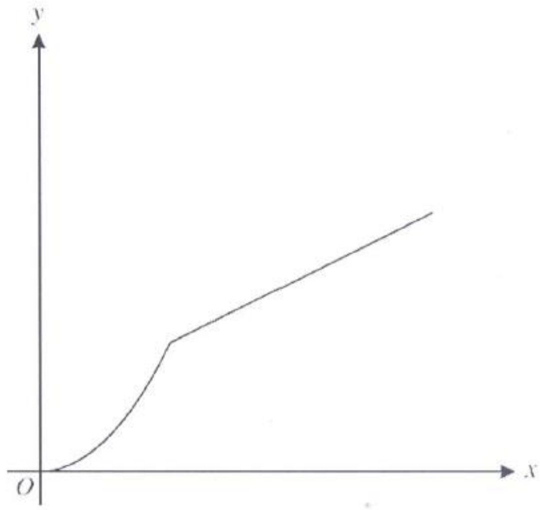
$\frac{1}{y} = x - 3$

$x = 3 + \frac{1}{y}$

$f^{-1}(x) = 3 + \frac{1}{x}$

$D_{f^{-1}} = R_f = \mathbb{R} \setminus \{0\}$

(Q4) [Cambridge/9709/2010]



The diagram shows the function f defined for $0 \leqslant x \leqslant 6$ by

$$x \mapsto \tfrac{1}{2}x^2 \quad \text{for } 0 \leqslant x \leqslant 2,$$
$$x \mapsto \tfrac{1}{2}x + 1 \quad \text{for } 2 < x \leqslant 6.$$

(i) State the range of f. [1]

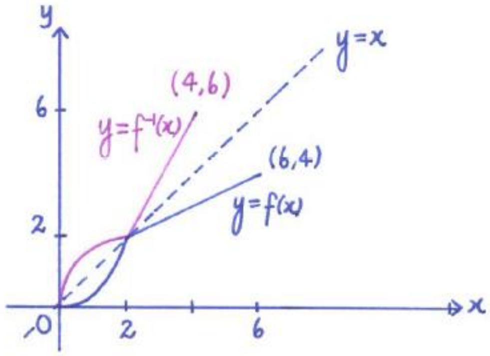
(ii) Copy the diagram and on your copy sketch the graph of $y = \mathrm{f}^{-1}(x)$. [2]

(iii) Obtain expressions to define $\mathrm{f}^{-1}(x)$, giving the set of values of x for which each expression is valid. [4]

(i) When $x=6$, $y = \frac{1}{2}(6)+1 = 4$

$\therefore R_f = [0,4]$

(ii)



(iii) Let $y=\frac{1}{2}x^2$, $0 \leqslant x \leqslant 2$

$x = \pm\sqrt{2y}$

Since $x \geqslant 0$, $x=\sqrt{2y}$

Let $y=\frac{1}{2}x+1$, $2<x\leqslant 6$

$x = 2(y-1)$

$$\therefore f^{-1}: x \mapsto \begin{cases} \sqrt{2x}, & 0 \leqslant x \leqslant 2 \\ 2(x-1), & 2 < x \leqslant 4 \end{cases}$$

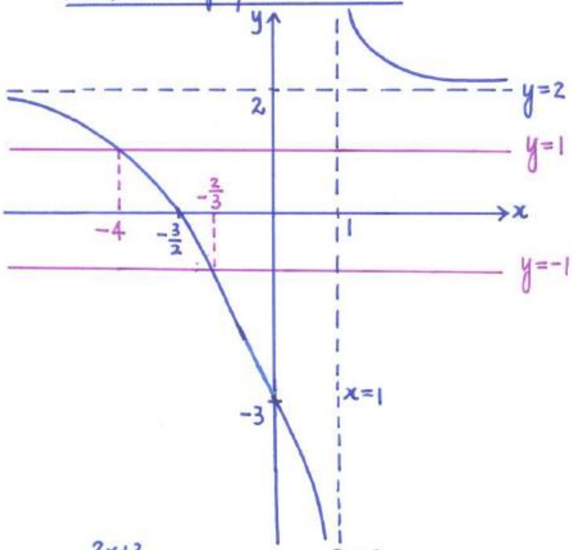
3. Inequalities

(Q1) [Cambridge/2002 modified]

Find the set of values of x such that $-1<\frac{2x+3}{x-1}<1$.

Deduce the range of values of x such that $-1<\frac{2x+1}{x-2}<1$.

Method 1 : Graphical method



$\frac{2x+3}{x-1}=-1$

$2x+3=-x+1$

$3x=-2$

$x=-\frac{2}{3}$

$\frac{2x+3}{x-1}=1$

$2x+3=x-1$

$x=-4$

$\therefore$ Set of values of $x=\{x\in\mathbb{R}:-4<x<-\frac{2}{3}\}$

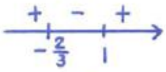
Method 2 : Analytical method

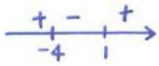
$-1<\frac{2x+3}{x-1}<1$

$-1<\frac{2x+3}{x-1}$ and $\frac{2x+3}{x-1}<1$

$\frac{2x+3+x-1}{x-1}>0$ and $\frac{2x+3-(x-1)}{x-1}<0$

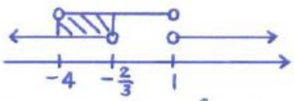
$\frac{3x+2}{x-1}>0$ and $\frac{x+4}{x-1}<0$





$\therefore x<-\frac{2}{3}$ or $x>1$

$-4<x<1$



$\therefore$ Set of values of $x=\{x\in\mathbb{R}:-4<x<-\frac{2}{3}\}$

Replace x by $x-1$:

$-1<\frac{2(x-1)+3}{(x-1)-1}<1$

$-1<\frac{2x+1}{x-2}<1$

$\therefore -4<x-1<-\frac{2}{3}$

$-3<x<\frac{1}{3}$

(Q2) [Cambridge/9709/2007]

(i)	Solve the inequality $|y-5|<1$.	[2]
(ii)	Hence solve the inequality $|3^x-5|<1$, giving 3 significant figures in your answer.	[3]

(i) $|y-5|<1$

$-1<y-5<1$

$4<y<6$

(ii) Replace y by 3^x

$4<3^x<6$

$\ln 4<x\ln 3<\ln 6$

$\frac{\ln 4}{\ln 3}<x<\frac{\ln 6}{\ln 3}$

$1.26<x<1.63$ (3sf)

(Q3 [Cambridge/H2 Maths 2007]

Show that $\dfrac{2x^2 - x - 19}{x^2 + 3x + 2} - 1 = \dfrac{x^2 - 4x - 21}{x^2 + 3x + 2}$. [1]

Hence, without using a calculator, solve the inequality

$$\frac{2x^2 - x - 19}{x^2 + 3x + 2} > 1. \qquad [4]$$

$$\frac{2x^2-x-19}{x^2+3x+2} - 1 = \frac{2x^2-x-19-(x^2+3x+2)}{x^2+3x+2}$$

$$= \frac{x^2-4x-21}{x^2+3x+2}$$

$$\frac{2x^2-x-19}{x^2+3x+2} > 1$$

$$\frac{2x^2-x-19}{x^2+3x+2} - 1 > 0$$

$$\frac{x^2-4x-21}{x^2+3x+2} > 0$$

$$\frac{(x-7)(x+3)}{(x+2)(x+1)} > 0$$

$$\underset{-3}{+\ |}\ -\ \underset{-2}{|}\ +\ \underset{-1}{|}\ -\ \underset{7}{|}\ +\ \rightarrow x$$

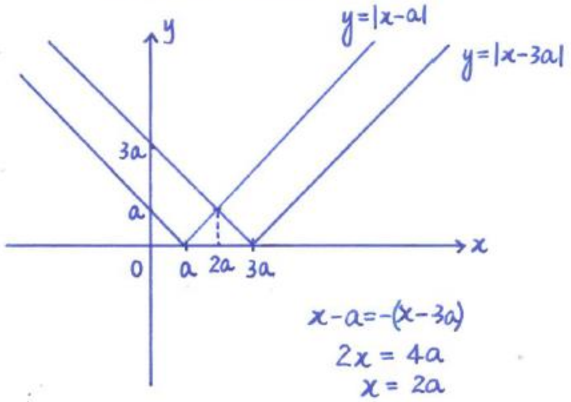
$\therefore\quad x < -3 \quad \text{OR} \quad -2 < x < -1 \quad \text{OR} \quad x > 7$

(Q4) [Cambridge/9709/2005]

Given that a is a positive constant, solve the inequality

$$|x - 3a| > |x - a|. \qquad [4]$$

Graphical method



$$x - a = -(x - 3a)$$
$$2x = 4a$$
$$x = 2a$$

$\therefore\quad x < 2a.$

4. System of Linear Equations

(Q1) [Cambridge/H2 Maths 2007]

Four friends buy three different kinds of fruit in the market. When they get home they cannot remember the individual prices per kilogram, but three of them can remember the total amount that they each paid. The weights of fruit and the total amounts paid are shown in the following table.

	Suresh	Fandi	Cindy	Lee Lian
Pineapples (kg)	1.15	1.20	2.15	1.30
Mangoes (kg)	0.60	0.45	0.90	0.25
Lychees (kg)	0.55	0.30	0.65	0.50
Total amount paid in $	8.28	6.84	13.05	

Assuming that, for each variety of fruit, the price per kilogram paid by each of the friends is the same, calculate the total amount that Lee Lian paid. [6]

Let P, M, L be the price per kg of pineapples, mangoes and lychees resp.

$$1.15P + 0.60M + 0.55L = 8.28$$
$$1.20P + 0.45M + 0.30L = 6.84$$
$$2.15P + 0.90M + 0.65L = 13.05$$

From GC,

$\therefore$ $P = \$3.50$
$M = \$2.60$
$L = \$4.90$

Total amt Lee Lian paid

$= 1.30(3.50) + 0.25(2.60) + 0.50(4.90)$

$= \$7.65$

(Q2)

A curve has equation $y=\sqrt{ax}+(bx)^2+cx^3+d$ where a, b, c and d are constants. The curve cuts the x-axis at $x=4$ and has a turning point at $(1,\ 127)$. The line $y=-193x+4$ intersects the curve at $x=9$. Find the possible values of a, b, c and d. [6]

$y=\sqrt{ax}+(bx)^2+cx^3+d$

① Curve cuts x-axis at $x=4$

$\therefore$ $(4,0)$ lies on the curve

$0=2\sqrt{a}+16b^2+64c+d$

② Turning pt at $(1,127)$

$\therefore$ $(1,127)$ lies on the curve

$127=\sqrt{a}+b^2+c+d$

$\frac{dy}{dx}=\frac{1}{2}\sqrt{a}x^{-\frac{1}{2}}+2b^2x+3cx^2$

At turning pt, $\frac{dy}{dx}=0$

$0=\frac{1}{2}\sqrt{a}+2b^2+3c$

③ $y=-193x+4$ intersects curve at $x=9$

$\Rightarrow$ $y=-193(9)+4=-1733$

$\therefore$ $(9,-1733)$ lies on the curve

$-1733=3\sqrt{a}+81b^2+729c+d$

From GC,

$\therefore$ $\sqrt{a}=2 \Rightarrow a=4$

$b^2=4 \Rightarrow b=\pm 2$

$c=-3$

$d=124$

5. Maclaurin and Binomial Series

(Q1) [Cambridge/2000]

(i) Expand $\dfrac{1-x^2}{\sqrt[3]{1+3x}}$ in ascending powers of x up to and including the term in x^3.

(ii) State the set of values of x for which the series expansion is valid.

(iii) Write down the equation of the tangent at the point (0,1) on the curve

$$y=\frac{1-x^2}{\sqrt[3]{1+3x}}.$$

(i)

$$\begin{aligned}\frac{1-x^2}{\sqrt[3]{1+3x}} &= (1-x^2)(1+3x)^{-\frac{1}{3}}\\ &= (1-x^2)\left[1+(-\tfrac{1}{3})(3x)+\frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}(3x)^2+\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!}(3x)^3+\cdots\right]\\ &= (1-x^2)\left[1-x+2x^2-\tfrac{14}{3}x^3+\cdots\right]\\ &= 1-x+2x^2-\tfrac{14}{3}x^3-x^2+x^3+\cdots\\ &= 1-x+x^2-\tfrac{11}{3}x^3+\cdots\end{aligned}$$

(ii) Expansion is valid for

$$|3x|<1$$
$$|x|<\tfrac{1}{3}$$
$$-\tfrac{1}{3}<x<\tfrac{1}{3}$$

$\therefore \{x\in\mathbb{R} : -\tfrac{1}{3}<x<\tfrac{1}{3}\}$

(iii) Eqn of tangent at $(0,1)$ is $y=1-x$.

(Q2) [Cambridge/2004]

Find the first two terms in the expansion of $\dfrac{(9+x)^{\frac{1}{2}}}{1+2x}$ in ascending powers of x.

$$\begin{aligned}\frac{(9+x)^{\frac{1}{2}}}{1+2x} &= \left[9(1+\tfrac{x}{9})\right]^{\frac{1}{2}}(1+2x)^{-1}\\ &= 3(1+\tfrac{x}{9})^{\frac{1}{2}}(1+2x)^{-1}\\ &= 3\left[1+(\tfrac{1}{2})(\tfrac{x}{9})+\cdots\right]\left[1+(-1)(2x)+\cdots\right]\\ &= 3\left[1+\tfrac{x}{18}+\cdots\right]\left[1-2x+\cdots\right]\\ &= 3\left(1-2x+\tfrac{x}{18}-\cdots\right)\\ &= 3-\tfrac{35}{6}x+\cdots\end{aligned}$$

(Q3) [Cambridge/9231/2007]

Find the first negative coefficient in the expansion of $(4+3x)^{\frac{5}{2}}$ in a series of ascending powers of x, where $|x| < \frac{4}{3}$. Give your answer as a fraction in its lowest terms. [3]

$$(4+3x)^{\frac{5}{2}} = \left[4\left(1+\frac{3x}{4}\right)\right]^{\frac{5}{2}}$$

$$= 4^{\frac{5}{2}}\left(1+\frac{3x}{4}\right)^{\frac{5}{2}}$$

$$= 32\left[1+\left(\frac{5}{2}\right)\left(\frac{3x}{4}\right)+\frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)}{2!}\left(\frac{3x}{4}\right)^2+\frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{3!}\left(\frac{3x}{4}\right)^3 + \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{4!}\left(\frac{3x}{4}\right)^4+\ldots\right]$$

First negative coeff. $= 32\left[\frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{4!}\left(\frac{3}{4}\right)^4\right]$

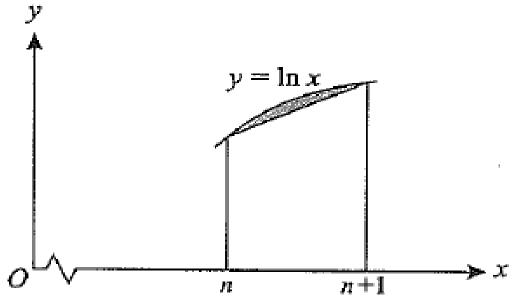
$= -\frac{405}{1024}$

(Q4) [2007 Specimen Paper]

(i) Show that

$$\int_n^{n+1} \ln x \, dx = (n+1)\ln(n+1) - n\ln n - 1. \qquad [4]$$

(ii)



The diagram shows the graph of $y = \ln x$ between $x = n$ and $x = n+1$. The area of the shaded region represents the error when the value of the integral in part (i) is approximated by using a single trapezium. Show that the area of the shaded region is

$$\left(n+\tfrac{1}{2}\right)\boxed{\ln\left(1+\frac{1}{n}\right)} - 1. \qquad [4]$$

(iii) Use a series expansion to show that if n is large enough for $\frac{1}{n^3}$ and higher powers of $\frac{1}{n}$ to be neglected, then the area in part (ii) is approximately equal to $\frac{k}{n^2}$, where k is a constant to be determined. [3]

(i) Integration by parts indicated with $u = \ln x$, $dv = 1$

$$x \ln x - \int x \times \frac{1}{x}\, dx$$

$x \ln x - x\ (+c)$

Limits used correctly producing $(n+1)\ln(n+1) - n\ln n - 1$ (AG)

(ii) Area = (i) – area of trapezium

Area of trapezium $= \frac{1}{2}[\ln n + \ln(n+1)]$

$\left(n+\frac{1}{2}\right)\ln(n+1) - \left(n+\frac{1}{2}\right)\ln n - 1$ or $\left(n+\frac{1}{2}\right)[\ln(n+1) - \ln n] - 1$

$\left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1 = \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) - 1$ (AG)

[N.B. Must be sufficient indication to obtain the AG]

(iii) Use $\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots$

part (ii) $= 1 - \frac{1}{2n} + \frac{1}{3n^2} + \frac{1}{2n} - \frac{1}{4n^2} - 1$

$= \frac{1}{12n^2}$ ('k' need not be stated)

(Q5) [Cambridge/9795/Specimen Paper]

Given that $y = \cos\{\ln(1+x)\}$, prove that

(i) $(1+x)\frac{dy}{dx} = -\sin\{\ln(1+x)\}$, [1]

(ii) $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 0.$ [2]

Obtain an equation relating $\frac{d^3y}{dx^3}$, $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$. [2]

Hence find Maclaurin's series for y, up to and including the term in x^3. [4]

Verify that the same result is obtained if the standard series expansions for $\ln(1+x)$ and $\cos x$ are used. [3]

$$y=\cos[\ln(1+x)]$$

(i) $$\frac{dy}{dx}=-\sin[\ln(1+x)]\,\frac{1}{1+x}$$

$$(1+x)\frac{dy}{dx}=-\sin[\ln(1+x)]$$

(ii) $$(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}=-\cos[\ln(1+x)]\,\frac{1}{1+x}$$

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}=-y$$

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=0$$

$$(1+x)^2\frac{d^3y}{dx^3}+2(1+x)\frac{d^2y}{dx^2}+(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}=0$$

$$(1+x)^2\frac{d^3y}{dx^3}+3(1+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

When $x=0$, $y=\cos[\ln(1)]=1$

$$\frac{dy}{dx}=0$$

$$\frac{d^2y}{dx^2}=-1$$

$$\frac{d^3y}{dx^3}=3$$

$$\therefore\quad y=1-\frac{1}{2}x^2+\frac{3}{3!}x^3+\cdots=1-\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

$$\cos x=1-\frac{x^2}{2!}+\cdots$$

$$\begin{aligned}\cos[\ln(1+x)]&=\cos\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right]\\&=1-\frac{\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right)^2}{2}+\cdots\\&=1-\frac{1}{2}(x^2-x^3)+\cdots\\&=1-\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots\quad\text{(verified)}\end{aligned}$$

(Q6) [Cambridge/9233/2004]

The equation $x\sin x+\cos x=1.015$ has a positive root α close to zero. Use small-angle approximations for sin x and cos x to obtain an approximation to α. Give your answer correct to 3 significant figures. [3]

$x\sin x+\cos x=1.015$

Since α is a root and close to zero, $\alpha.\alpha+\left(1-\dfrac{\alpha^2}{2}\right)\approx 1.015$

$\Rightarrow \dfrac{1}{2}\alpha^2\approx 0.015$

$\Rightarrow \alpha\approx+\sqrt{2(0.015)}=0.173$ (to 3 s. f.)

(Q7) [Cambridge/Nov 1983/I/3]

Use the binomial series to expand $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$ as a series of ascending powers of x up to and including the terms in x^2, where $|x| < 1$.

By putting $x = \dfrac{1}{10}$ in your result, show that $\sqrt{11} \approx \dfrac{663}{200}$.

$$\sqrt{\left(\frac{1+x}{1-x}\right)}$$
$$= (1+x)^{1/2}(1-x)^{-1/2}$$
$$\approx \left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)\left(1+\frac{1}{2}x+\frac{3}{8}x^2\right)$$
$$\approx 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2$$
$$= 1+x+\frac{1}{2}x^2$$

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1+x+\frac{1}{2}x^2$$
$$\sqrt{\left(\frac{\frac{11}{10}}{\frac{9}{10}}\right)} \approx 1+\frac{1}{10}+\frac{1}{200}$$

(substitute $x = \dfrac{1}{10}$ on both sides of eqn)

$$\sqrt{\left(\frac{11}{9}\right)} = \frac{221}{200}$$
$$\sqrt{(11)} = 3\left(\frac{221}{200}\right) = \frac{663}{200}$$

6. Trigonometrical Equations (try these qns without using GC)

(Q1) [Cambridge/Jun 73/II/4]

(a) Find the values of θ between 0° and 180° for which $\tan^2\theta = 5 - \sec\theta$.
(b) Find the values of θ between 0° and 360° which satisfy the equation $6\cos\theta + 7\sin\theta = 4$.

(a) Since $1+\tan^2\theta = \sec^2\theta$,

$\sec^2\theta - 1 = 5 - \sec\theta$

$\sec^2\theta + \sec\theta - 6 = 0$

$(\sec\theta + 3)(\sec\theta - 2) = 0$

$\cos\theta = -\frac{1}{3}$ or $\cos\theta = \frac{1}{2}$

$\theta = 60°$ or $109.5°$

(b) Let $R\sin(\theta+\alpha) = 6\cos\theta + 7\sin\theta$

$\left.\begin{aligned} R\cos\alpha &= 7 \\ R\sin\alpha &= 6 \end{aligned}\right\}$ $R^2 = 7^2+6^2 = 85 \Rightarrow R = \sqrt{85}$

$\tan\alpha = \frac{6}{7} \Rightarrow \alpha = \tan^{-1}\left(\frac{6}{7}\right) = 40.6°$

$\sin(\theta+\alpha) = \frac{4}{\sqrt{85}}$

$\theta = 113.7°$ or $345.1°$

(Q2) [Cambridge/Jun 1980/I/4]

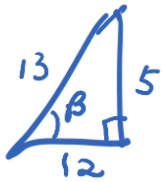
Given that $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{12}{13}$, show that one possible value of $\cos(\alpha+\beta)$ is $\frac{33}{65}$, and find all the other possible values.

$\sin\alpha = \frac{3}{5}$

$\cos\alpha = \frac{4}{5}$

$\cos\beta = \frac{12}{13}$

$\sin\beta = \frac{5}{13}$



If α, β is in the 1st quadrant,

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)$

$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$

If α is in 2nd quadrant, β is in 1st quadrant,

$\cos(\alpha+\beta) = \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) = -\frac{63}{65}$

If α is in 2nd quadrant, β is in 4th quadrant.

$\cos(\alpha+\beta) = \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) = -\frac{33}{65}$

If α is in 1st quadrant, β is in 4th quadrant,

$\cos(\alpha+\beta) = \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) = \frac{63}{65}$

(Q3) [Cambridge/Jun 1984/II/4]

Express $2\cos\theta + 9\sin\theta$ in the form of $R\cos(\theta-\alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving the values of R and α.

Hence

(i) Write down the maximum value of $2\cos\theta + 9\sin\theta$ and the smallest positive value of θ at which this maximum value occurs;

(ii) Find the smallest positive solution of the equation $2\cos\theta + 9\sin\theta = 1$.

$R\cos(\theta-\alpha) = 2\cos\theta + 9\sin\theta$

$R(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = 2\cos\theta + 9\sin\theta$

$\left.\begin{array}{l} R\cos\alpha = 2 \\ R\sin\alpha = 9 \end{array}\right\}$ $R^2 = 2^2 + 9^2 \Rightarrow R = \sqrt{85}$

$\tan\alpha = \frac{9}{2} \Rightarrow \alpha = \tan^{-1}\frac{9}{2} = 1.3521$ radians

(i) Max of $2\cos\theta + 9\sin\theta$ is $\sqrt{85}$ when $\theta = \alpha = 1.3521$ radians

(ii) $\sqrt{85}\cos(\theta-\alpha) = 1 \Rightarrow \theta = 2.81$ radians

7. AP & GP

(Q1) [Cambridge/2002]

The nth term of a series is $2^{n-2}+3n$. Find the sum of the first N terms.

$$\sum_{n=1}^{N}(2^{n-2}+3n) = \sum_{n=1}^{N}\frac{1}{4}\cdot 2^n + \sum_{n=1}^{N}3n$$

$$= \frac{1}{4}\left[\frac{2(2^N-1)}{2-1}\right] + 3\left[\frac{N}{2}(1+N)\right]$$

$$= \frac{1}{2}(2^N-1) + \frac{3N}{2}(N+1)$$

(Q2) [Cambridge/2006]

The sum, S_n, of the first n terms of a geometric progression is given by $S_n = 6-\frac{2}{3^{n-1}}$. Find the first term and the common ratio.

$$S_n = 6 - \frac{2}{3^{n-1}}$$

$$T_n = S_n - S_{n-1}$$
$$= \left[6-\frac{2}{3^{n-1}}\right]-\left[6-\frac{2}{3^{n-2}}\right]$$
$$= \frac{2}{3^{n-2}} - \frac{2}{3^{n-1}}$$
$$= \frac{18}{3^n} - \frac{6}{3^n}$$
$$= \frac{12}{3^n}$$

First term, $T_1 = \frac{12}{3} = 4$

or $S_1 = 6 - \frac{2}{3^0}$
$= 6 - 2 = 4$

Common ratio $= \frac{T_{n+1}}{T_n}$
$= \frac{12}{3^{n+1}} \div \frac{12}{3^n}$
$= \frac{3^n}{3^{n+1}}$
$= \frac{1}{3}$

(Q3) [Cambridge/9709/2007]

The 1st term of an arithmetic progression is a and the common difference is d, where $d \neq 0$.

(i) Write down expressions, in terms of a and d, for the 5th term and the 15th term. [1]

The 1st term, the 5th term and the 15th term of the arithmetic progression are the first three terms of a geometric progression.

(ii) Show that $3a = 8d$. [3]

(iii) Find the common ratio of the geometric progression. [2]

(i) $T_5 = a + 4d$

$T_{15} = a + 14d$

(ii) $ar = a + 4d$

$ar^2 = a + 14d$

$$\frac{a+4d}{a} = \frac{a+14d}{a+4d}$$

$$a^2 + 8ad + 16d^2 = a^2 + 14ad$$

$$6ad = 16d^2$$

$$3a = 8d$$

(iii) $$r = \frac{a + \frac{3a}{2}}{a}$$

$$= \frac{5}{2}$$

(Q4) [Cambridge/H2 Maths 2007]

A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the first, fourth and sixth terms respectively of the arithmetic series.

(i) Show that $3r^2 - 5r + 2 = 0$. [4]

(ii) Deduce that the geometric series is convergent and find, in terms of a, the sum to infinity. [5]

(iii) The sum of the first n terms of the arithmetic series is denoted by S. Given that $a > 0$, find the set of possible values of n for which S exceeds $4a$. [5]

(i) $ar = a + 3d \Rightarrow d = \frac{ar-a}{3}$

$ar^2 = a + 5d \Rightarrow d = \frac{ar^2-a}{5}$

$\therefore \quad \frac{a(r^2-1)}{5} = \frac{a(r-1)}{3}$

$3r^2 - 3 = 5(r-1)$

$3r^2 - 5r + 2 = 0$

(ii) $(3r-2)(r-1) = 0$

$r = \frac{2}{3}$ or $r = 1$ (rejected $\because$ if $r=1$, $d = \frac{a-a}{3} = 0$ but $d \neq 0$)

Since $|r| = \frac{2}{3} < 1$, the geometric series is convergent

and $S_\infty = \frac{a}{1-\frac{2}{3}} = 3a$

(iii) $S = \frac{n}{2}[2a + (n-1)d] > 4a$

$d = \frac{\frac{2}{3}a - a}{3} = -\frac{a}{9}$

$\frac{n}{2}\left[2a + (n-1)\left(-\frac{a}{9}\right)\right] > 4a$

$\frac{n}{18}[18 - n + 1] > 4$

$n^2 - 19n + 72 < 0$

$$\xrightarrow[\quad 5.23 \qquad 13.8 \quad]{+ \qquad - \qquad +}$$

$\therefore \quad 5.23 < n < 13.8$

Since $n \in \mathbb{Z}^+$, $\{n \in \mathbb{Z}^+ : 6 \leq n \leq 13\}$

(Q5) [Cambridge/9233/2004]

Find how many positive integers, less than 1000, are

(i) odd numbers, [1]

(ii) odd numbers which are **not** divisible by 5. [3]

Find the sum of the odd numbers, less than 1000, which are **not** divisible by 5. [4]

(i) No. of odd numbers = 500

(ii) Odd numbers divisible by 5: 5, 15, 25, …, 995

$$5+(n-1)(10)=995 \Rightarrow n=100$$

No. of odd numbers not divisible by 5 = 500 – 100 = 400

Sum of all odd numbers = $\frac{500}{2}(1+999)=250000$

Sum of all odd numbers divisible by 5 = $\frac{100}{2}(5+995)=50000$

Required sum = 250000 – 50000 = 200000

(Q6) [Cambridge/9233/2005]

It is given that a, b, c are the first three terms of a geometric progression. It is also given that a, c, b are the first three terms of an arithmetic progression.

(i) Show that

$$b^2 = ac \text{ and } c=\frac{a+b}{2}.$$ [2]

(ii) Hence show that

$$2\left(\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)-1=0.$$ [1]

(iii) Given that the sum to infinity of the geometric progression is S, find S in terms of a. [4]

(i) a, b, c are in G.P. $\Rightarrow \frac{b}{a}=\frac{c}{b} \Rightarrow b^2 = ac$ (shown)

a, c, b are in A.P. $\Rightarrow b-c=c-a \Rightarrow 2c=a+b \Rightarrow c=\frac{a+b}{2}$ (shown)

(ii) $b^2 = ac = a\left(\frac{a+b}{2}\right) \Rightarrow 2b^2 = a^2+ab$

Divide throughout by a^2: $2\left(\frac{b}{a}\right)^2 = 1+\left(\frac{b}{a}\right) \Rightarrow 2\left(\frac{b}{a}\right)^2-\left(\frac{b}{a}\right)-1=0$ (shown) ❖

(iii) Factorizing, $\left(\frac{b}{a}-1\right)\left(2.\frac{b}{a}+1\right)=0 \Rightarrow \frac{b}{a}=1 \text{ or } \frac{b}{a}=-\frac{1}{2}$ (Note: $r=\frac{b}{a}$)

Since sum to infinity exists, $\frac{b}{a}\neq 1, \therefore \frac{b}{a}=-\frac{1}{2}$

$$\therefore S=\frac{a}{1-\frac{b}{a}}=\frac{a}{1+\frac{1}{2}}=\frac{2}{3}a$$

(Q7) [Cambridge/9709/2005]

A small trading company made a profit of \$250 000 in the year 2000. The company considered two different plans, plan A and plan B, for increasing its profits.

Under plan A, the annual profit would increase each year by 5% of its value in the preceding year. Find, for plan A,

(i) the profit for the year 2008, [3]

(ii) the total profit for the 10 years 2000 to 2009 inclusive. [2]

Under plan B, the annual profit would increase each year by a constant amount \$$D$.

(iii) Find the value of D for which the total profit for the 10 years 2000 to 2009 inclusive would be the same for both plans. [3]

(i) Profit for year 2008 $= 250000(1.05)^8$ (ar^8)

$= \$369\,000$ (3sf)

2000	2001	...	2008
$n=1$	2	...	9
250000	250000(1.05)	...	

(ii) Total profit from 2000 to 2009 inclusive

$= 250\,000 + 250\,000(1.05) + \dots + 250\,000(1.05)^9$

$= \dfrac{250\,000\left[1.05^{10}-1\right]}{1.05-1}$

$= \$\,3140\,000$ (3sf)

(iii) $\dfrac{10}{2}\left[2(250\,000)+9D\right] = \dfrac{250000\left[1.05^{10}-1\right]}{0.05}$

$D = 14300$ (3sf)

8. Sigma notation and sequences

(Q1) [Cambridge/9795/Specimen Paper]

Given that

$$\sum_{n=1}^{N} n^3 = \left[\frac{N}{2}(N+1)\right]^2,$$

Find

$$1^2 - 2^3 + 3^3 - 4^3 + \ldots - (2N)^3,$$

In terms of N, simplifying your answer.

[Solution]

$$1^3 - 2^3 + 3^3 - 4^3 + \ldots + (2N)^3$$

$$= \left[1^3 + 2^3 + 3^3 + 4^3 + \ldots + (2N)^3\right] - 2\left[2^3 + 4^3 + \ldots + (2N)^3\right]$$

$$= \sum_{n=1}^{2N} n^3 - 2\cdot 2^3\left[1^3 + 2^3 + \cdots + N^3\right]$$

$$= N^2(2N+1)^2 - 16\sum_{n=1}^{N} n^3$$

$$= N^2(2N+1)^2 - 16\left[\frac{N^2(N+1)^2}{4}\right]$$

$$= N^2(2N+1)^2 - 4N^2(N+1)^2$$

$$= N^2\left[4N^2 + 4N + 1 - 4(N^2 + 2N + 1)\right]$$

$$= N^2\left[-4N - 3\right]$$

$$= -N^2(4N+3)$$

(Q2) [EJC/9758/2020/I/7]

The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = a\left(3^n\right) + bn + c$, where a, b and c are constants and $n \geq 1$.

(i) Given that $u_1 = 5, u_2 = 9$ and $u_3 = 33$, find a, b and c. [4]

(ii) Show that $u_{n+1} = A\left(3^n\right) + B$, where A and B are constants to be determined. [3]

Using your answer in part **(ii)**, find $\sum_{r=2}^{n} u_{r+1}$ in terms of n. (You need not simplify your answer.)

[3]

[Solution]

(i)	$S_1 = 5 = 3a + b + c$ $S_2 = 14 = 9a + 2b + c$ $S_3 = 47 = 27a + 3b + c$ OR $u_1 = S_1 \Rightarrow 5 = 3a + b + c$ $u_2 = S_2 - S_1 \Rightarrow 9 = 6a + b$ $u_3 = S_3 - S_2 \Rightarrow 33 = 18a + b$ Using GC, $a = 2, b = -3$ and $c = 2$ Hence $S_n = 2\left(3^n\right) - 3n + 2$
(ii)	$u_{n+1} = S_{n+1} - S_n$ $= 2\left(3^{n+1}\right) - 3(n+1) + 2 - \left[2\left(3^n\right) - 3n + 2\right]$ $= 2\left(3^{n+1}\right) - 3n - 3 + 2 - 2\left(3^n\right) + 3n - 2$ $= 2(3-1)\left(3^n\right) - 3$ $= 4\left(3^n\right) - 3$
	$\sum_{r=2}^{n} u_{r+1} = \sum_{r=2}^{n}\left[4\left(3^r\right) - 3\right]$ $= 4\sum_{r=2}^{n}\left(3^r\right) - 3\sum_{r=2}^{n} 1$ $= 4\left[\frac{3^2\left(3^{n-1} - 1\right)}{3-1}\right] - 3(n - 2 + 1)$ $= 18\left(3^{n-1} - 1\right) - 3(n-1)$ $= 6\left(3^n\right) - 3n - 15$

(Q3) [VJC/9758/2018/Promo/4]

(i) Given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, find $\sum_{r=1}^{n}(3r+1)^2$ in terms of n. [3]

(ii) Use your answer to part **(i)** to find $\sum_{r=1}^{N+1}(3r-2)^2$. [3]

[Solution]

4(i)	$\sum_{r=1}^{n}(3r+1)^2 = \sum_{r=1}^{n}(9r^2+6r+1)$ $= \sum_{r=1}^{n}(9r^2) + \sum_{r=1}^{n}(6r+1)$ $= 9\times\frac{n(n+1)(2n+1)}{6} + \frac{n}{2}(7+6n+1)$ $= \frac{n}{2}(3(n+1)(2n+1)+6n+8)$ $= \frac{n}{2}(6n^2+15n+11)$
4(ii)	$\sum_{r=1}^{N+1}(3r-2)^2 = \sum_{r+1=1}^{r+1=N+1}(3r+3-2)^2$ (Replace r with $r+1$) $= \sum_{r=0}^{N}(3r+1)^2$ $= \sum_{r=1}^{N}(3r+1)^2 + 1$ $= \frac{N}{2}(6N^2+15N+11)+1$

(Q4) [NYJC/9758/2012/I/2a]
A sequence of negative real numbers $x_1, x_2, x_3, ...$ satisfies the relation

$$x_{n+1} = -\sqrt{1-2x_n}, \text{ for } n \geq 1.$$

Given that the sequence converges to l, find the exact value of l. **[2]**
[Solution]

When $n \to \infty$, $x_n \to l$ and $x_{n+1} \to l$,

$$l = -\sqrt{1-2l}$$
$$l^2 = 1-2l$$
$$l^2+2l-1=0$$
$$l = \frac{-2\pm\sqrt{4-4(-1)}}{2}$$
$$= \frac{-2\pm 2\sqrt{2}}{2} \quad = -1\pm\sqrt{2}$$

Since $x_n < 0$, $l = -1-\sqrt{2}$

9. Vectors

(Q1) [Cambridge/1999]

The angle between the vector $\lambda\mathbf{i}+3\mathbf{j}-6\mathbf{k}$ and the vector $\mathbf{i}$ is 120°. Find the exact value of the constant λ.

$$\cos 120^\circ = \frac{\begin{pmatrix}\lambda\\3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{\lambda^2+3^2+6^2}\sqrt{1}}$$

$$-\frac{1}{2} = \frac{\lambda}{\sqrt{\lambda^2+45}} \quad \text{— } (*)$$

$$\sqrt{\lambda^2+45} = -2\lambda$$

$$\lambda^2+45 = 4\lambda^2$$

$$3\lambda^2 = 45$$

$$\lambda^2 = 15$$

$\lambda = -\sqrt{15}$ OR $\sqrt{15}$ (rejected since $\lambda < 0$, $\sqrt{\lambda^2+45} > 0$ for $(*)$)

(Q2) [Cambridge/2002]

(i) Show that the lines given by

$$\mathbf{r} = (5\mathbf{i}+2\mathbf{j}+4\mathbf{k})+\lambda(\mathbf{i}+3\mathbf{j}+\mathbf{k}) \quad \text{and} \quad \mathbf{r} = (3\mathbf{i}+\mathbf{j}+\mathbf{k})+\mu(4\mathbf{i}+7\mathbf{j}+5\mathbf{k})$$

intersect, and find their point of intersection.

(ii) Calculate the acute angle between the lines.

$l_1: \underset{\sim}{r} = \begin{pmatrix}5\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\1\end{pmatrix}, \quad \lambda\in\mathbb{R}$

$l_2: \underset{\sim}{r} = \begin{pmatrix}3\\1\\1\end{pmatrix}+\mu\begin{pmatrix}4\\7\\5\end{pmatrix}, \quad \mu\in\mathbb{R}$

(i) Let $\begin{pmatrix}5+\lambda\\2+3\lambda\\4+\lambda\end{pmatrix} = \begin{pmatrix}3+4\mu\\1+7\mu\\1+5\mu\end{pmatrix}$

$$4\mu-\lambda = 2 \quad \text{— } ①$$

$$7\mu-3\lambda = 1 \quad \text{— } ②$$

$$5\mu-\lambda = 3 \quad \text{— } ③$$

② − ① : $\mu = 1$

$\lambda = 2$

LHS of ③ $= 5(1)-2 = 3 =$ RHS of ③

$\therefore$ l_1 & l_2 intersect and pt of intersection is $(7, 8, 6)$.

(ii) Let θ be the required angle.

$$\cos\theta = \left|\frac{\begin{pmatrix}1\\3\\1\end{pmatrix}\cdot\begin{pmatrix}4\\7\\5\end{pmatrix}}{\sqrt{1+9+1}\sqrt{16+49+25}}\right|$$

$$= \frac{4+21+5}{\sqrt{11}\sqrt{90}}$$

$$\theta = 17.5^\circ \text{ (1 dp)}$$

(Q3) [Cambridge/9709/2007]

The straight line l has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$. The plane p has equation $(\mathbf{r} - 3\mathbf{i}).(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 0$. The line l intersects the plane p at the point A.

(i) Find the position vector of A. [3]

(ii) Find the acute angle between l and p. [4]

(iii) Find a vector equation for the line which lies in p, passes through A and is perpendicular to l. [5]

$l: \underset{\sim}{r} = \begin{pmatrix}1\\6\\-3\end{pmatrix} + s\begin{pmatrix}1\\-2\\2\end{pmatrix}, \quad s \in \mathbb{R}$

$p: \left[\underset{\sim}{r} - \begin{pmatrix}3\\0\\0\end{pmatrix}\right] \cdot \begin{pmatrix}2\\-3\\6\end{pmatrix} = 0 \quad$ i.e. $\underset{\sim}{r} \cdot \begin{pmatrix}2\\-3\\6\end{pmatrix} = 6$

(i) At point A,

$$\begin{pmatrix}1+s\\6-2s\\-3+2s\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\6\end{pmatrix} = 6$$

$$2 + 2s - 18 + 6s - 18 + 12s = 6$$

$$20s = 40$$

$$s = 2$$

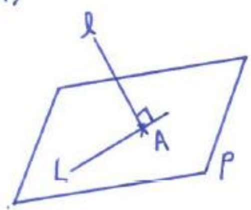
$$\therefore \overrightarrow{OA} = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

(ii) Let required angle be θ.

$$\sin\theta = \left| \frac{\begin{pmatrix}1\\-2\\2\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\6\end{pmatrix}}{\sqrt{1+4+4}\sqrt{4+9+36}} \right| = \frac{20}{\sqrt{9}\sqrt{49}}$$

$$\theta = 72.2° \ (1 \text{ dp})$$

(iii)



Note. The required line is $\perp$ to both l and the normal of the plane p.

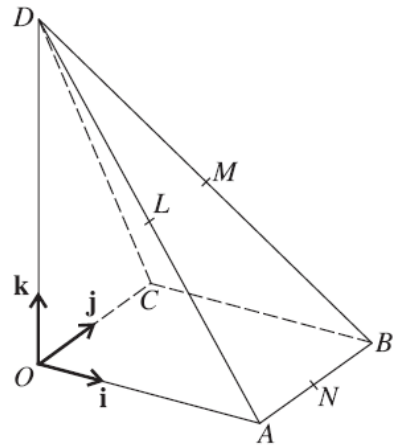
Let L represent the required line.

$$\underset{\sim}{d}_L = \begin{pmatrix}1\\-2\\2\end{pmatrix} \times \begin{pmatrix}2\\-3\\6\end{pmatrix}$$

$$= \begin{pmatrix}-6\\-2\\1\end{pmatrix}$$

$$\therefore L: \underset{\sim}{r} = \begin{pmatrix}3\\2\\1\end{pmatrix} + \alpha\begin{pmatrix}6\\2\\-1\end{pmatrix}, \quad \alpha \in \mathbb{R}$$

(Q4) [Cambridge/9794/Specimen Paper for 2010]



The diagram shows a pyramid $DOABC$. Taking unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as shown, the position vectors of A, B, C, D are given by

$$\overrightarrow{OA} = 4\mathbf{i}, \quad \overrightarrow{OB} = 4\mathbf{i} + 2\mathbf{j}, \quad \overrightarrow{OC} = 2\mathbf{j}, \quad \overrightarrow{OD} = 6\mathbf{k}.$$

The mid-points of AD, BD and AB are L, M and N respectively.

(i) Find the vector $\overrightarrow{MN}$ and the angle between the directions of $\overrightarrow{MN}$ and $\overrightarrow{OB}$. [4]

(ii) The point P lying on OD has position vector $p\mathbf{k}$. Determine the value of p for which the line through P and B intersects the line through C and L. [7]

(i) Using Mid-pt Theorem,

$\vec{OL} = \frac{1}{2}(\vec{OA}+\vec{OD}) = \begin{pmatrix}2\\0\\3\end{pmatrix}$, $\vec{OM} = \frac{1}{2}(\vec{OB}+\vec{OD}) = \begin{pmatrix}2\\1\\3\end{pmatrix}$, $\vec{ON} = \frac{1}{2}(\vec{OA}+\vec{OB}) = \begin{pmatrix}4\\1\\0\end{pmatrix}$

$\vec{MN} = \vec{ON} - \vec{OM} = \begin{pmatrix}2\\0\\-3\end{pmatrix}$

Let θ be the required angle.

$$\cos\theta = \frac{\vec{MN}\cdot\vec{OB}}{|\vec{MN}||\vec{OB}|} = \frac{\begin{pmatrix}2\\0\\-3\end{pmatrix}\cdot\begin{pmatrix}4\\2\\0\end{pmatrix}}{\sqrt{4+9}\sqrt{16+4}} = \frac{8}{\sqrt{13}\sqrt{20}}$$

$\theta = 60.3°$ (1 dp)

(ii) $\vec{OP} = \begin{pmatrix}0\\0\\p\end{pmatrix}$

l_{PB} : $\underline{r} = \vec{OB} + \lambda\vec{PB}$
$= \begin{pmatrix}4\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}4\\2\\-p\end{pmatrix}$, $\lambda \in \mathbb{R}$

l_{CL} : $\underline{r} = \vec{OC} + \mu\vec{CL}$
$= \begin{pmatrix}0\\2\\0\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\3\end{pmatrix}$, $\mu \in \mathbb{R}$

When l_{PB} & l_{CL} intersect,

$$\begin{pmatrix}4+4\lambda\\2+2\lambda\\-\lambda p\end{pmatrix} = \begin{pmatrix}2\mu\\2-2\mu\\3\mu\end{pmatrix}$$

$2\mu - 4\lambda = 4$ — ①
$-2\mu - 2\lambda = 0$ — ②
$3\mu = -\lambda p$

① + ② : $\lambda = -\frac{2}{3}$
$\mu = \frac{2}{3}$ $\quad\therefore\quad p = 3.$

(Q5) [Cambridge/9795/Specimen Paper]

With respect to an origin O, the points A, B, C, D have position vectors

$$2\mathbf{i}-\mathbf{j}+\mathbf{k}, \quad \mathbf{i}-2\mathbf{k}, \quad -\mathbf{i}+3\mathbf{j}+2\mathbf{k}, \quad -\mathbf{i}+\mathbf{j}+4\mathbf{k},$$

respectively. Find

(i) a vector perpendicular to the plane OAB, [2]

(ii) the acute angle between the planes OAB and OCD, correct to the nearest $0.1°$, [3]

$\vec{OA}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\vec{OB}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$, $\vec{OC}=\begin{pmatrix}-1\\3\\2\end{pmatrix}$, $\vec{OD}=\begin{pmatrix}-1\\1\\4\end{pmatrix}$

(i)

$$\vec{OA}\times\vec{OB}=\begin{pmatrix}2\\-1\\1\end{pmatrix}\times\begin{pmatrix}1\\0\\-2\end{pmatrix}$$
$$=\begin{pmatrix}2\\5\\1\end{pmatrix}$$

$\therefore$ A vector perpendicular to plane OAB is $\begin{pmatrix}2\\5\\1\end{pmatrix}$.

(ii) A vector $\perp$ to plane OCD

$$=\begin{pmatrix}-1\\3\\2\end{pmatrix}\times\begin{pmatrix}-1\\1\\4\end{pmatrix}$$
$$=\begin{pmatrix}10\\2\\2\end{pmatrix}$$
$$=2\begin{pmatrix}5\\1\\1\end{pmatrix}$$

Let required angle be θ.

$$\cos\theta=\left|\frac{\begin{pmatrix}2\\5\\1\end{pmatrix}\cdot\begin{pmatrix}5\\1\\1\end{pmatrix}}{\sqrt{4+25+1}\sqrt{25+1+1}}\right|$$
$$=\frac{16}{\sqrt{30}\sqrt{27}}$$

$$\theta = 55.8° \text{ (1 dp)}$$

(Q6) [Cambridge/9231/2007]

The points A, B and C have position vectors $2\mathbf{i}$, $3\mathbf{j}$ and $4\mathbf{k}$ respectively. Find a vector which is perpendicular to the plane Π_1 containing A, B and C. [3]

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} - \mathbf{k}).$$

Find the acute angle between the planes Π_1 and Π_2. [5]

$\vec{OA} = \begin{pmatrix}2\\0\\0\end{pmatrix}$, $\vec{OB} = \begin{pmatrix}0\\3\\0\end{pmatrix}$, $\vec{OC} = \begin{pmatrix}0\\0\\4\end{pmatrix}$

$\vec{AB} = \begin{pmatrix}-2\\3\\0\end{pmatrix}$, $\vec{AC} = \begin{pmatrix}-2\\0\\4\end{pmatrix} = 2\begin{pmatrix}-1\\0\\2\end{pmatrix}$

$\underset{\sim}{n}_1 = \begin{pmatrix}-2\\3\\0\end{pmatrix} \times \begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}6\\4\\3\end{pmatrix}$ (required vector)

$\underset{\sim}{n}_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

Let required angle be θ.

$$\cos\theta = \left| \frac{\begin{pmatrix}6\\4\\3\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{36+16+9}\sqrt{3}} \right|$$

$$= \frac{13}{\sqrt{61}\sqrt{3}}$$

$\theta = 16.1°$ (1 dp)

(Q7) [Cambridge/H2 Maths 2007]

Referred to the origin O, the position vectors of the points A and B are

$$\mathbf{i} - \mathbf{j} + 2\mathbf{k} \qquad \text{and} \qquad 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

respectively.

(i) Show that OA is perpendicular to OB. [2]

(ii) Find the position vector of the point M on the line segment AB such that $AM : MB = 1 : 2$. [3]

(iii) The point C has position vector $-4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Use a vector product to find the exact area of triangle OAC. [4]

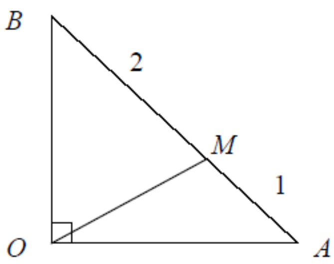
6(i) Given: $\overrightarrow{OA}=\begin{pmatrix}1\\-1\\2\end{pmatrix}, \overrightarrow{OB}=\begin{pmatrix}2\\4\\1\end{pmatrix}$

Consider $\overrightarrow{OA}\cdot\overrightarrow{OB}=\begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\4\\1\end{pmatrix}=2-4+2=0$

[Recall that $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3$ and $\mathbf{a}\cdot\mathbf{b}=0$ is the condition for perpendicular vectors **a** and **b**]

$\therefore OA \perp OB$ (shown) ❖

(ii)

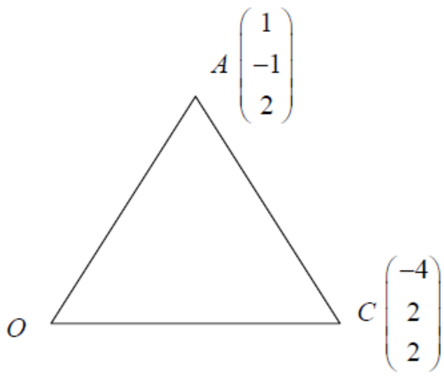


By Ratio Theorem, $\overrightarrow{OM}=\dfrac{2\overrightarrow{OA}+\overrightarrow{OB}}{3}$

$$=\frac{1}{3}\left[2\begin{pmatrix}1\\-1\\2\end{pmatrix}+\begin{pmatrix}2\\4\\1\end{pmatrix}\right]$$

$$=\begin{pmatrix}4/3\\2/3\\5/3\end{pmatrix}$$ ❖

(iii)



[Recall that $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-b_2a_3\\-(a_1b_3-b_1a_3)\\a_1b_2-b_1a_2\end{pmatrix}$]

Area of triangle $OAC=\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OC}\right|$

$$=\frac{1}{2}\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}-4\\2\\2\end{pmatrix}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}-2-4\\-(2+8)\\2-4\end{pmatrix}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}-6\\-10\\-2\end{pmatrix}\right|$$

$$=\frac{1}{2}\sqrt{6^2+10^2+2^2}$$

$$=\frac{1}{2}\sqrt{140}$$

$$=\sqrt{35}\text{ units}^2$$ ❖

(Q8) [Cambridge/H2 Maths 2007]

The line l passes through the points A and B with coordinates $(1, 2, 4)$ and $(-2, 3, 1)$ respectively. The plane p has equation $3x - y + 2z = 17$. Find

(i) the coordinates of the point of intersection of l and p, [5]

(ii) the acute angle between l and p, [3]

(iii) the perpendicular distance from A to p. [3]

$l: \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}, \ \lambda \in \mathbb{R}$

$p: \underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 17$

(i) At the point of intersection of l and p, S

$$\begin{pmatrix} 1+3\lambda \\ 2-\lambda \\ 4+3\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 17$$

$$3 + 9\lambda - 2 + \lambda + 8 + 6\lambda = 17$$

$$16\lambda = 8$$

$$\lambda = \frac{1}{2}$$

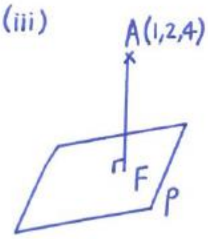
$\therefore \ S\left(\frac{5}{2}, \frac{3}{2}, \frac{11}{2}\right)$

(ii) Let required angle be θ.

$$\sin\theta = \left| \frac{\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}}{\sqrt{9+1+9}\sqrt{9+1+4}} \right| = \frac{16}{\sqrt{19}\sqrt{14}}$$

$$\theta = 78.8^\circ \ (1 \text{ dp})$$

(iii)



$l_{AF}: \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \ \mu \in \mathbb{R}$

To find F (pt of intersection of l_{AF} & p),

$$\begin{pmatrix} 1+3\mu \\ 2-\mu \\ 4+2\mu \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 17$$

$$3 + 9\mu - 2 + \mu + 8 + 4\mu = 17$$

$$14\mu = 8$$

$$\mu = \frac{4}{7}$$

$\therefore$ Perpendicular distance from A to p

$\therefore$ Perpendicular distance from A to p

$$= |\vec{AF}|$$

$$= \left| \begin{pmatrix} 1+3(\frac{4}{7}) \\ 2-\frac{4}{7} \\ 4+2(\frac{4}{7}) \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right|$$

$$= \frac{1}{7} \left| \begin{pmatrix} 12 \\ -4 \\ 8 \end{pmatrix} \right| = 2.14$$

Alt.

$$\text{Dist} = \frac{|d - \underset{\sim}{a} \cdot \underset{\sim}{n}|}{|\underset{\sim}{n}|}$$

$$= \frac{\left|17 - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\right|}{\sqrt{3^2+1^2+2^2}}$$

$$= \frac{8}{\sqrt{14}} = 2.14$$

(Q9) [Cambridge/9233/2004]

Referred to an origin O, the position vectors of four non-collinear points A, B, C and D are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively. Given that $\mathbf{a} - \mathbf{b} = \mathbf{d} - \mathbf{c}$, show that $ABCD$ is a parallelogram. [2] Given also that $\mid \mathbf{a} - \mathbf{c} \mid = \mid \mathbf{b} - \mathbf{d} \mid$, identify the shape of the parallelogram $ABCD$, justifying your answer. [2]

$\mathbf{a} - \mathbf{b} = \mathbf{d} - \mathbf{c} \Rightarrow \overrightarrow{BA} = \overrightarrow{CD} \Rightarrow BA \,//\, CD$ and $BA = CD$.
$\therefore ABCD$ has to be a parallelogram. (shown)

$|\mathbf{a} - \mathbf{c}| = |\mathbf{b} - \mathbf{d}| \Rightarrow \left|\overrightarrow{CA}\right| = \left|\overrightarrow{DB}\right|$ i.e. diagonals are of the same length

Since $ABCD$ is a parallelogram and the diagonals are of the same length, the shape has to be a square or a rectangle.

(Q10) [Cambridge/9233/2005]

The planes $\prod_1$ and $\prod_2$ have vector equations $\mathbf{r} = \lambda_1(\mathbf{i}+\mathbf{j}-\mathbf{k}) + \mu_1\ (2\mathbf{i}-\mathbf{j}+\mathbf{k})$ and $\mathbf{r} = \lambda_2(\mathbf{i}+2\mathbf{j}+\mathbf{k}) + \mu_2(3\mathbf{i}+\mathbf{j}-\mathbf{k})$ respectively. The line l passes through the point with position vector $4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ and is parallel to both $\prod_1$ and $\prod_2$. Find a vector equation for l. [4] Find also the shortest distance between l and the line of intersection of $\prod_1$ and $\prod_2$. [4]

$$\prod_1 : \underset{\sim}{n}_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\-3\\-3\end{pmatrix} = -3\begin{pmatrix}0\\1\\1\end{pmatrix} \qquad \prod_2 : \underset{\sim}{n}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}-3\\4\\-5\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \times \begin{pmatrix}-3\\4\\-5\end{pmatrix} = \begin{pmatrix}-9\\-3\\3\end{pmatrix} = -3\begin{pmatrix}3\\1\\-1\end{pmatrix}$$

$$l : \underset{\sim}{r} = \begin{pmatrix}4\\5\\6\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\-1\end{pmatrix}, \lambda \in \Re$$

Line of intersection: $\underset{\sim}{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-1\end{pmatrix}, \mu \in \Re$

METHOD 1:

Let P be the foot of perpendicular from O to l.

$$\overrightarrow{OP} = \begin{pmatrix}4+3\lambda\\5+\lambda\\6-\lambda\end{pmatrix}$$

$$\overrightarrow{OP} \bullet \begin{pmatrix}3\\1\\-1\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}4+3\lambda\\5+\lambda\\6-\lambda\end{pmatrix} \bullet \begin{pmatrix}3\\1\\-1\end{pmatrix} = 0$$

$$\Rightarrow 12 + 9\lambda + 5 + \lambda - 6 + \lambda = 0$$

$$\Rightarrow \lambda = -1$$

$$\overrightarrow{OP} = \begin{pmatrix}1\\4\\7\end{pmatrix}$$

Distance required = $OP = \sqrt{1^2 + 4^2 + 7^2} = \sqrt{66}$

METHOD 2:

Distance required

$$= \frac{\left|\begin{pmatrix}4\\5\\6\end{pmatrix} \times \begin{pmatrix}3\\1\\-1\end{pmatrix}\right|}{\sqrt{3^2 + 1^2 + 1^2}}$$

$$= \frac{\left|\begin{pmatrix}-11\\22\\-11\end{pmatrix}\right|}{\sqrt{11}}$$

$$= \frac{11\sqrt{6}}{\sqrt{11}} = \sqrt{66}$$

(Q11) [Cambridge/9231/2008]

The plane Π_1 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \theta(2\mathbf{j} - \mathbf{k}) + \phi(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

Find a vector normal to Π_1 and hence show that the equation of Π_1 can be written as $2x + 3y + 6z = 14$. [4]

The line l has equation

$$\mathbf{r} = 3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k} + t(4\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}).$$

The point on l where $t = \lambda$ is denoted by P. Find the set of values of λ for which the perpendicular distance of P from Π_1 is not greater than 4. [4]

The plane Π_2 contains l and the point with position vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find the acute angle between Π_1 and Π_2. [4]

$$\underset{\sim}{n}_1 = \begin{pmatrix}0\\2\\-1\end{pmatrix} \times \begin{pmatrix}3\\2\\-2\end{pmatrix} = \begin{pmatrix}-2\\-3\\-6\end{pmatrix}$$

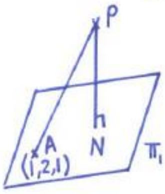
$$\Pi_1: \quad \underset{\sim}{r} \cdot \begin{pmatrix}2\\3\\6\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}2\\3\\6\end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix}2\\3\\6\end{pmatrix} = 2+6+6 = 14$$

$$\therefore \quad 2x+3y+6z = 14$$

$$l: \quad \underset{\sim}{r} = \begin{pmatrix}3\\8\\2\end{pmatrix} + t\begin{pmatrix}4\\6\\5\end{pmatrix}$$

$$\vec{OP} = \begin{pmatrix}3+4\lambda\\8+6\lambda\\2+5\lambda\end{pmatrix}$$



$$PN = \left| \vec{PA} \cdot \hat{\underset{\sim}{n}}_1 \right|$$

$$= \left| -\begin{pmatrix}2+4\lambda\\6+6\lambda\\1+5\lambda\end{pmatrix} \cdot \frac{\begin{pmatrix}2\\3\\6\end{pmatrix}}{\sqrt{4+9+36}} \right|$$

$$= \left| \frac{4+8\lambda+18+18\lambda+6+30\lambda}{\sqrt{49}} \right|$$

$$= \left| \frac{56\lambda+28}{7} \right|$$

$$= |8\lambda+4|$$

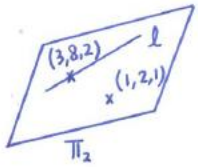
Given $PN \leqslant 4$

$$|8\lambda+4| \leqslant 4$$

$$-4 \leqslant 8\lambda+4 \leqslant 4$$

$$-8 \leqslant 8\lambda \leqslant 0$$

$$-1 \leqslant \lambda \leqslant 0$$



$$\underset{\sim}{n}_2 = \left[\begin{pmatrix}3\\8\\2\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} \right] \times \begin{pmatrix}4\\6\\5\end{pmatrix}$$

$$= \begin{pmatrix}2\\6\\1\end{pmatrix} \times \begin{pmatrix}4\\6\\5\end{pmatrix}$$

$$= \begin{pmatrix}24\\-6\\-12\end{pmatrix}$$

$$= 6\begin{pmatrix}4\\-1\\-2\end{pmatrix}$$

Let required angle be θ.

$$\cos\theta = \left| \frac{\begin{pmatrix}2\\3\\6\end{pmatrix} \cdot \begin{pmatrix}4\\-1\\-2\end{pmatrix}}{\sqrt{4+9+36}\sqrt{16+1+4}} \right|$$

$$= \frac{7}{\sqrt{49}\sqrt{21}}$$

$$\theta = 77.4° \text{ (1 dp)}$$

(Q12) [Cambridge/9231/2002]

The planes Π_1 and Π_2, which meet in the line l, have vector equations

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_1(2\mathbf{i} + 3\mathbf{k}) + \phi_1(-4\mathbf{j} + 5\mathbf{k}),$$
$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k} + \theta_2(3\mathbf{j} + \mathbf{k}) + \phi_2(-\mathbf{i} + \mathbf{j} + 2\mathbf{k}),$$

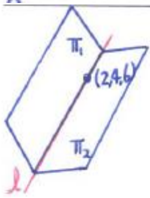
respectively. Find a vector equation of the line l in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. [5]

Find a vector equation of the plane Π_3 which contains l and which passes through the point with position vector $4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. Find also the equation of Π_3 in the form $ax + by + cz = d$. [4]

Deduce, or prove otherwise, that the system of equations

$$6x - 5y - 4z = -32,$$
$$5x - y + 3z = 24,$$
$$9x - 2y + 5z = 40,$$

has an infinite number of solutions. [3]



Consider $\begin{pmatrix}2\\0\\3\end{pmatrix} \times \begin{pmatrix}0\\-4\\5\end{pmatrix} = \begin{pmatrix}12\\-10\\-8\end{pmatrix} = -2\begin{pmatrix}-6\\5\\4\end{pmatrix}$

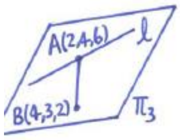
$\therefore\ \underset{\sim}{n}_1 = \begin{pmatrix}-6\\5\\4\end{pmatrix}$

Consider $\begin{pmatrix}0\\3\\1\end{pmatrix} \times \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}5\\-1\\3\end{pmatrix}$

$\therefore\ \underset{\sim}{n}_2 = \begin{pmatrix}5\\-1\\3\end{pmatrix}$

$\underset{\sim}{b} = \begin{pmatrix}-6\\5\\4\end{pmatrix} \times \begin{pmatrix}5\\-1\\3\end{pmatrix} = \begin{pmatrix}19\\38\\-19\end{pmatrix} = 19\begin{pmatrix}1\\2\\-1\end{pmatrix}$

$\therefore\ l : \underset{\sim}{r} = \begin{pmatrix}2\\4\\6\end{pmatrix} + t\begin{pmatrix}1\\2\\-1\end{pmatrix},\ t \in \mathbb{R}$



$\vec{AB} = \begin{pmatrix}2\\-1\\-4\end{pmatrix}$

Consider $\begin{pmatrix}2\\-1\\-4\end{pmatrix} \times \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}9\\-2\\5\end{pmatrix}$

$\Pi_3 : \underset{\sim}{r} \cdot \begin{pmatrix}9\\-2\\5\end{pmatrix} = \begin{pmatrix}4\\3\\2\end{pmatrix} \cdot \begin{pmatrix}9\\-2\\5\end{pmatrix}$

$= 36 - 6 + 10$

$= 40$

$\therefore\ 9x - 2y + 5z = 40$

$\Pi_1 : \underset{\sim}{r} \cdot \begin{pmatrix}-6\\5\\4\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix} \cdot \begin{pmatrix}-6\\5\\4\end{pmatrix}$

$-6x + 5y + 4z = -12 + 20 + 24$

$= 32$

$6x - 5y - 4z = -32$

$\Pi_2 : \underset{\sim}{r} \cdot \begin{pmatrix}5\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix} \cdot \begin{pmatrix}5\\-1\\3\end{pmatrix}$

$= 10 - 4 + 18$

$= 24$

$5x - y + 3z = 24$

Since Π_1, Π_2, Π_3 contains the line l, they intersect at l. Thus the system of eqns has an infinite no. of solns.

(Q13) [Cambridge/9709/2009]

The plane p has equation $2x - 3y + 6z = 16$. The plane q is parallel to p and contains the point with position vector $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

(i) Find the equation of q, giving your answer in the form $ax + by + cz = d$. [2]

(ii) Calculate the perpendicular distance between p and q. [3]

(iii) The line l is parallel to the plane p and also parallel to the plane with equation $x - 2y + 2z = 5$. Given that l passes through the origin, find a vector equation for l. [5]

$p: \ \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 16$

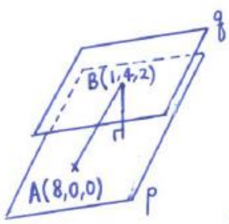
(i) $q: \ \underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

$= 2 - 12 + 12$

$\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 2$

$2x - 3y + 6z = 2$

(ii) Since p & q are parallel, perpendicular distance bet. p & q is equivalent to perpendicular distance from a pt on q to the plane p.



Required distance $= \left| \vec{AB} \cdot \dfrac{\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}}{\sqrt{4+9+36}} \right|$

$= \left| \dfrac{\begin{pmatrix} -7 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}}{\sqrt{49}} \right|$

$= \dfrac{14}{7}$

$= 2$

Alt.

$\text{Dist} = \dfrac{|d - \underset{\sim}{q} \cdot \underset{\sim}{n}|}{\underset{\sim}{n}}$

$= \dfrac{\left| 2 - \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right|}{\sqrt{2^2 + 3^2 + 6^2}}$

$= \dfrac{|-14|}{7} = 2$

(iii) l is parallel to $p \Rightarrow l$ is $\perp$ to normal of p

l is parallel to plane $x - 2y + 2z = 5 \Rightarrow l$ is $\perp$ to normal i.e. $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$\therefore \ \underset{\sim}{d}_l = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}$

$l: \ \underset{\sim}{r} = \lambda \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R}$

10. Differentiation & Applications of differentiation

(Q1) [Cambridge/9709/2007]

The curve with equation $y = e^{-x}\sin x$ has one stationary point for which $0 \leqslant x \leqslant \pi$.

(i) Find the x-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

(i) $y = e^{-x}\sin x$

$\frac{dy}{dx} = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x - \sin x)$

$\frac{dy}{dx} = 0 \Rightarrow \cos x - \sin x = 0$

$\sin x = \cos x$

$\tan x = 1$

For $0 \leqslant x \leqslant \pi$, $x = \frac{\pi}{4}$

$\therefore$ x-coordinate of stationary pt is $\frac{\pi}{4}$.

(ii) $\frac{d^2y}{dx^2} = -e^{-x}(\cos x - \sin x) + e^{-x}(-\sin x - \cos x) = -2e^{-x}\cos x$

When $x = \frac{\pi}{4}$, $\frac{d^2y}{dx^2} = -2e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} < 0$

$\therefore$ stationary pt is a max. pt.

(Q2) [Cambridge/9709/2007]

The equation of a curve is $y = 2x - \tan x$, where x is in radians. Find the coordinates of the stationary points of the curve for which $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. [5]

$y = 2x - \tan x$

$\frac{dy}{dx} = 2 - \sec^2 x$

$\frac{dy}{dx} = 0 \Rightarrow 2 - \sec^2 x = 0$

$\sec^2 x = 2$

$\cos^2 x = \frac{1}{2}$

$\cos x = \pm\frac{1}{\sqrt{2}}$

For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $x = -\frac{\pi}{4}, \frac{\pi}{4}$

When $x = -\frac{\pi}{4}$, $y = -\frac{\pi}{2} + 1$ $\therefore$ $(-\frac{\pi}{4}, 1 - \frac{\pi}{2})$

When $x = \frac{\pi}{4}$, $y = \frac{\pi}{2} - 1$ $\therefore$ $(\frac{\pi}{4}, \frac{\pi}{2} - 1)$

(Q3) [Cambridge/9231/2007]

The positive variables x and y are related by

$$y = x^2 + 2\ln(xy).$$

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when both x and y are equal to 1. [7]

$$y = x^2 + 2\ln(xy) = x^2 + 2(\ln x + \ln y)$$

$$\frac{dy}{dx} = 2x + 2\left(\frac{1}{x} + \frac{1}{y}\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = 2 + 2\left[-\frac{1}{x^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{y}\frac{d^2y}{dx^2}\right]$$

When $x=1$, $y=1$, $\frac{dy}{dx} = 2 + 2\left(1 + \frac{dy}{dx}\right)$

$$\frac{dy}{dx} = -4$$

$$\frac{d^2y}{dx^2} = 2 + 2\left[-1 - (-4)^2 + \frac{d^2y}{dx^2}\right]$$

$$= 2 + 2\left[-17 + \frac{d^2y}{dx^2}\right]$$

$$\frac{d^2y}{dx^2} = 32.$$

(Q4) [Cambridge/9231/2006]

The variables x and y are such that

$$y^3 + \left(\frac{dy}{dx}\right)^3 = x^4 + 6.$$

Given that $y = -1$ when $x = 1$, find the value of $\frac{d^2y}{dx^2}$ when $x = 1$. [5]

$$y^3 + \left(\frac{dy}{dx}\right)^3 = x^4 + 6$$

$$3y^2\frac{dy}{dx} + 3\left(\frac{dy}{dx}\right)^2\frac{d^2y}{dx^2} = 4x^3$$

When $x=1$, $y=-1$, $-1 + \left(\frac{dy}{dx}\right)^3 = 1 + 6$

$$\frac{dy}{dx} = 2$$

$$\therefore \quad 3(-1)^2(2) + 3(2)^2\frac{d^2y}{dx^2} = 4(1)^3$$

$$\frac{d^2y}{dx^2} = -\frac{1}{6}$$

(Q5) [Cambridge/9709/2009]

The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [4]

(ii) Find the equation of the tangent to the curve at the point (2, 1), giving your answer in the form $ax + by + c = 0$. [2]

(i)
$$x^3 - x^2y - y^3 = 3$$
$$3x^2 - x^2\frac{dy}{dx} - 2xy - 3y^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{3x^2-2xy}{x^2+3y^2}$$

(ii) At (2,1), $\frac{dy}{dx} = \frac{3(2)^2 - 2(2)(1)}{2^2+3(1)^2}$
$$= \frac{8}{7}$$

Eqn of tangent is
$$y-1 = \frac{8}{7}(x-2)$$
$$7y-7 = 8x-16$$
$$8x-7y-9=0$$

(Q6) [Cambridge/9709/2006]

The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\frac{dy}{dx} = \frac{y-x^2}{2y^2-x}$. [4]

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the x-axis. [5]

(i)
$$x^3 + 2y^3 = 3xy$$
$$3x^2 + 6y^2\frac{dy}{dx} = 3\left(x\frac{dy}{dx} + y\right)$$
$$\frac{dy}{dx} = \frac{3y-3x^2}{6y^2-3x}$$
$$= \frac{y-x^2}{2y^2-x}$$

(ii) When tangent is // x-axis, $\frac{dy}{dx} = 0$
$$\Rightarrow y = x^2$$
$$\therefore \quad x^3 + 2x^6 = 3x^3$$
$$2x^3 - 2x^6 = 0$$
$$2x^3(1-x^3) = 0$$
$$x^3 = 1$$
$$x = 1 \quad \therefore y = 1 \quad \therefore (1,1)$$

(Q7) [Cambridge/9709/2010]

The parametric equations of a curve are

$$x = 1 + \ln(t-2), \quad y = t + \frac{9}{t}, \quad \text{for } t > 2.$$

(i) Show that $\frac{dy}{dx} = \frac{(t^2-9)(t-2)}{t^2}$. [3]

(ii) Find the coordinates of the only point on the curve at which the gradient is equal to 0. [3]

(i) $x = 1 + \ln(t-2)$, $y = t + \frac{9}{t}$

$\frac{dx}{dt} = \frac{1}{t-2}$ $\quad \frac{dy}{dt} = 1 - \frac{9}{t^2} = \frac{t^2-9}{t^2}$

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{(t^2-9)(t-2)}{t^2}$

(ii) When $\frac{dy}{dx} = 0$, $\quad (t^2-9)(t-2) = 0$

$t^2 - 9 = 0$ OR $t = 2$ (rejected $\because t > 2$)

$t = 3 \quad (t > 2)$

$\therefore$ coordinates $(1, 6)$

(Q8) [Cambridge/2003]

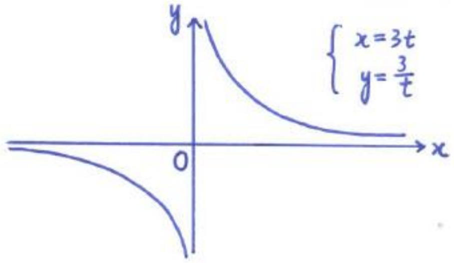
Sketch the curve with parametric equations

$$x = 3t, y = \frac{3}{t}.$$

The point P on the curve has parameter $t = 2$. The normal at P meets the curve again at the point Q.

(i) Show that the normal at P has equation $2y = 8x - 45$.

(ii) Find the value of t at Q.



(i) $P(6, \frac{3}{2})$

$$\left.\begin{aligned} \frac{dx}{dt} &= 3 \\ \frac{dy}{dt} &= -\frac{3}{t^2} \end{aligned}\right\} \quad \frac{dy}{dx} = -\frac{1}{t^2}$$

At P, $\frac{dy}{dx} = -\frac{1}{4}$ $\therefore$ gradient of normal $= 4$

Eqn of normal at P is

$$y - \frac{3}{2} = 4(x-6)$$
$$2y - 3 = 8x - 48$$
$$2y = 8x - 45$$

(ii) Subst. $x = 3t$, $y = \frac{3}{t}$ into $2y = 8x - 45$:

$$\frac{6}{t} = 8(3t) - 45$$
$$6 = 24t^2 - 45t$$
$$8t^2 - 15t - 2 = 0$$
$$(8t+1)(t-2) = 0$$
$$t = -\frac{1}{8} \quad \text{or} \quad t = 2 \ (\text{at } P)$$

$\therefore$ At Q, $t = -\frac{1}{8}$.

(Q9) [Cambridge/1998]

A spherical balloon is being inflated and at the instant when its radius is 3 m, its surface area is increasing at the rate of 2 m^2s^{-1}. Find the rate of increase, at the same instant of,

(i) the radius, **(ii)** the volume.

[The formulae for the surface area and the volume of a sphere are $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$.

(i) Surface area of spherical balloon $A = 4\pi r^2$

$$\frac{dA}{dt} = 4\pi\left(2r\frac{dr}{dt}\right)$$

When $r = 3$m,

$$2 = 4\pi(2)(3)\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{12\pi} \text{ m/s}$$

(ii) Vol. of spherical balloon $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

When $r = 3$m,

$$\frac{dV}{dt} = 4\pi(3)^2\left(\frac{1}{12\pi}\right)$$

$$= 3 \text{ m}^3\text{/s}$$

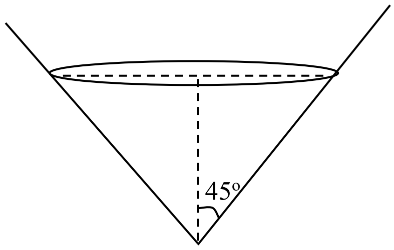
(Q10) [Cambridge/2006]

(a) Find the coordinates of the points on the curve

$$3x^2 + xy + y^2 = 33$$

at which the tangent is parallel to the x-axis. [4]

(b)



A hollow cone of semi-vertical angle 45° is held with its axis vertical and vertex downwards (see diagram). At the beginning of an experiment, it is filled with 390 cm^3 of liquid. The liquid runs out through a small hole at the vertex at a constant rate of 2 cm^3 s^{-1}. Find the rate at which the depth of the liquid is decreasing 3 minutes after the start of the experiment. [5]

(a) $3x^2+xy+y^2=33$

Diff. wrt x : $6x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-6x-y}{x+2y}$

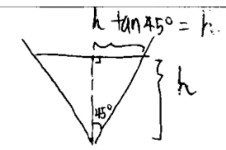
Tangent // to x-axis $\Rightarrow \dfrac{-6x-y}{x+2y} = 0 \Rightarrow y = -6x$.

$3x^2 + x(-6x) + (-6x)^2 = 33$

$x = \pm 1$

$x=1,\ y=-6$
$x=-1,\ y=6$ } $(1,-6)$ and $(-1,6)$ #

(b)



$V = \frac{1}{3}\pi\ (h)^2(h) = \frac{1}{3}\pi h^3$

$\dfrac{dV}{dh} = \pi h^2 \Rightarrow \dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt} = \dfrac{1}{\pi h^2}(-2) = \dfrac{-2}{\pi h^2}$

After 3 min, $V = 390 - 2\times3\times60 = 30$

$30 = \dfrac{\pi h^3}{3} \Rightarrow h = \left(\dfrac{90}{\pi}\right)^{1/3}$

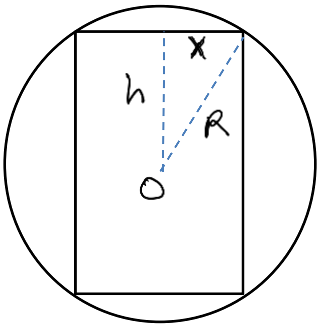
$\therefore \dfrac{dh}{dt} = \dfrac{-2}{\pi(90/\pi)^{2/3}} = -0.0680$

$\therefore$ Rate at which the depth is ↓ $= 0.0680$ cm s^{-1} #.

(Q11) [Cambridge/N75/II/5]

A cylinder of radius x is inscribed in a fixed sphere of internal radius R such that the circumferences of the circular ends of the cylinder are in contact with the inner surface of the sphere. Show that $A^2 = 16\pi^2 x^2 \left(R^2 - x^2\right)$, where A is the curved surface area of the cylinder.

Prove that, if x is made to vary, the maximum value of A is obtained when the height of the cylinder is equal to its diameter.



from the diagram,

$$h^2 + x^2 = R^2 \quad — (1)$$

$$A = 2\pi x (2h)$$
$$= 4\pi x h$$
$$A^2 = 16\pi^2 x^2 h^2$$
$$= 16\pi^2 x^2 (R^2 - x^2) \quad \text{from (1)}$$

Differentiate w.r.t x,

$$2A \frac{dA}{dx} = 16\pi^2 R^2 \left(2x - \frac{4x^3}{R^2}\right)$$

for max A, $\frac{dA}{dx} = 0$,

$$2x - \frac{4x^3}{R^2} = 0 \Rightarrow x\left(1 - \frac{2x^2}{R^2}\right) = 0$$
$$x = 0 \text{ or } x = \frac{R}{\sqrt{2}} \quad (\because x > 0)$$

$$2A\left(\frac{d^2A}{dx^2}\right) + 2\left(\frac{dA}{dx}\right)^2 = 16\pi^2 R^2 \left(2 - \frac{12x^2}{R^2}\right)$$

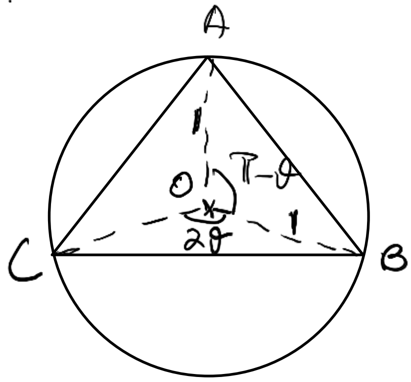
when $x = \frac{R}{\sqrt{2}}$, $\frac{dA}{dx} = 0$, $\frac{d^2A}{dx^2} < 0 \Rightarrow A$ is max at $x = \frac{R}{\sqrt{2}}$

$$h^2 = R^2 - x^2 = \frac{R^2}{2} \Rightarrow h = \frac{R}{\sqrt{2}} = x$$

$\therefore$ A is max when diameter of cylinder $(2x)$ is the same as height of cylinder $(2h)$.

(Q12) [Cambridge/J78/II/6]

An isosceles triangle ABC with AB = AC is inscribed in a fixed circle, centre O, whose radius is 1 unit. Angle $BOC = 2\theta$, where θ is acute. Express the area of the triangle ABC in terms of θ, and hence show that, as θ varies, the area is a maximum when the triangle is equilateral.



Area of $\triangle ABC$

$$A = 2\left(\tfrac{1}{2}\right)(1)(1)\sin(\pi-\theta) + \left(\tfrac{1}{2}\right)(1)(1)(\sin 2\theta)$$

$$= \sin\theta + \tfrac{1}{2}\sin 2\theta$$

$$\frac{dA}{d\theta} = \cos\theta + \cos 2\theta$$

when $\frac{dA}{d\theta} = 0$, $\cos\theta + 2\cos^2\theta - 1 = 0$

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\cos\theta = \tfrac{1}{2} \text{ or } \cos\theta = -1$$

(rejected as $2\theta > \pi$)

$$\theta = \frac{\pi}{3}$$

$$\frac{d^2A}{d\theta^2} = -\sin\theta - 2\sin 2\theta < 0 \quad \text{at } \theta = \frac{\pi}{3}$$

$\therefore$ A is max when $\theta = \frac{\pi}{3}$

$\angle BOC = \frac{2\pi}{3} = \angle BOA = \angle AOC \Rightarrow \triangle ABC$ is equilateral

as $\triangle AOC \equiv \triangle BOA \equiv \triangle BOC$

11. Integration Techniques

(Q1) [Cambridge/9709/2007]

Find the exact value of the constant k for which $\displaystyle\int_1^k \frac{1}{2x-1}\,dx = 1.$ [4]

$$\int_1^k \frac{1}{2x-1}\,dx = 1$$

$$\frac{1}{2}\Big[\ln|2x-1|\Big]_1^k = 1$$

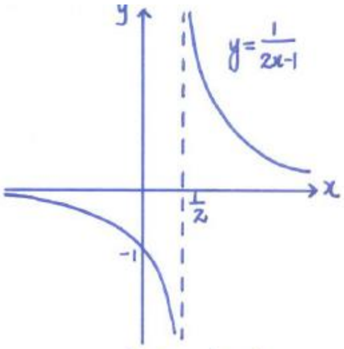
$$\frac{1}{2}\Big[\ln|2k-1| - \ln|1|\Big] = 1$$

$$\ln|2k-1| = 2$$

$$|2k-1| = e^2$$

$$2k-1 = e^2$$

$$k = \frac{1}{2}(1+e^2)$$



Note : $k > 1$

(Q2) [Cambridge/9709/2007]

Use integration by parts to show that

$$\int_2^4 \ln x\,dx = 6\ln 2 - 2.$$ [4]

$$\int_2^4 \ln x\,dx$$

$$= \Big[x\ln x\Big]_2^4 - \int_2^4 x\cdot\frac{1}{x}\,dx$$

$$= 4\ln 4 - 2\ln 2 - \Big[x\Big]_2^4$$

$$= 8\ln 2 - 2\ln 2 - 2$$

$$= 6\ln 2 - 2$$

$u = \ln x \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \qquad v = x$

(Q3) [Cambridge/9794/Specimen Paper for 2010]

Find the value of $\displaystyle\int_0^2 \frac{x-1}{x^2-2x-3}\,\mathrm{d}x$.	[6]

$$\int_0^2 \frac{x-1}{x^2-2x-3}\,dx$$

$$= \frac{1}{2}\int_0^2 \frac{2x-2}{x^2-2x-3}\,dx$$

$$= \frac{1}{2}\Big[\ln|x^2-2x-3|\Big]_0^2$$

$$= \frac{1}{2}\Big[\ln|2^2-2(2)-3| - \ln|-3|\Big]$$

$$= \frac{1}{2}\Big[\ln 3 - \ln 3\Big]$$

$$= 0$$

(Q4) [Cambridge/2003 modified]

Use the substitution $x = \tan\theta$ to find

$$\int \frac{1-x^2}{(1+x^2)^2}\mathrm{d}x \,.$$

$$\int \frac{1-x^2}{(1+x^2)^2}\,dx \quad ; \qquad x = \tan\theta, \quad \frac{dx}{d\theta} = \sec^2\theta$$

$$= \int \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2}(\sec^2\theta)\,d\theta$$

$$= \int \frac{1-\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \int (\cos^2\theta - \sin^2\theta)\,d\theta$$

$$= \int \cos 2\theta\,d\theta$$

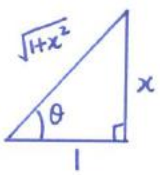
$$= \frac{1}{2}\sin 2\theta + C$$

$$= \frac{1}{2}(2\sin\theta\cos\theta) + C$$

$$= \sin\theta\cos\theta + C$$

$$= \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}} + C$$

$$= \frac{x}{1+x^2} + C$$



(Q5) [Cambridge/1997]

Use the substitution $y = x^3$ to find the value of

$$\int_0^2 \frac{x^2}{1+x^6} \mathrm{d}x .$$

$\int_0^2 \frac{x^2}{1+x^6}\,dx$;

$y = x^3$

$\frac{dy}{dx} = 3x^2$

When $x = 0,\ y = 0$

$x = 2,\ y = 8$

$\frac{x^2}{1+y^2} \cdot \frac{dx}{dy}$

$= \frac{x^2}{1+y^2} \cdot \frac{1}{3x^2}$

$= \frac{1}{3}\int_0^8 \frac{1}{1+y^2}\,dy$

$= \frac{1}{3}\left[\tan^{-1} y\right]_0^8$

$= \frac{1}{3}\left(\tan^{-1}(8) - \tan^{-1}(0)\right)$

$= \frac{1}{3}\tan^{-1}(8)$

$= 0.482$

(Q6) [Cambridge/9233/2004]

Use partial fractions to evaluate $\int_2^3 \frac{9x^2}{(x-1)^2(x+2)}\,dx$, giving your answer in an exact form. [8]

Let $\frac{9x^2}{(x-1)^2(x+2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

$9x^2 \equiv A(x-1)(x+2) + B(x+2) + C(x-1)^2$

Put $x = 1$, $3B = 9 \Rightarrow B = 3$

Put $x = -2$, $9C = 36 \Rightarrow C = 4$

Compare coefficients of x^2, $A + C = 9 \Rightarrow A = 5$

$$\int_2^3 \frac{9x^2}{(x-1)^2(x+2)}\mathrm{d}x = \int_2^3 \left[\frac{5}{x-1} + \frac{3}{(x-1)^2} + \frac{4}{x+2}\right]\mathrm{d}x$$

$$= \left[5\ln(x-1) - \frac{3}{x-1} + 4\ln(x+2)\right]_2^3$$

$$= \left(5\ln 2 - \frac{3}{2} + 4\ln 5\right) - \left(5\ln 1 - 3 + 4\ln 4\right)$$

$$= \ln 32 + \ln 625 - \ln 256 + \frac{3}{2}$$

$$= \ln\frac{(32)(625)}{256} + \frac{3}{2}$$

$$= \ln\frac{625}{8} + \frac{3}{2}$$

(Q7) [Cambridge/9233/2004]

Use the substitution $t=\sqrt{(x+1)}$ to find the exact value of $\int_0^3 x\sqrt{(x+1)}\,dx$.	[4]

$$t=\sqrt{x+1}\Rightarrow\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{2\sqrt{x+1}}=\frac{1}{2t}$$

$$x=0, t=\sqrt{1}=1$$

$$x=3, t=\sqrt{4}=2$$

$$\therefore\int_0^3 x\sqrt{x+1}\,\mathrm{d}x=\int_1^2\left(t^2-1\right)t\frac{\mathrm{d}x}{\mathrm{d}t}.\mathrm{d}t$$

$$=\int_1^2\left(t^2-1\right)t\left(2t\right)\mathrm{d}t=\int_1^2\left(2t^4-2t^2\right)\mathrm{d}t$$

$$=\left[\frac{2t^5}{5}-\frac{2t^3}{3}\right]_1^2=\left(12\frac{4}{5}-5\frac{1}{3}\right)-\left(\frac{2}{5}-\frac{2}{3}\right)=7\frac{11}{15}\text{ or }\frac{116}{15}$$

(Q8) [Cambridge/9233/2004]

Write down constants A and B such that, for all values of x,

$$6x+16=A(2x+4)+B. \qquad [1]$$

Hence find the exact value of $\displaystyle\int_{-2}^{1}\frac{6x+16}{x^2+4x+13}\,dx$. [6]

$$6x+16=3\left(2x+4\right)+4\Rightarrow A=3\text{ and }B=4$$

$$\int_{-2}^{1}\frac{6x+16}{x^2+4x+13}\mathrm{d}x=\int_{-2}^{1}\frac{3\left(2x+4\right)+4}{\left(x+2\right)^2+9}\mathrm{d}x=3\int_{-2}^{1}\frac{2x+4}{\left(x+2\right)^2+9}\mathrm{d}x+4\int_{-2}^{1}\frac{1}{\left(x+2\right)^2+3^2}\mathrm{d}x$$

$$=3\left[\ln\left\{\left(x+2\right)^2+9\right\}\right]_{-2}^{1}+4\left[\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\right]_{-2}^{1}$$

$$=3\left[\ln 18-\ln 9\right]+4\left[\frac{1}{3}\left(\frac{\pi}{4}\right)-0\right]=3\ln 2+\frac{\pi}{3}$$

(Q9) [Cambridge/9233/2005]

(i) State the derivative of $\sin\left(x^2\right)$. [1]

(ii) Find $\int x^3\cos\left(x^2\right)\,\mathrm{dx}$. [3]

(i) $\dfrac{d}{dx}\sin\left(x^2\right)=2x\cos\left(x^2\right)$

(ii) $\displaystyle\int x^3\cos\left(x^2\right)\,\mathrm{d}x=\int\left(\frac{1}{2}x^2\right)2x\cos\left(x^2\right)\,\mathrm{d}x$

$$=\frac{1}{2}x^2\sin\left(x^2\right)-\int x\sin\left(x^2\right)\,\mathrm{d}x$$

$$=\frac{1}{2}x^2\sin\left(x^2\right)+\frac{1}{2}\cos\left(x^2\right)+C$$

(Q10) [Cambridge/N78/I/20]

Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\left(x^2\right)\right)$ and hence , or otherwise, find $\int x\tan^{-1}\left(x^2\right)\mathrm{d}x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\left(x^2\right)\right)$$

$$=\frac{2x}{1+x^4}$$

$$\int x\tan^{-1}\left(x^2\right)\mathrm{d}x$$

$$=\frac{x^2}{2}\tan^{-1}\left(x^2\right)-\int\frac{x^2}{2}\left(\frac{2x}{1+x^4}\right)\mathrm{d}x$$

$$=\frac{x^2}{2}\tan^{-1}\left(x^2\right)-\frac{1}{4}\int\frac{4x^3}{1+x^4}\mathrm{d}x$$

$$=\frac{x^2}{2}\tan^{-1}\left(x^2\right)-\frac{1}{4}\ln(1+x^4)+C$$

12. Definite Integrals, Area and Volume

(Q1) [Cambridge/H2 Maths 2007]

(i) Find the exact value of $\int_0^{\frac{5}{3}\pi} \sin^2 x\,dx$. Hence find the exact value of $\int_0^{\frac{5}{3}\pi} \cos^2 x\,dx$. [6]

(ii) The region R is bounded by the curve $y = x^2 \sin x$, the line $x = \frac{1}{2}\pi$ and the part of the x-axis between 0 and $\frac{1}{2}\pi$. Find

(a) the exact area of R, [5]

(b) the numerical value of the volume of revolution formed when R is rotated completely about the x-axis, giving your answer correct to 3 decimal places. [2]

(i)

$$\int_0^{\frac{5}{3}\pi} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{5\pi}{3}} (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{5\pi}{3}}$$

$$= \frac{1}{2}\left[\frac{5\pi}{3} - \frac{1}{2}\sin\frac{10\pi}{3}\right] \qquad \sin\left(\frac{10\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right) = -\sin\frac{\pi}{3}$$

$$= \frac{5\pi}{6} + \frac{\sqrt{3}}{8}$$

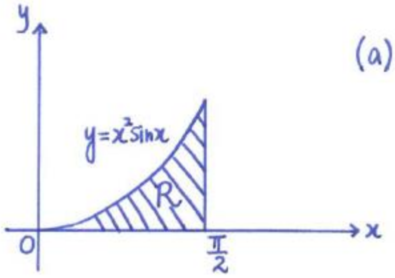
$$\int_0^{\frac{5\pi}{3}} \cos^2 x\,dx = \int_0^{\frac{5\pi}{3}} (1-\sin^2 x)\,dx$$

$$= [x]_0^{\frac{5\pi}{3}} - \int_0^{\frac{5\pi}{3}} \sin^2 x\,dx$$

$$= \frac{5\pi}{3} - \left[\frac{5\pi}{6} + \frac{\sqrt{3}}{8}\right]$$

$$= \frac{5\pi}{6} - \frac{\sqrt{3}}{8}$$

(ii)



(a) Area of R

$u = x^2 \quad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2x \quad v = -\cos x$

$$= \int_0^{\frac{\pi}{2}} x^2 \sin x\,dx$$

$$= \left[-x^2\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 2x\cos x\,dx$$

$$= 2\left\{\left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx\right\}$$

$$= 2\left\{\frac{\pi}{2} - \left[-\cos x\right]_0^{\frac{\pi}{2}}\right\}$$

$$= \pi - 2 \text{ units}^2$$

(b) Volume $= \pi\int_0^{\frac{\pi}{2}} (x^2\sin x)^2\,dx$

$= 5.391$ units^3 using GC.

(Q2) [Cambridge/H1 Maths 2007]

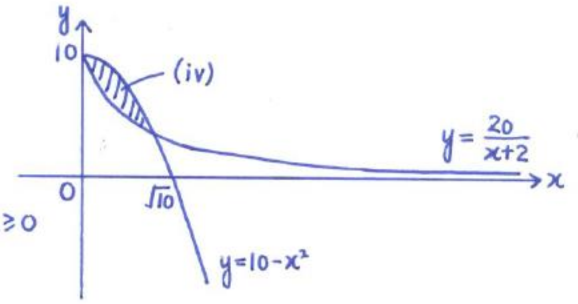
(i) Sketch, for $x \geqslant 0$, the graphs of $y = \dfrac{20}{x+2}$ and $y = 10 - x^2$ on the same axes. [2]

(ii) The graphs intersect on the y-axis. Find, correct to 3 decimal places, the x-coordinate of the point of intersection for which $x > 0$. [1]

(iii) Find $\displaystyle\int \frac{20}{x+2}\,\mathrm{d}x$ and $\displaystyle\int (10 - x^2)\,\mathrm{d}x$. [3]

(iv) Use your answers to parts **(ii)** and **(iii)** to find the area of the region, in the first quadrant, between the two graphs. [2]

(i)



Note : $x \geq 0$

(ii) Using GC, x-coordinate of pt of intersection for $x > 0$ is $2.31662 = 2.317$ (3 dp)

(iii) $\displaystyle\int \frac{20}{x+2}\,dx = 20\ln|x+2| + C_1$

$\displaystyle\int (10 - x^2)\,dx = 10x - \frac{x^3}{3} + C_2$

(iv) Required area $= \displaystyle\int_0^{2.31662} \left[(10 - x^2) - \frac{20}{x+2}\right] dx$

$= \left[10x - \dfrac{x^3}{3} - 20\ln|x+2|\right]_0^{2.31662}$

$= \left[10(2.31662) - \frac{1}{3}(2.31662)^3 - 20\ln(4.31662)\right] + 20\ln 2$

$= 3.64$ units^2

(Q3) [Cambridge/2006]

Given that $z = \dfrac{x}{(x^2+32)^{\frac{1}{2}}}$, show that $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{32}{(x^2+32)^{\frac{3}{2}}}$. [3]

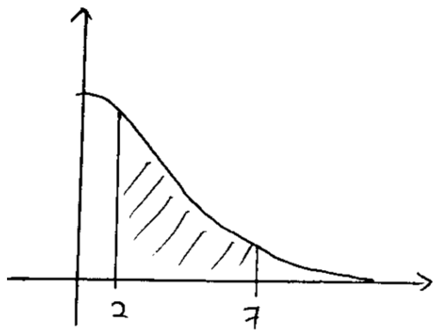
Find the exact value of the area of the region bounded by the curve $y = \dfrac{1}{(x^2+32)^{\frac{3}{2}}}$,

the x-axis and the lines $x = 2$ and $x = 7$. [3]

$$z = \frac{x}{(x^2+32)^{\frac{1}{2}}} \Rightarrow \frac{dz}{dx} = \frac{(x^2+32)^{\frac{1}{2}} - x(\frac{1}{2})(x^2+32)^{-\frac{1}{2}}(2x)}{x^2+32}$$

$$= \frac{(x^2+32)^{-\frac{1}{2}}[x^2+32-x^2]}{x^2+32} = \frac{32}{(x^2+32)^{3/2}}$$

(shown)



$$\text{Area} = \int_2^7 \frac{1}{(x^2+32)^{3/2}}\,dx$$

$$= \frac{1}{32}\left[\frac{x}{(x^2+32)^{\frac{1}{2}}}\right]_2^7$$

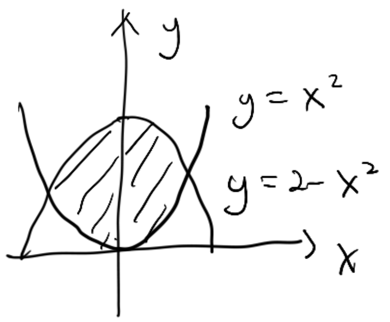
$$= \frac{1}{32}\left[\frac{7}{9} - \frac{2}{6}\right] = \frac{1}{72}$$

(Q4) [Cambridge/1984]

The region R is bounded by the curves of $y = x^2$ and $y = 2 - x^2$. Calculate

(a) the area of R.

(b) the volume of the solid generated when R is rotated through π radians about the y-axis.



To find pt of intersection,

$$x^2 = 2 - x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\text{Area of } R = 2\int_0^1 (2-x^2) - (x^2)\,dx$$

$$= 2\int_0^1 2 - 2x^2\,dx$$

$$= 4\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{8}{3} \text{ units}^2$$

$$\text{Volume of solid} = \pi\int_0^1 x^2\,dy + \pi\int_1^2 x^2\,dy$$

$$= \pi\int_0^1 y\,dy + \pi\int_1^2 (2-y)\,dy$$

$$= \pi\left[\frac{1}{2}\right] + \pi\left[2y - \frac{y^2}{2}\right]_1^2$$

$$= \frac{\pi}{2} + \pi\left(2 - 2 + \frac{1}{2}\right) = \pi \text{ units}^3$$

(Q5) [Cambridge/9709/2010]

(i) $9 - x^3 = \frac{8}{x^3}$

$9x^3 - x^6 = 8$

$x^6 - 9x^3 + 8 = 0$

$\therefore$ $x = a$ & $x = b$ are roots of the eqn $x^6 - 9x^3 + 8 = 0$

$(x^3)^2 - 9x^3 + 8 = 0$

$(x^3 - 1)(x^3 - 8) = 0$

$x^3 = 1$ OR $x^3 = 8$

$x = 1$ $\quad$ $x = 2$

$\therefore$ $a = 1$, $b = 2$

(ii) Required area

$= \int_1^2 (9 - x^3 - \frac{8}{x^3})\,dx$

$= \left[9x - \frac{x^4}{4} + \frac{4}{x^2}\right]$

$= [18 - 4 + 1] - [9 - \frac{1}{4} + 4]$

$= \frac{9}{4}$ units^2

(ii) $y = 9 - x^3 \Rightarrow \frac{dy}{dx} = -3x^2$

$y = \frac{8}{x^3} \Rightarrow \frac{dy}{dx} = -\frac{24}{x^4}$

At $x = c$, $\quad -\frac{24}{c^4} = -3c^2$

$c^6 = 8$

$c = \sqrt{2}$

13. Differential Equations

(Q1) [Cambridge]

In a chemical reaction a compound X is formed from a compound Y. The masses in grams of X and Y present at time t seconds after the start of the reaction are x and y respectively. The sum of the two masses is equal to 100 grams throughout the reaction. At any time, the rate of formation of X is proportional to the mass of Y at that time. When $t = 0$, $x = 5$ and $\frac{dx}{dt} = 1.9$.

(i) Show that x satisfies the differential equation

$$\frac{dx}{dt} = 0.02(100 - x). \qquad [2]$$

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

(iii) State what happens to the value of x as t becomes very large. [1]

(i) $x + y = 100$

$\frac{dx}{dt} \propto y \Rightarrow \frac{dx}{dt} = ky = k(100-x)$

When $t=0$, $x=5$, $\frac{dx}{dt} = 1.9$,

$1.9 = k(100-5)$

$k = 0.02$

$\therefore \frac{dx}{dt} = 0.02(100-x)$

(ii) $\int \frac{1}{100-x}\, dx = \int 0.02\, dt$

$-\ln|100-x| = 0.02t + c$

$\ln|100-x| = -0.02t - c$

$|100-x| = e^{-0.02t} \cdot e^{-c}$

$100 - x = \pm e^{-c} \cdot e^{-0.02t}$

$= Ae^{-0.02t}$ where $A = \pm e^{-c}$

$x = 100 - Ae^{-0.02t}$

When $t=0$, $x=5$

$5 = 100 - A$

$\Rightarrow A = 95$

$\therefore x = 100 - 95e^{-0.02t}$

(iii) As $t \to \infty$, $e^{-0.02t} \to 0$

$\therefore x \to 100.$

(Q2) [Cambridge/H2 Maths 2007]

The current I in an electric circuit at time t satisfies the differential equation

$$4\frac{dI}{dt} = 2 - 3I.$$

Find I in terms of t, given that $I = 2$ when $t = 0$. [6]

State what happens to the current in this circuit for large values of t. [1]

$$4\frac{dI}{dt} = 2-3I$$

$$\int \frac{4}{2-3I}\,dI = \int dt$$

$$4\left(-\frac{1}{3}\right)\ln|2-3I| = t + C'$$

$$\ln|2-3I| = -\frac{3}{4}t + C$$

$$|2-3I| = e^{-\frac{3}{4}t}\cdot e^{C}$$

$$2-3I = \pm e^{C}\cdot e^{-\frac{3}{4}t}$$

$$= Ae^{-\frac{3}{4}t} \quad \text{where } A = \pm e^{C}$$

$$I = \frac{1}{3}\left(2 - Ae^{-\frac{3}{4}t}\right)$$

When $t = 0$, $I = 2$,

$$2 = \frac{1}{3}(2 - A)$$

$$A = -4$$

$$\therefore\ I = \frac{1}{3}\left(2 + 4e^{-\frac{3}{4}t}\right)$$

$$= \frac{2}{3}\left(1 + 2e^{-\frac{3}{4}t}\right)$$

As $t \to \infty$, $I \to \frac{2}{3}$ since $e^{-\frac{3}{4}t} \to 0$.

(Q3) [Cambridge/9709/2004]

A rectangular reservoir has a horizontal base of area $1000\ \text{m}^2$. At time $t = 0$, it is empty and water begins to flow into it at a constant rate of $30\ \text{m}^3\ \text{s}^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where h m is the depth of the water at time t s. When $h = 1$, $\frac{dh}{dt} = 0.02$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = 0.01(3 - \sqrt{h}).$$ [3]

It is given that, after making the substitution $x = 3 - \sqrt{h}$, the equation in part **(i)** becomes

$$(x - 3)\frac{dx}{dt} = 0.005x.$$

(ii) Using the fact that $x = 3$ when $t = 0$, solve this differential equation, obtaining an expression for t in terms of x. [5]

(iii) Find the time at which the depth of water reaches 4 m. [2]

(i) Volume of reservoir, $V = 1000h$

$\therefore \frac{dV}{dt} = 1000\frac{dh}{dt}$ — ①

Also, $\frac{dV}{dt} = \frac{dV_{in}}{dt} - \frac{dV_{out}}{dt}$

$\frac{dV}{dt} = 30 - k\sqrt{h}$ — ②

① = ② : $1000\frac{dh}{dt} = 30 - k\sqrt{h}$

$\Rightarrow \frac{dh}{dt} = 0.03 - \frac{k}{1000}\sqrt{h}$

When $h = 1$, $\frac{dh}{dt} = 0.02$

$\therefore 0.02 = 0.03 - \frac{k}{1000} \Rightarrow k = 10$

$\frac{dh}{dt} = 0.03 - 0.01\sqrt{h}$

$= 0.01(3 - \sqrt{h})$

$x = 3 - \sqrt{h}$

$\frac{dx}{dt} = -\frac{1}{2}h^{-\frac{1}{2}}\frac{dh}{dt}$

$\frac{dh}{dt} = -2\sqrt{h}\frac{dx}{dt}$

$\therefore 2(x-3)\frac{dx}{dt} = 0.01(x)$

$(x-3)\frac{dx}{dt} = 0.005x$

(ii) $\int \frac{x-3}{x}\,dx = \int 0.005\,dt$

$\int 1 - \frac{3}{x}\,dx = \int 0.005\,dt$

$x - 3\ln|x| = 0.005t + c$

When $t = 0$, $x = 3$,

$3 - 3\ln 3 = c$

$\therefore 0.005t = x - 3\ln|x| - 3 + 3\ln 3$

$= (x-3) + 3\ln\left|\frac{3}{x}\right|$

$t = 200\left[x - 3 + 3\ln\left|\frac{3}{x}\right|\right]$

(iii) When $h = 4\text{m}$,

$x = 3 - \sqrt{4}$

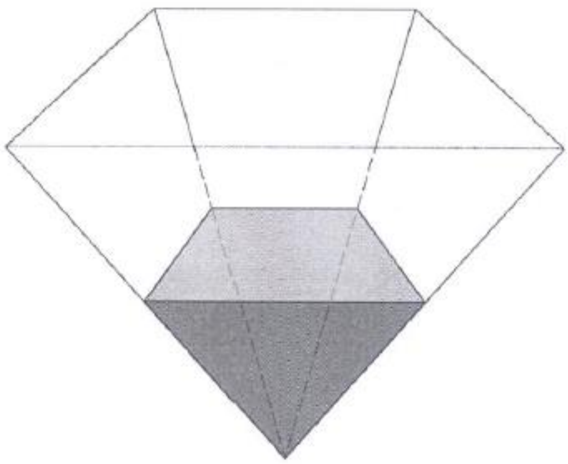
$= 1$

$t = 200[1 - 3 + 3\ln 3]$

$= 200(3\ln 3 - 2)$

$= 259\text{s}$

(Q4) [Cambridge/9709/2009]



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is V m^3 and the depth of liquid is h m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m^3 per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\frac{20h^2}{100 - h^2} \equiv -20 + \frac{2000}{(10-h)(10+h)}$. [1]

(iii) Hence solve the differential equation in part **(i)**, obtaining an expression for t in terms of h. [5]

(i) $V = \frac{4}{3}h^3$

$$\frac{dV}{dt} = 4h^2\frac{dh}{dt}$$

$$20 - kh^2 = 4h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{20}{4h^2} - \frac{kh^2}{4h^2}$$

$$= \frac{5}{h^2} - \frac{k}{4}$$

When $h=1$, $\frac{dh}{dt} = 4.95$

$$4.95 = 5 - \frac{k}{4}$$

$$k = \frac{1}{5}$$

$$\therefore \frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}$$

(ii) $$\frac{20h^2}{100-h^2} = \frac{(-20)(100-h^2) + 2000}{100-h^2}$$

$$= -20 + \frac{2000}{(10-h)(10+h)}$$

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}$$

$$= \frac{100-h^2}{20h^2}$$

$$\int \frac{20h^2}{100-h^2}\,dh = \int dt$$

$$\int -20 + \frac{2000}{(10-h)(10+h)}\,dh = \int dt$$

$$\int -20 + \frac{100}{(10-h)} + \frac{100}{(10+h)}\,dh = \int dt$$

$$-20h - 100\ln|10-h| + 100\ln|10+h| = t + c$$

$$-20h + 100\ln\left|\frac{10+h}{10-h}\right| = t + c$$

When $t=0$, $h=0$,

$$c = 0$$

$$\therefore t = 100\ln\left|\frac{10+h}{10-h}\right| - 20h$$

14. Complex Numbers

(Q1) [Cambridge/9794/Specimen Paper for 2010] (do not attempt (ii), out of syllabus)

(i) Determine the roots p and q of the equation $x^2 - 6x + 13 = 0$. If p is the root whose argument is an acute angle, find an equation whose roots are $\mathrm{i}p$ and $-\mathrm{i}q$. [5]

(i) $x^2-6x+13=0$

$$x = \frac{6 \pm \sqrt{36-4(13)}}{2}$$

$$= \frac{6 \pm 4i}{2}$$

$$= 3 \pm 2i$$

Since p is the root whose argument is an acute angle,

$p = 3+2i$ & $q = 3-2i$

Required eqn whose roots are ip & $-iq$ is

$$(x - i(3+2i))(x-(-i)(3-2i)) = 0$$

$$(x+2-3i)(x+2+3i) = 0$$

$$(x+2)^2 - (3i)^2 = 0$$

$$x^2 + 4x + 13 = 0$$

(Q2) [Cambridge/H2 Maths 2007]

The complex number w is such that $ww^* + 2w = 3 + 4\mathrm{i}$, where w^* is the complex conjugate of w. Find w in the form $a + \mathrm{i}b$, where a and b are real. [4]

Let $w = a+bi$

$$(a+bi)(a-bi) + 2(a+bi) = 3+4i$$

$$(a^2+b^2+2a) + 2bi = 3+4i$$

$$2b = 4 \Rightarrow b = 2$$

$$a^2 + 4 + 2a = 3$$

$$a^2 + 2a + 1 = 0$$

$$(a+1)^2 = 0 \Rightarrow a = -1$$

$\therefore w = -1+2i$

(Q3) [Cambridge]

(b) Given that

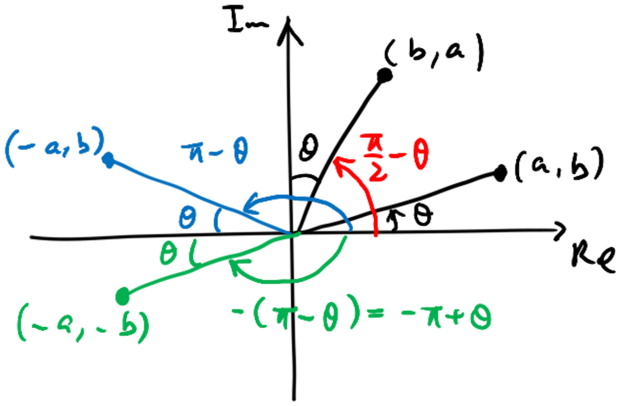
$$\arg(a + ib) = \theta,$$

where $a > 0$, $b > 0$, find, in terms of θ and π, the value of

(i) $\arg(-a + ib)$, [1]

(ii) $\arg(-a - ib)$, [1]

(iii) $\arg(b + ia)$. [1]



(i) $\arg(-a+ib) = \pi - \theta$

(ii) $\arg(-a-ib) = -\pi + \theta$

(iii) $\arg(b+ia) = \frac{\pi}{2} - \theta$

(Q4) [Cambridge/2004]

The use of calculator is not allowed in this question.

Express $(3-\mathrm{i})^2$ in the form $a + ib$. [1]

Hence or otherwise find the roots of the equation $(z+\mathrm{i})^2 = -8 + 6\mathrm{i}$. [3]

$$(3-\mathrm{i})^2 = 9 - 6\mathrm{i} + \mathrm{i}^2 = 8 - 6\mathrm{i}$$

$$(z+\mathrm{i})^2 = -8 + 6\mathrm{i} = -(8-6\mathrm{i}) = \mathrm{i}^2(8-6\mathrm{i}) = \mathrm{i}^2(3-\mathrm{i})^2$$

$$\Rightarrow (z+\mathrm{i})^2 - \mathrm{i}^2(3-\mathrm{i})^2 = 0 \qquad [\mathrm{i}(3-\mathrm{i})]^2 = 0$$

$$\Rightarrow (z+\mathrm{i})^2 - (3\mathrm{i}+1)^2 = 0$$

$$\Rightarrow (z+\mathrm{i}+3\mathrm{i}+1)(z+\mathrm{i}-3\mathrm{i}-1) = 0$$

$$\Rightarrow (z+1+4\mathrm{i})(z-1-2\mathrm{i}) = 0$$

$$\Rightarrow z = -1-4\mathrm{i} \text{ or } 1+2\mathrm{i}$$

(Q5) [Cambridge/9709/2009]

The complex number $-2+\mathrm{i}$ is denoted by u.

(i) Given that u is a root of the equation $x^3 - 11x - k = 0$, where k is real, find the value of k. [3]

(ii) Write down the other complex root of this equation. [1]

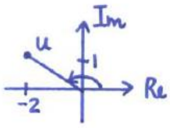
(iii) Find the modulus and argument of u. [2]

(i) Since $-2+i$ is a root,

$$(-2+i)^3 - 11(-2+i) - k = 0$$
$$-2+11i + 22-11i - k = 0$$
$$k = 20$$

(ii) The other complex root is the conjugate of $-2+i$, i.e. $-2-i$.

(iii) $|u| = \sqrt{(-2)^2+1} = \sqrt{5}$



$\arg(u) = \pi - \tan^{-1}\left(\frac{1}{2}\right)$

$= 2.68$ rad