Mathematical Formulae

Compound interest

$$\text{Total amount} = P\left(1+\frac{r}{100}\right)^n$$

Mensuration

$$\text{Curved surface area of a cone} = \pi rl$$

$$\text{Surface area of a sphere} = 4\pi r^2$$

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h$$

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

$$\text{Area of triangle } ABC = \frac{1}{2}ab\sin C$$

$$\text{Arc length} = r\theta\text{ , where } \theta \text{ is in radians}$$

$$\text{Sector area} = \frac{1}{2}r^2\theta\text{, where } \theta \text{ is in radians}$$

Trigonometry

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Statistics

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f}$$

$$\text{Standard deviation} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$$

Candidate Name ____________________

Class	Index Number

ST ANDREW'S SECONDARY SCHOOL

PRELIMINARY EXAMINATION 2020

SECONDARY 4 EXPRESS/ 5 NORMAL ACADEMIC

MATHEMATICS **4048/02**

Paper 2

FRIDAY 21 AUGUST 2020 **2 hours 30 minutes**

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
If working is needed for any question it must be shown with the answer.
Omission of essential working will result in loss of marks.
The use of an approved scientific calculator is expected, where appropriate.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures. Give answers in degrees to one decimal place.
For π, use either your calculator value or 3.142, unless the question requires the answer in terms of π.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total of the marks for this paper is 100.

This question paper consists of 27 printed pages.

2 The first three terms in a sequence of numbers are given below.

$$T_1 = \frac{1}{1\times2} = \frac{1}{2}$$

$$T_2 = \frac{1}{1\times2} + \frac{1}{2\times3} = \frac{2}{3}$$

$$T_3 = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} = \frac{3}{4}$$

(a) Write down an expression for T_n in terms of n.

Answer $\frac{n}{n+1}$ [1]

(b) Hence, find the exact value of $\frac{1}{3\times4} + \frac{1}{4\times5} + \frac{1}{5\times6} + \ldots + \frac{1}{53\times54}$.

$\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \frac{1}{5\times6} + \ldots + \frac{1}{53\times54} - \left(\frac{1}{1\times2} + \frac{1}{2\times3}\right)$

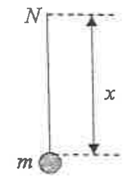
$= \frac{53}{54} - \frac{2}{3}$

$= \frac{17}{54}$

Answer $\frac{17}{54}$ [2]

1 An elastic string hangs from a nail N. When a mass of m grams is attached to its lower end, the elastic string is stretched so that its total length is x cm, as shown in the diagram. The table below shows the results of two experiments.

Length (x cm)	43	49
Mass (m grams)	50	80



It is known that x and m are connected by the equation $x = c + dm$, where c and d are constants.

(a) Use this information to write down two equations in c and d.

Answer $43 = c + 50d$

$49 = c + 80d$ [1]

(b) Solve the equations to find the value of c and the value of d.

$43 = c + 50d$ —— (1)

$49 = c + 80d$ —— (2)

(2) − (1) $6 = 30d$

$d = \frac{1}{5}$

subst $d = \frac{1}{5}$ into (1)

$43 = c + 50\left(\frac{1}{5}\right)$

$c = 33$

Answer $c =$ 33

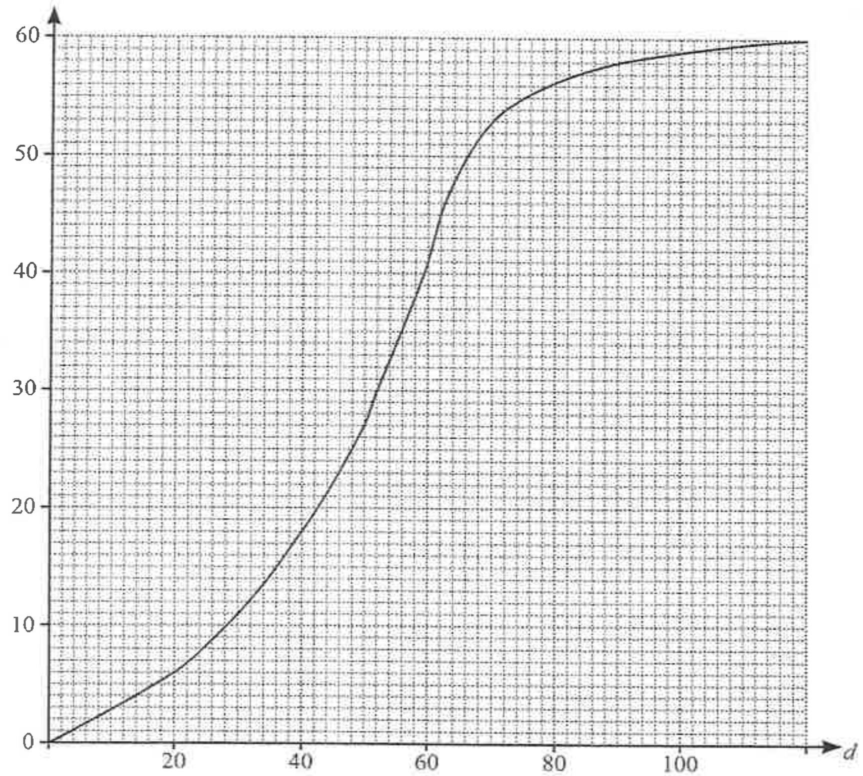
$d =$ $\frac{1}{3}$ [3]

(c) What does the value of c represent? [1]

Answer The length of the elastic string before any weight is added

[Turn Over

3 The cumulative frequency curve shows information about the distance, d km, travelled by 60 female cyclists in one weekend.



(a) Use the cumulative frequency curve to find an estimate of

(i) the median,

Answer 52 km [1]

(ii) the interquartile range,

Interquartile Range $= Q_3 - Q_1$

$= 62 - 36$

$= 26$

Answer 26 km [2]

(iii) the eightieth percentile.

$\frac{80}{100} \times 60$

$= 48$

$\therefore P_{80} = 64$

Answer 64 km [1]

(c) The first three terms of another sequence is $\frac{1}{2}$, $\frac{4}{3}$ and $\frac{9}{4}$. Find an expression for P_n, in terms of n, the nth term of this sequence.

Answer $\frac{n^2}{n+1}$ [1]

(d) Determine and explain whether there is a value of n for which $\frac{P_n}{T_n} = \frac{5}{3}$. [2]

Answer

$\frac{P_n}{T_n} = \frac{n^2}{n+1} \div \frac{n}{n+1}$

$\frac{5}{3} = \frac{n^2}{n+1} \times \frac{n+1}{n}$

$= n$

Since n is not a positive integer, there is no values of n for which $\frac{P_n}{T_n} = \frac{5}{3}$

4 A tank holds 2000 litres of oil.
A small pump can add oil to the tank at a rate of x litres per minute. A large pump can add oil to the tank at a rate of $(x + 10)$ litres per minute.

(a) Write down an expression, in terms of x, for the number of minutes the small pump takes to fill the empty tank.

Answer $\frac{2000}{x}$ [1]

(b) Write down an expression, in terms of x, for the number of minutes the large pump takes to fill the empty tank.

Answer $\frac{2000}{x+10}$ [1]

(c) It takes 15 minutes longer to fill the empty tank using the small pump than it does with the large pump. Form an equation in x and show that it simplifies to $3x^2 + 30x - 4000 = 0$ [3]

Answer

$$\frac{2000}{x} - \frac{2000}{x+10} = 15$$

$$\frac{2000(x+10) - 2000x}{x(x+10)} = 15$$

$$2000 = 15x^2 + 150x$$

$$3x^2 + 30x - 4000 = 0$$

(b) Use the cumulative frequency curve to find the number of female cyclists who travelled

(i) $20 < d \le 40$,

No of cyclists: $18 - 6 = 12$

Answer 12 [1]

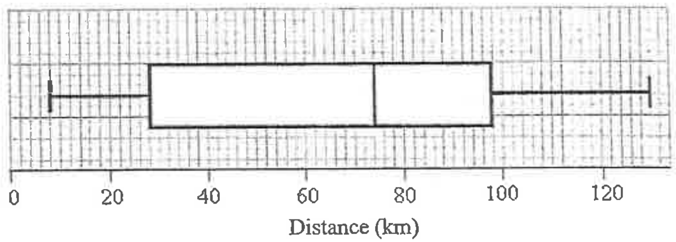
(ii) $d \geq 90$.

No of cyclists $= 60 - 58$

$= 2$

Answer 2 [1]

(c) The distribution of the distances travelled by the male cyclists, for the same weekend, is shown in the box and whisker plot below.



(i) If 75% of the male cyclists travelled s km or more, find s.

$s = Q_1$

$= 28$

Answer 28 [1]

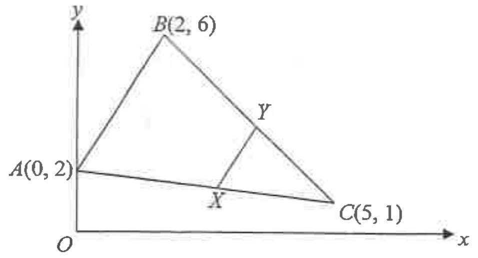
(ii) Make two comments comparing the distribution of the distances travelled by the males with the distribution of the distances travelled by the females. [2]

Answer

On average the males travelled more distance because their median is larger (74 vs 52)
The spread of distance travelled is larger for the males because their interquartile range is larger (70 vs 26)

[Turn Over

5 In the diagram, the coordinates of A, B and C are (0, 2), (2, 6) and (5, 1) respectively. X and Y lie on AC and BC respectively. XY is parallel to AB.



(a) Find BC.

$BC = \sqrt{(2-5)^2 + (6-1)^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34}$

$= 5.8309 = 5.83$

Answer 5.83 units [2]

(b) Show that the equation of BC is $3y = 28 - 5x$. [2]

Answer

grad $= \frac{6-1}{2-5}$

$= -\frac{5}{3}$

$y = -\frac{5}{3}x + c$

Sub (2, 6)

$6 = -\frac{5}{3}(2) + c$

$c = \frac{28}{3}$

$\therefore y = -\frac{5}{3}x + \frac{28}{3}$

$3y = 28 - 5x$

(c) Find the coordinates of the point P on the line segment BC such that the shortest distances from P to each of the axes are the same.

Let $x = y$

$3y = 28 - 5y$

$8y = 28$

$y = 3\frac{1}{2}$

$\therefore (3\frac{1}{2}, 3\frac{1}{2})$

Answer ($3\frac{1}{2}$, $3\frac{1}{2}$) [2]

(d) Solve the equation $3x^2 + 30x - 4000 = 0$. Give your answers correct to three decimal places.

$x = \frac{-30 \pm \sqrt{30^2 - 4(3)(-4000)}}{2(3)}$

$= 31.8555$ or -41.8555

$= 31.856$ or -41.856

Answer 31.856 or -41.856 [3]

(e) Hence, find the length of time the large pump takes to fill the empty tank. Give your answer in minutes and seconds, correct to the nearest second.

Time $= \frac{2000}{31.856 + 10}$

$= 47.7828$ min

$= 47$ min 0.7828×60 s

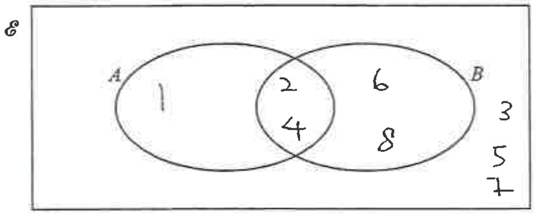
$= 47$ min 46.9 s

$= 47$ min 47 s

Answer 47 minutes 47 seconds [2]

6 **(a)** $\mathscr{E} = \{x: x \text{ is an integer } 1 \le x \le 8\}$
$A = \{x: x \text{ is a factor of } 4\}$
$B = \{x: x \text{ is a multiple of } 2\}$

(i) Complete the Venn diagram. [2]



(ii) List the elements in $(A \cup B)'$.

Answer $\{3, 5, 7\}$ [1]

(iii) List all the subsets of $A \cap B$.

Answer ϕ, $\{2\}$, $\{4\}$, $\{2, 4\}$ [1]

(iv) Describe in words the set B'. [1]

They are odd numbers

(v) Two integers are selected randomly from $\mathscr{E}$. Find the probability that one is from B and the other is from B'. Give your answer as a fraction in its lowest terms.

$P(B \text{ and } B') = P(BB') + P(B'B)$

$= \frac{4}{8} \times \frac{4}{7} \times 2$

$= \frac{4}{7}$

Answer $\frac{4}{7}$ [1]

(d) Given that $5CY = 2CB$, find, as a fraction in its lowest terms, the value of

(i) $\dfrac{\text{area of triangle } XCY}{\text{area of triangle } AXY}$,

$\frac{CY}{CB} = \frac{2}{5}$

$\frac{\frac{1}{2}(2)(h)}{\frac{1}{2}(3)(h)} = \frac{2}{3}$

XC AX

$\frac{\text{area } \triangle XCY}{\text{area } \triangle AXY} = \frac{b_1}{b_2} = \frac{XC}{AX} = \frac{2}{3}$

Answer $\frac{2}{3}$ [1]

(ii) $\dfrac{\text{area of triangle } XCY}{\text{area of quadrilateral } AXYB}$.

$\frac{\text{area } \triangle XCY}{\text{area } \triangle ACB} = \left(\frac{l_1}{l_2}\right)^2$

$= \left(\frac{2}{5}\right)^2$

$= \frac{4}{25}$

$\frac{\text{area } \triangle XCY}{\text{area } AXYB} = \frac{4}{21}$

Answer $\frac{4}{21}$ [2]

(e) $ABDC$ is a trapezium where AB is parallel to CD. The area of trapezium $ABDC$ is 5 times the area of triangle ABC. Show that $CD = 4AB$. [2]

Answer

$\frac{\text{area } \triangle ABC}{\text{area } \triangle BCD} = \frac{1}{4}$

$\frac{\frac{1}{2} \times AB \times h}{\frac{1}{2} \times CD \times h} = \frac{1}{4}$

$\frac{AB}{CD} = \frac{1}{4}$

$4AB = CD$

[Turn Over

(b) Peter plays a board game.
In the game, a **throw** is rolling the **two** fair 6-sided dice, each with faces numbered 1, 2, 3, 4, 5 and 6, and then adding the numbers on their top faces. This total is the number of spaces to move on the board.

(i) Determine the most likely number of spaces that Peter will move with his next throw using a **possibility diagram**.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Answer .... 7 .... [1]

(ii) To win the game Peter must move **exactly** to the 100th space.

95	96	97	98	99 Go back 3 spaces	100 WIN

Peter is on the 97th space.
If his next throw takes him to 99, he has to move back to 96.
If his next throw takes him over 100, he stays on 97.

Giving your answer as a fraction in its lowest terms, find the probability that

(a) his next throw takes him to 99,

Answer .... $\frac{1}{36}$ .... [1]

[Turn Over

(b) his next throw takes him over 100 and reaches 100 in his second throw,

P (1st throw over 100 and 100 in 2nd throw)

$= \left(1 - \frac{3}{36}\right) \times \frac{2}{36}$

$= \frac{33}{36} \times \frac{2}{36}$

$= \frac{11}{216}$

Answer .... $\frac{11}{216}$ .... [2]

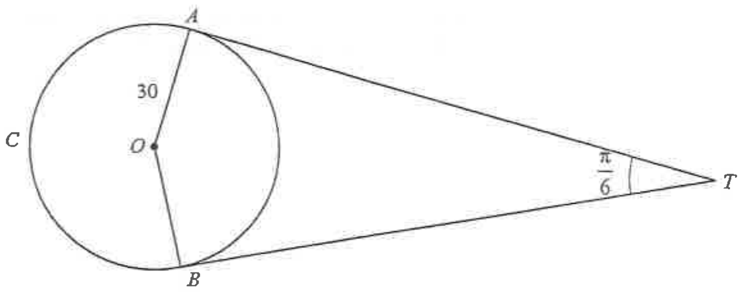
(c) he reaches 100 in either of his next two throws.

$P(100) = \frac{2}{36} + \frac{11}{216} + \left(\frac{1}{36} \times \frac{3}{36}\right)$

$= \frac{47}{432}$

Answer .... $\frac{47}{432}$ .... [2]

(b) The diagram shows the cross-section of a circular disc, centre O. The radius of the disc is 30 cm. A string $TACB$ is wrapped tightly round the disc from T such that TA and TB are straight lines. Angle $ATB = \frac{\pi}{6}$ radians.



Calculate

(i) the area of the major sector ACB,

$\therefore \angle OAT = \angle OBT = \frac{\pi}{2}$ (rad ⊥ tan)

$\angle AOB = 2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

reflex $\angle AOB = 2\pi - \frac{5}{6}\pi$ (∠s at a pt)

$= \frac{7}{6}\pi$

area of major sector $ACB = \frac{1}{2}(30)^2(\frac{7}{6}\pi)$

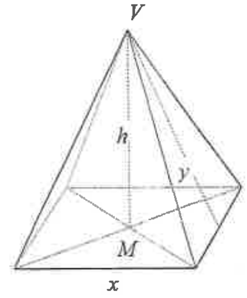
$= 525\pi$

$= 1649.33$

$= 1650$

Answer 1650 cm^2 [2]

7 (a) A pyramid tent has a square base of side x m. The diagonals of the square base intersect at M. The height of the pyramid, VM, is h m. The distance between the top of the pyramid and the midpoint of each edge of the base is y m.



If 8 m^2 of material is used to make the four triangular faces of each tent, show that

(i) $y = \frac{4}{x}$, [2]

Answer

area of 4 △s $= 4 \times \frac{1}{2}xy$

$8 = 2xy$

$y = \frac{8}{2x}$

$= \frac{4}{x}$

(ii) $h^2 = \frac{16}{x^2} - \frac{x^2}{4}$. [2]

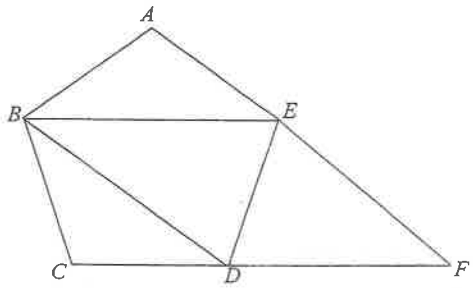
Answer

$h^2 = y^2 - (\frac{1}{2}x)^2$ (Pythagoras' Theorem)

$= (\frac{4}{x})^2 - \frac{x^2}{4}$

$= \frac{16}{x^2} - \frac{x^2}{4}$

8 In the diagram, $ABCDE$ is a regular pentagon. AE and CD produced meet at F.



(a) Calculate $\angle BAE$.

$\angle BAE = \frac{(5-2) \times 180°}{5}$

$= 108°$

Answer 108 ° [2]

(b) Calculate $\angle ABE$.

$\angle ABE = \frac{180° - 108°}{2}$ (isos $\triangle$)

$= 36°$

Answer 36 ° [1]

(c) Calculate $\angle BED$.

$\angle BED = 108° - 36°$

$= 72°$

Answer 72 ° [1]

(ii) the length of the string.

arc ACB $= 30(\frac{7}{6}\pi)$

$= 35\pi$

$\tan\frac{\pi}{12} = \frac{30}{AT}$

$AT = \frac{30}{\tan\frac{\pi}{12}}$

$= 111.96$

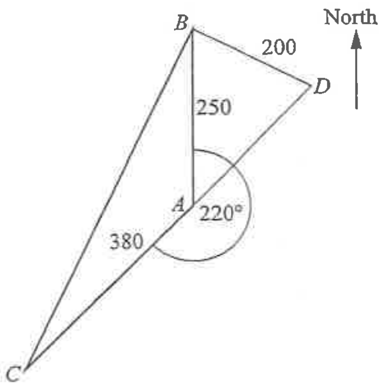
length $= 35\pi + 111.96 \times 2$

$= 333.878$

$= 334$

Answer 334 cm [3]

9 The diagram shows the paths AB, AC, AD and BD in a park. D lies on CA produced. B is due north of A and the bearing of C from A is 220°. $BD = 200$ m, $AB = 250$ m and $AC = 380$ m.



(a) Find the bearing of A from C.

$\angle CAB = 360° - 220°$ (∠s at a pt)
$= 140°$
Bearing $= 180° - 140°$ (int ∠s)
$= 040°$

Answer 040 ° [1]

(b) Find the area of triangle ABC.

area $= \frac{1}{2} \times 380 \times 250 \times \sin 140°$
$= 30532$
$= 30500$

Answer 30500 m² [1]

(c) Find BC.

$BC^2 = 250^2 + 380^2 - 2(250)(380)\cos 140°$
$= 352448$
$BC = 593.67$
$= 594$

Answer 594 m [2]

(d) Explain why BE is parallel to CF. [1]

Answer
$\angle CDE = 108°$
$\angle BED = 72°$
since $\angle CDE + \angle BED = 180°$, by int ∠s, $BE // CF$

(e) Prove that triangle BED and triangle FDE are congruent. [3]

Answer
$\angle BED = \angle FDE$ (alt ∠s)
$\angle BDE = \angle FED$ (alt ∠s)
DE is common.
$\triangle BED \equiv \triangle FDE$ (ASA)

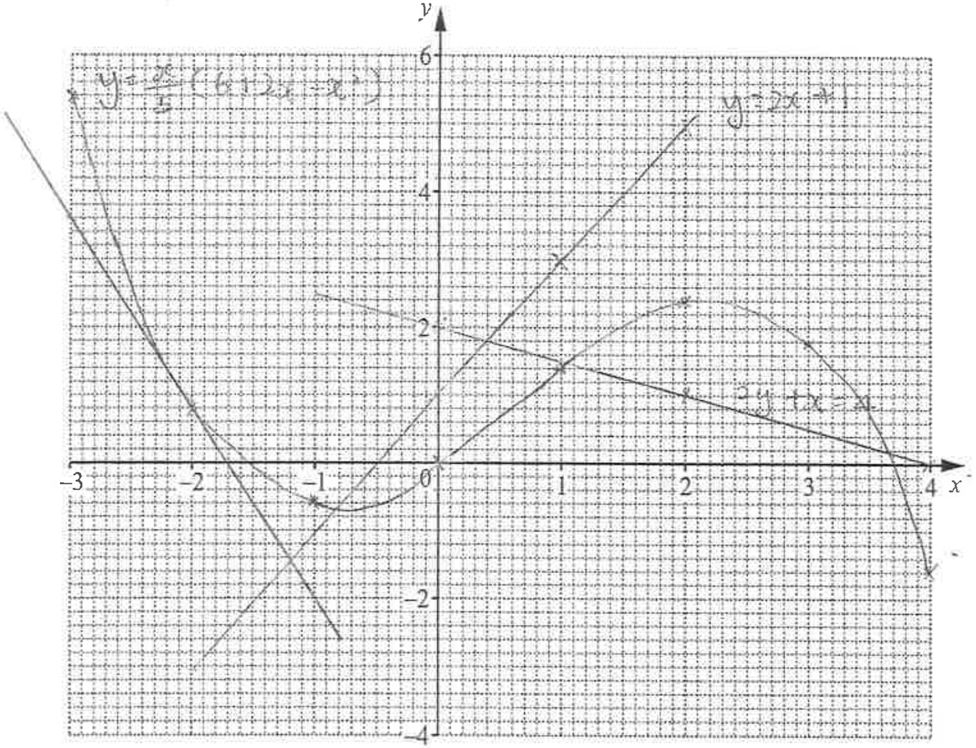
(f) Hence, explain briefly why BF and DE bisect each other at 90°. [1]

Answer
Since $\triangle BED \equiv \triangle FDE$ and $BE = BD$
$BDFE$ is a rhombus.
The diagonals BF and DE bisect each other at 90°

10 The table below shows some values of x and the corresponding values of y, for $y=\frac{x}{5}(6+2x-x^2)$.

x	−3	−2	−1	0	1	2	3	4
y	5.4	0.8	−0.6	0	1.4	2.4	1.8	−1.6

(a) Draw the graph of $y=\frac{x}{5}(6+2x-x^2)$ for $-3 \le x \le 4$. [3]



(b) By drawing a tangent, estimate the gradient of $y=\frac{x}{5}(6+2x-x^2)$ at (–2, 0.8).

$(-2, 0.8)$, $(-3, 3.6)$

grad $= \frac{3.6-0.8}{-3+2}$

$= -2.8$

Answer −2.8 [2]

(d) Find angle ABD.

$\frac{\sin \angle ADB}{250} = \frac{\sin(220° - 180°)}{200}$

$\angle ADB = \sin^{-1}\left[\frac{\sin 40°}{200} \times 250\right]$

$= 53.464°$

$\angle ABD = 180° - 40° - 53.464°$

$= 86.535°$

$= 86.5°$

Answer 86.5 ° [2]

(e) A mast of height 50 m is erected at B. Find the largest angle of elevation of the top of the mast, M, as seen from the path AD.

$\sin 40° = \frac{\text{short dist}}{250}$

short dist $= 250 \sin 40°$

$= 160.6969$

$\tan \theta = \frac{50}{160.6969}$

$\theta = \tan^{-1} \frac{50}{160.6969}$

$= 17.283°$

$= 17.3°$

Answer 17.3 ° [3]

[Turn Over

11 (a) P is the point (1, 1), $\overrightarrow{PQ} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$.

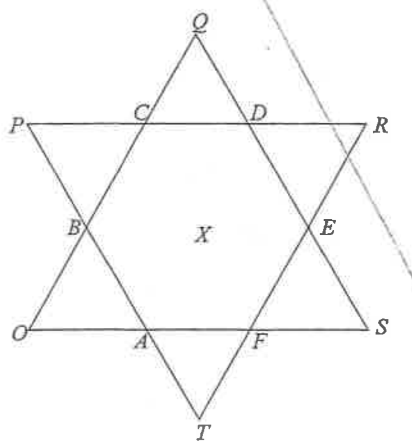
(i) Find $|\overrightarrow{PQ}|$.

Answer .... [1]

(ii) Express $\overrightarrow{QR}$ as a column vector.

Answer .... [1]

(b) In the diagram, the star *OATFSERDQCPB* is made up of a regular hexagon *ABCDEF* with centre *X* and 6 equilateral triangles *OAB, TFA, SEF, RDE, QCD* and *PBC*. $\overrightarrow{OA}$ = **a** and $\overrightarrow{OC}$ = **b**.



(c) Use your graph to solve the equation

(i) $x(6+2x-x^2)=10$,

$\frac{x(6+2x-x^2)}{5} = \frac{10}{5}$

$y = 2$

$x = -2.35,\ 1.4,\ 2.8$

Answer -2.35, 1.4, 2.8 [1]

(ii) $x(6+2x-x^2)-5=10x$.

$\frac{x}{5}(6+2x-x^2) - \frac{5}{5} = \frac{10x}{5}$

x	0	1	2
y	1	3	5

$y - 1 = 2x$

$y = 2x + 1$

$x = -0.85$

Answer -0.85 [2]

(d) Using the grid on page 22, draw the graph of $2y+x=4$. [1]

$y = -\frac{1}{2}x + 2$

x	0	2	4
y	2	1	0

(e) A cubic equation is satisfied by the value(s) of the x-coordinate(s) of the point(s) where the graphs drawn in **(a)** and **(d)** intersect.
Find the cubic equation, in the form $2x^3+bx^2+cx+d=0$, where b, c and d are integers.

$y = \frac{x}{5}(6+2x-x^2)$ — (1)

$y = -\frac{1}{2}x + 2$ — (2)

subst (1) into (2)

$\frac{x}{5}(6+2x-x^2) = -\frac{1}{2}x + 2$

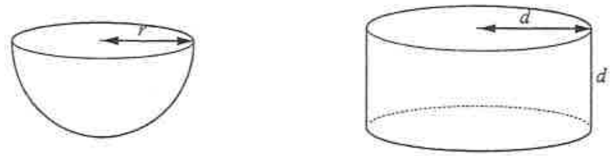
$12x + 4x^2 - 2x^3 = -5x + 20$

$2x^3 - 4x^2 - 17x + 20 = 0$

Answer $2x^3 - 4x^2 - 17x + 20 = 0$ [2]

[Turn Over

12 John wants to build a pond for his fishes.
The ponds can be either hemispherical or cylindrical as shown in the diagram.
A hemispherical pond has radius r m.
A cylindrical pond has a radius of d m and a depth of d m.
The cost of building the pond is proportional to the surface area of the pond and is $200 per square metres.



(a) John has 15 fishes. The average length of the fishes is 18 cm. A fish needs 5 litres of water for every 0.1 cm of its length. Find the number of cubic metres of water needed for these fishes.

Total length $= 18 \times 15$
$= 270$ cm

0.1 cm $-$ 5ℓ
270 cm $-$ $\frac{5 \times 270}{0.1}$
$= 13500\ \ell$

$1\ell = 1000 m^3$
$13500\ \ell = 13.5 m^3$

Answer 13.5 m^3 [2]

(i) Write the following vectors in terms of **a** and/ or **b**, giving your answers in their simplest form.

(a) $\overrightarrow{AC}$,

Answer .... [1]

(b) $\overrightarrow{RO}$,

Answer .... [1]

(c) $\overrightarrow{DF}$.

Answer .... [1]

(ii) Show that O, X and R lie on a straight line

Answer

[Turn Over

(b) John wants enough room for his fishes to swim, grow and even reproduce.
He has a budget of $5000 to build the pond.
Which design will you recommend to John?
Justify your recommendation and show your calculation clearly. [7]

Answer

Hemisphere

min vol = 13.5

$\frac{1}{2} \times \frac{4}{3}\pi r^3 = 13.5$

$r = \sqrt[3]{\frac{13.5 \times 3}{2\pi}}$

min r = 1.86105

min Surface Area $= \frac{1}{2} \times 4\pi r^2$

$= 2\pi r^2$

$= 2\pi (1.86105)^2$

$= 21.7618$

min cost $= 21.7618 \times 200$

$= \$4352.37$

$< \$5000$ (within budget)

Cylinder

min vol = 13.5

$\pi d^2 d = 13.5$

$\pi d^3 = 13.5$

$d = \sqrt[3]{\frac{13.5}{\pi}}$

min d = 1.62577

min surface area $= 2\pi r h + \pi r^2$

$= 2\pi d^2 + \pi d^2$

$= 3\pi d^2$

$= 3\pi (1.62577)^2$

$= 24.9111$

min cost $= 24.9111 \times 200$

$= 4982.22$

(within budget)

Recommend hemispherical pond because it is cheaper