ANS

CEDAR GIRLS' SECONDARY SCHOOL
Preliminary Examination
Secondary Four

CANDIDATE NAME: **Worked Solutions**

CLASS:

INDEX NUMBER:

CENTRE/ INDEX NO: /

ADDITIONAL MATHEMATICS

4049/01

Paper 1

13 September 2021

2 hours 15 minutes

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your centre number, index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 84.

For Examiner's Use
84

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

Answer **all** the questions.

1 P and Q are the points of intersection of the line $\frac{x}{a}+\frac{y}{b}=1$, $(a>0, b>0)$, with the x and y axes respectively. The distance PQ is 10 and the gradient of PQ is -2.
Find the exact value of a and of b. [5]

Since P lies on x –axis, y = 0, $\frac{x}{a}+\frac{0}{b}=1$, $x=a$

$P(a,0)$

Since Q lies on y –axis, $x=0$, $\frac{0}{a}+\frac{y}{b}=1$, $y=b$

$Q(0,b)$

Since gradient of $PQ = -2$, $\frac{b-0}{0-a}=-2$

$\therefore b=2a$..........................[1]

Since distance $PQ = 10$

$\sqrt{(a-0)^2+(0-b)^2}=10$

$\sqrt{(a-0)^2+(0-2a)^2}=10$

$5a^2=100$

$a=\sqrt{\frac{100}{5}}=\sqrt{20}=2\sqrt{5}$ $(a>0, b>0))$

$\therefore b=4\sqrt{5}$

2 The polynomial $P(x)=ax^3+bx^2-19x+4$, where a and b are constants, has a factor $x+4$ and is such that $2P(1)=5P(0)$.

(a) Find the value of a and of b. [4]

Since $x+4$ is a factor, $P(-4)=0$

$a(-4)^3+b(-4)^2-19(-4)+4=0$

$-4a+b--5$ ....(1)

Since $2P(1)=5P(0)$

$2(a+b-19+4)=20$

$a+b=25$ ....(2)

$\therefore a=6$ and $b=19$

(b) Hence, factorise $P(x)$ completely. [2]

By long division or factorization, $P(x)=(x+4)\left(6x^2-5x+1\right)$

$P(x)=(x+4)(3x-1)(2x-1)$

3 A function is defined by the equation $y=\dfrac{2x^2}{x-1}$, $x\neq 1$.

(a) Show that y is increasing for $x>2$. [3]

$$\frac{dy}{dx}=\frac{(x-1)4x-2x^2}{(x-1)^2}=\frac{2x(x-2)}{(x-1)^2}$$

Since $(x-1)^2>0$, and for $\dfrac{dy}{dx}>0$, $2x(x-2)>0 \Rightarrow x<0$ or >2

Hence function is increasing for $x>2$.

(b) If y is increasing at the rate of 2 units per second, find the rate of change of x when $x=3$. [2]

$$\frac{dx}{dt}=\frac{dx}{dy}\times\frac{dy}{dt}$$

$$=\frac{(x-1)^2}{2x(x-2)}\times 2$$

When $x=3$, $\dfrac{dx}{dt}=\dfrac{4}{6}\times 2=\dfrac{4}{3}$ units /s

4 Consider the expansion $\left(\frac{a^2}{\sqrt{x}}-\frac{\sqrt{x}}{a}\right)^8$ where a is a positive constant.

Find, **in terms of a,**

(a) the term independent of x. [2]

General Term $= \binom{8}{r}\left(\frac{a^2}{\sqrt{x}}\right)^{8-r}\left(-\frac{\sqrt{x}}{a}\right)^r$

$= \binom{8}{r}a^{16-3r}x^{r-4}(-1)^r$

For independent term, $r-4=0 \Rightarrow r=4$

Term independent of $x = \binom{8}{4}a^4(-1)^4 = 70\,a^4$

(b) the coefficient of x^2 in the expansion of $\left(\frac{3x^4-4x^2}{x^2}\right)\left(\frac{a^2}{\sqrt{x}}-\frac{\sqrt{x}}{a}\right)^8$. [4]

$\left(\frac{3x^4-4x^2}{x^2}\right)\left(\frac{a^2}{\sqrt{x}}-\frac{\sqrt{x}}{a}\right)^8 = (3x^2-4)(\text{...Term in } x^2 + \text{Independent Term} + \text{...})$

For term in x^2, $r-4=2 \Rightarrow r=6$

Term in $x^2 = \binom{8}{6}a^{16-18}x^2(-1)^6 = \frac{28x^2}{a^2}$

$(3x^2-4)\left(\ldots\ \frac{28x^2}{a^2}+70a^4+\ldots\right)$

$= \ldots + 210a^4x^2 - \frac{112}{a^2}x^2 + \ldots$

Coefficient of $x^2 = 210a^4 - \frac{112}{a^2}$

5 There are approximately 5 times as many animals of Species A as animals of Species B in a nature reserve.
The population of animals of Species A decreases at a rate of 5% per year and the population of animals of Species B increases at a rate of 10% per year.

It is given that N is the number of animals of Species B in the nature reserve.

(a) Write down, **in terms of N**,

(i) an expression, P_A, to model the population of animals of Species A after t years, [1]

$$P_L = 5N(0.95)^t$$

(ii) write down an expression, P_B, to model the population of animals of Species B after t years. [1]

$$P_B = N(1.1)^t$$

(b) Using your models, find the number of years that will elapse before the population of animals from Species A is equal to the population of animals from Species B. [3]

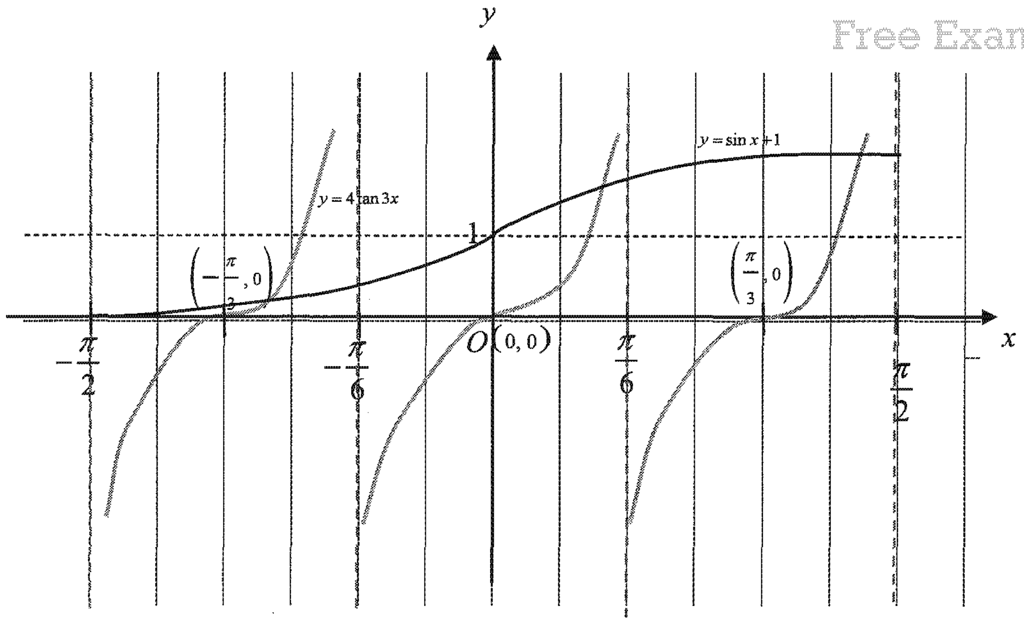
$$5N(0.95)^t = N(1.1)^t$$
$$\lg 5 + t\lg 0.95 = t\lg 1.1$$
$$t = \frac{\lg 5}{\lg 1.1 - \lg 0.95} = 10.978$$

No. of years $= 11.0$

6 **(a)** Solve $\tan 3x = 0$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ giving your answers in terms of π. [2]

$\tan 3x = 0$
$3x = -\pi, 0, \pi$
$x = -\frac{\pi}{3}, 0, \frac{\pi}{3}$

(b) Use your answers to part **(a)** to sketch the graph of $y = 4\tan 3x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ on the axes below. Show the coordinates of the points where the curve meets the axes. [3]



Period = $\frac{\pi}{3}$

(c) On the same diagram given in part **(b),** sketch the graph of $y = \sin x + 1$ for the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Hence state the number of solutions of the equation $4\tan 3x - \sin x = 1$ in the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. [3]

Correct shape of $y = \sin x + 1$

Correct points of $(0,1)$, $(\frac{\pi}{2},0)$, and $(-\frac{\pi}{2},0)$

No. of Solutions = 3

7 **(a)** Show that $2\sin(A+45°)\cos(A+45°)=\cos 2A$. [3]

$LHS = 2\sin(A+45°)\cos(A+45°)$
$= \sin(2A+90°)$
$= \sin 2A\cos 90° + \cos 2A\sin 90°$
$= \sin 2A(0) + \cos 2A(1)$
$= \cos 2A$

Alternative :
LHS = $2(\sin A\cos 45° + \sin 45°\cos A)(\cos A\cos 45° - \sin A\sin 45°)$

$$= 2\left(\sin A\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\cos A\right)\left(\cos A\left(\frac{\sqrt{2}}{2}\right)-\sin A\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$= 2\left[\left(\frac{\sqrt{2}}{2}\cos A\right)^2-\left(\frac{\sqrt{2}}{2}\sin A\right)^2\right]$$

$$= 2\left(\frac{1}{2}\cos^2 A-\frac{1}{2}\sin^2 A\right)$$

$= \cos 2A$

(b) Hence, solve the equation $\sin(A+45°)=\frac{1}{8}\sec(A+45°)$, for $0° \le A \le 360°$. [4]

$$\sin(A+45°)=\frac{1}{8\cos(A+45°)}$$

$$2\sin(A+45°)\cos(A+45°)=\frac{1}{4}$$

$$\cos 2A=\frac{1}{4},\ 0°\le 2A\le 720°$$

Basic Angle = $\cos^{-1}\frac{1}{4}=75.522°$

$2A$ can be in first or fourth quadrant.
$2A = 75.522°, 284.478°, 435.522°, 644.478°$
$A = 37.8°, 142.2°, 217.8°, 322.2°$

8 A curve is such that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 5\cos 2x$. This curve has a gradient of $\dfrac{3}{4}$ at the point $\left(-\dfrac{\pi}{12}, \dfrac{5\pi}{4}\right)$.

Show that the equation of the curve is $y = -\dfrac{5}{4}\cos 2x + 2x + \dfrac{17\pi}{12} + \dfrac{5\sqrt{3}}{8}$. [6]

When $x = -\dfrac{\pi}{12}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3}{4}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{5\sin 2x}{2} + c$

$\dfrac{5}{2}\sin 2\left(-\dfrac{\pi}{12}\right) + c = \dfrac{3}{4}$

$c = \dfrac{3}{4} + \dfrac{5}{2}\left(\dfrac{1}{2}\right) = 2$

$y = \displaystyle\int \left[\dfrac{5}{2}\sin 2x + 2\right] \mathrm{d}x$

$= -\dfrac{5}{4}\cos 2x + 2x + d$

When $x = -\dfrac{\pi}{12}$, $y = \dfrac{5\pi}{4}$

$\dfrac{5\pi}{4} = -\dfrac{5}{4}\cos 2\left(-\dfrac{\pi}{12}\right) + 2\left(-\dfrac{\pi}{12}\right) + d$ [

$d = \dfrac{17\pi}{12} + \dfrac{5\sqrt{3}}{8}$

Equation of curve: $y = -\dfrac{5}{4}\cos 2x + 2x + \dfrac{17\pi}{12} + \dfrac{5\sqrt{3}}{8}$. (proved)

n

9 **(a)** Factorise x^3+y^3. [1]

$x^3+y^3=(x+y)(x^2-xy+y^2)$ [B1]

(b) Hence, prove the identity $\dfrac{\sin^3\theta+\cos^3\theta}{2\sin^2\theta-1}=\dfrac{\sec\theta-\sin\theta}{\tan\theta-1}$. [4]

$$\frac{\sin^3\theta+\cos^3\theta}{2\sin^2\theta-1}=\frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{2\sin^2\theta-(\sin^2\theta+\cos^2\theta)}$$

$$=\frac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{\sin^2\theta-\cos^2\theta}$$

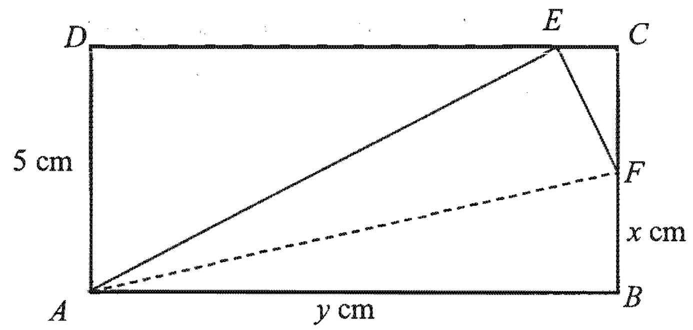
$$=\frac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}$$

$$=\frac{\dfrac{(1-\sin\theta\cos\theta)}{\cos\theta}}{\dfrac{(\sin\theta-\cos\theta)}{\cos\theta}}$$

$$=\frac{\sec\theta-\sin\theta}{\tan\theta-1}\ \text{(proved)}$$

Continuation of working space for Question (9b).

10



The diagram shows a rectangular piece of paper $ABCD$ such that $AD = 5$ cm, $BF = x$ cm and $AB = y$ cm. The paper is folded along AF such that B meets E on DC.

(a) Express EC and AE in terms of x. [4]

$$CF = (5-x)\text{cm}$$
$$EC = \sqrt{x^2-(5-x)^2} = \sqrt{10x-25}\ \text{cm}$$

$$AE = y \text{ and } DE = y-\sqrt{10x-25}\ \text{cm}$$

$$5^2+\left(y-\sqrt{10x-25}\right)^2 = y^2$$
$$25+y^2-2y\sqrt{10x-25}+10x-25 = y^2$$
$$y = \frac{5x}{\sqrt{10x-25}}$$

$$AE = \frac{5x}{\sqrt{10x-25}}\ \text{cm}$$

(b) Hence show that the area, A cm^2, of triangle AEF is given by

$$A = \frac{5x^2}{2\sqrt{10x-25}}.$$ [2]

$$A = \frac{1}{2}xy = \frac{1}{2}x \times \frac{5x}{\sqrt{10x-25}}$$
$$= \frac{5x^2}{2\sqrt{10x-25}}.$$

(c) Show that $x=\dfrac{10}{3}$ for which $\dfrac{dA}{dx}=0.$ [3]

$$\frac{dA}{dx}=\frac{5}{2}\left[\frac{\sqrt{10-2x}(2x)-x^2\left(\frac{1}{2}(10-2x)^{-\frac{1}{2}}(10)\right)}{10x-25}\right]$$

$$=\frac{5}{2}\left[\frac{(10x-25)^{-\frac{1}{2}}\left(2x(10x-25)-5x^2\right)}{10x-25}\right]$$

$$=\frac{5}{2}\left[\frac{15x^2-50x}{(10x-25)^{\frac{3}{2}}}\right]$$

For $\dfrac{dA}{dx}=0,\quad 15x^2-50x=0$

$5x(3x-10)=0$

$x=0$ (rej) or $x=\dfrac{10}{3}$

(d) By considering the change in the sign of $\dfrac{dA}{dx}$, show that A is a minimum at $x=\dfrac{10}{3}$. Hence find the minimum value of A. [3]

	$\frac{10}{3}^-$	$\frac{10}{3}$	$\frac{10}{3}^+$
$\frac{dA}{dx}$	Negative	Zero	Positive
Slope	$\diagdown$	—	$\diagup$

Hence from the table, A is a minimum.

Minimum value of A in surd form $=\dfrac{5}{2}\left(\dfrac{\left(\frac{10}{3}\right)^2}{\sqrt{10\left(\frac{10}{3}\right)-25}}\right)=\dfrac{50\sqrt{3}}{9}=9.62$

11 The height, in metres, of a ball above the ground t seconds after it is thrown from a tower is given by the function $h(t) = -5t^2 + 10t + c$, where c is a constant.

(a) **(i)** Given that $h(0) = 6$, find the value of c. [1]

$-5(0)^2 + 10(0) + c = 6$
$c = 6$

(ii) Interpret the meaning of the answer in **(a) (i)**. [1]

When $t = 0$, height of ball = 6m. The ball was thrown from a tower which is 6 m tall.

(b) **Without the use of differentiation** and using the value of c from **(a) (i)**, determine the maximum height reached by the ball and the corresponding value of t. [4]

$h(t) = -5t^2 + 10t + 6$
$= -5\left(\left(t-1\right)^2 - 1\right) + 6$
$= -5\left(t-1\right)^2 + 11$

Maximum height = 11 m
Corresponding value of t = 1

Continuation of working space for Question (11b).

(c) Find the time it takes for the ball to drop to the ground. [2]

When it drops to the ground, $h(t)=0$

$-5t^2+10t+6=0$

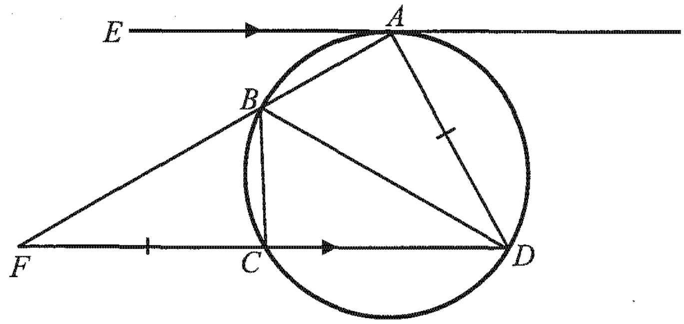
$-5(t-1)^2+11=0$

$(t-1)^2=\dfrac{11}{5}$

$t=\sqrt{2.2}+1=2.48$

Time = 2.48 s

12



In the diagram above, EA is the tangent to the circle at A.
FBA and FCD are straight lines.
EA is parallel to FD and $FC = AD$.

(a) Show that $\angle BAD = \angle BCF$. [2]

$\angle BAD + \angle BCD = 180°$ (Angles in the opposite segments)
$\angle BCF + \angle BCD = 180°$ (Adjacent angles on a straight line)
$\angle BCF + (180° - \angle BAD) = 180°$
$\angle BAD = \angle BCF$ (proved)

(b) Hence, prove that triangle ABD is congruent to triangle CBF. [4]

$\angle EAB = \angle AFC$ (alternate angles, $EA//FD$)
$\angle EAB = \angle ADB$ (Angle in alternate segment or Tangent Chord Theorem)

(1) $\angle ADB = \angle BFC$ (proved in above)
(2) $\angle BAD = \angle BCF$ (proved in part (a))
(3) $FC = AD$ (given)
$\Delta ABD \equiv \Delta CBF$ (ASA)

13 A particle starts from rest at a fixed point O and moves in a straight line with its acceleration, a m/s^2, is given by $a = 9 - kt$, where t seconds is the time since leaving O, and k is a constant. When $t = 2$, its velocity is 15 m/s.

(a) Find the value of k. [3]

$v = \int (9 - kt)\,\mathrm{d}t$

$v = 9t - \dfrac{kt^2}{2} + c$

When $t = 0,\ v = 0$

$c = 0$

When $t = 2,\ v = 15$

$18 - 2k = 15$

$\therefore k = \dfrac{3}{2}$

(b) The particle changes its direction of motion when $t = T$.
Find the value of T. [2]

When $v = 0$

$9T - \dfrac{3T^2}{4} = 0$

$T\left(9 - \dfrac{3}{4}T\right) = 0$

$T = 12$

(c) Show that the particle returns to O when $t = 18$. [3]

$$s = \int \left(9t - \frac{3t^2}{4}\right) \mathrm{d}t$$
$$= \frac{9t^2}{2} - \frac{t^3}{4} + d$$

When $t = 0,\ s = 0, d = 0$

When $s = 0$,

$$\frac{9t^2}{2} - \frac{t^3}{4} = 0$$
$$t^2\left(\frac{9}{2} - \frac{t}{4}\right) = 0$$
$$t = 0 \text{ or } t = 18$$

Hence particle returns to O when $t = 18$.

(d) Hence find the distance travelled by the particle between $t = 0$ and $t = 18$. [3]

When $t = 12$, $s = \frac{9}{2}(12)^2 - \frac{12^3}{4} = 216\text{ m}$.

Distance travelled between $t = 0$ and $t = 18 = 216 \times 2$
$= 432\text{ m}$

END OF PAPER

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