ANS

Name:		Index Number:		Class:	

CATHOLIC HIGH SCHOOL
2021 Preliminary Examination
Secondary 4 (O-Level Programme)

Full solutions

ADDITIONAL MATHEMATICS 4049/02

Paper 2

15 September 2021
2 hours 15 minutes

Additional materials: Answer booklets A, B and C

READ THESE INSTRUCTIONS FIRST

Write your name, register number and class on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.
DO **NOT** WRITE ON THE MARGINS.

Answer **all** questions.

If working is needed for any question, it must be shown with the answer.
Omission of essential working will result in loss of marks.
The use of an approved scientific calculator is expected, where appropriate.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to **three significant figures**. Give answers in **degrees to one decimal place**.
For π, use either your calculator value or 3.142, unless the question requires the answer in terms of π.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total of the marks for this paper is **90**.

This document consists of **8** printed pages, including 1 blank page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n ,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \tfrac{1}{2} ab \sin C$$

1 **(a)** Solve the simultaneous equations

$$2y-7x=5x^2,$$
$$3y=7-4x.$$
[3]

$$2y-7x=5x^2 \Rightarrow y=\frac{5}{2}x^2+\frac{7}{2}x \quad ...\ (1)$$

$$3y=7-4x \quad ...\ (2)$$

Sub (1) into (2):

$$3\left(\frac{5}{2}x^2+\frac{7}{2}x\right)=7-4x$$

$$15x^2+29x-14=0$$

$$(5x-2)(3x+7)=0$$

$$x=\frac{2}{5} \text{ or } -\frac{7}{3}$$

$$y=\frac{9}{5} \text{ or } \frac{49}{9}$$

(b) Given that the line $y=mx+1$ is a tangent to the curve $y=x^2+m(2x+1)$, find the value(s) of the constant m. [3]

Sub $y=mx+1$ into $y=x^2+m(2x+1)$:

$$x^2+m(2x+1)=mx+1$$

$$x^2+mx+m-1=0$$

$$D=0$$

$$m^2-4(1)(m-1)=0$$

$$m^2-4m+4=0$$

$$(m-2)^2=0$$

$$m=2$$

(c) Find the range of values of k such that $x^2+4x>kx-9$. [3]

$$x^2+4x>kx-9$$

$$x^2+(4-k)x+9>0$$

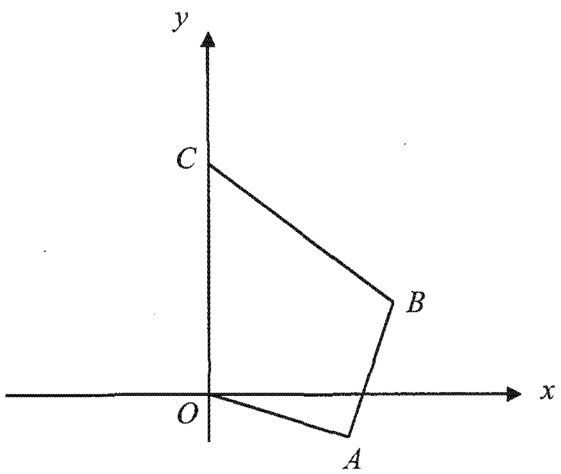
$$D<0$$

$$(4-k)^2-4(1)(9)<0$$

$$k^2-8k-20<0$$

$$-2<k<10$$

2 The diagram shows a quadrilateral with vertices $O(0, 0)$, $A(8, -2)$, B and C. The equation of AB is $y = 4x - 34$ and the equation of BC is $5y + 3x = 60$.



(i) Find the coordinates of B and C. [3]

Sub $x = 0$ into $5y + 3x = 60$:

$$y = 12$$
$$C(0,12)$$

Sub $y = 4x - 34$ into $5y + 3x = 60$:

$$5(4x - 34) + 3x = 60$$
$$23x = 230$$
$$x = 10$$
$$y = 6$$
$$B(10, 6)$$

(ii) Determine if $OABC$ is a kite. [2]

Mtd 1:

$$\text{Grad}_{OB} = \frac{6}{10} = \frac{3}{5}$$
$$\text{Grad}_{AC} = \frac{-2-12}{8-0} = -\frac{7}{4}$$
$$\text{Grad}_{OB} \times \text{Grad}_{AC} = \frac{3}{5} \times \left(-\frac{7}{4}\right)$$
$$\neq -1$$

$\Rightarrow OB$ is not perpendicular to AC.

Hence, $OABC$ is **NOT** a kite.

Mtd 2:

$$OC = 12$$

$$BC = \sqrt{(10-0)^2 + (6-12)^2}$$

$$= \sqrt{136} \text{ or } 11.7$$

$$BC \neq OC$$

Hence, *OABC* is **NOT** a kite.

(iii) A point D, above the x-axis, is such that OD is parallel to AB and $OD = 5\sqrt{17}$, find the coordinates of D. [4]

Since AB is $y = 4x - 34$

$\text{Grad}_{AB} = 4$

Eqn. of OD: $y = 4x$

Let $D(x, 4x)$:

$$OD = 5\sqrt{17}$$

$$\sqrt{x^2 + (4x)^2} = 5\sqrt{17}$$

$$17x^2 = 25(17)$$

$$x^2 = 25$$

$$x = 5 \quad (x > 0 \text{ since } D \text{ is above } x\text{-axis, } y > 0)$$

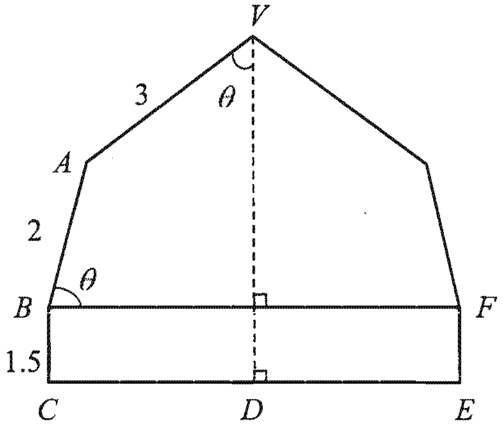
$$D(5, 20)$$

(iv) Find the area of the trapezium *OABD*. [2]

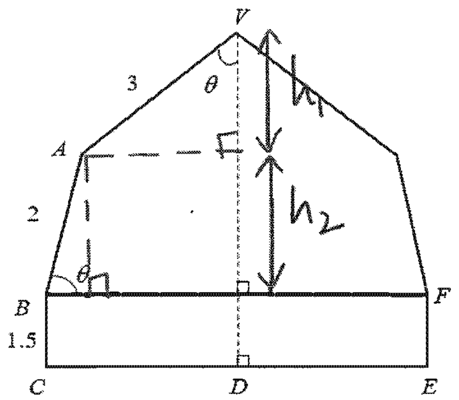
$$\text{area of the trapezium } OABD = \frac{1}{2}\begin{vmatrix} 0 & 8 & 10 & 5 & 0 \\ 0 & -2 & 6 & 20 & 0 \end{vmatrix}$$

$$= 119 \text{ units}^2$$

3 The diagram shows a structure such that $VA = 3$ m, $AB = 2$ m, $BC = 1.5$ m and angle $AVD =$ angle $ABF = \theta$ degrees. BF is parallel to CE, D is the midpoint of CE and the structure is symmetrical about VD.



(i) Show that $VD = 3\cos\theta + 2\sin\theta + 1.5$. [1]



$h_1 = 3\cos\theta$

$h_2 = 2\sin\theta$

$VD = h_1 + h_2 + 1.5$

$\quad = 3\cos\theta + 2\sin\theta + 1.5$

(ii) Express VD in the form $R\cos(\theta-\alpha)+1.5$, where $R>0$ and α is acute. [3]

Let $3\cos\theta+2\sin\theta = R\cos(\theta-\alpha)$

$$= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

$R\cos\alpha = 3$ ... (1)

$R\sin\alpha = 2$ ... (2)

$\frac{(2)}{(1)}$: $\tan\alpha = \frac{2}{3} \Rightarrow \alpha = \tan^{-1}\frac{2}{3} \approx 33.69°$

$(1)^2+(2)^2$: $R^2 = 3^2+2^2 \Rightarrow R=\sqrt{13}$

$$VD = \sqrt{13}\cos(\theta-33.69°)+1.5$$

Note: Some students gave angle in radian mode instead.

(iii) State the largest value of VD and find the corresponding value of θ. [2]

Large value of $VD = \sqrt{13}+1.5$ or 5.11

when $\theta = 33.69°$

(iv) Find the value(s) of θ for which $VD = 5$ m. [4]

$\sqrt{13}\cos(\theta-33.69°) = 3.5$

$\cos(\theta-33.69°) = \frac{3.5}{\sqrt{13}}$

Basic angle $= \cos^{-1}\frac{3.5}{\sqrt{13}}$

$\approx 13.8978°$

$\theta - 33.69° = 13.8978°, \quad -13.8978°$

$\theta \approx 47.6°, \quad 19.8°$

Note: Many students missed out the answer, 19.8°.

4 **(a)** Solve the equation $8e^{x+1} = \frac{15}{e^{x-1}} - 14e$. [4]

$$8e^{x+1} = \frac{15}{e^{x-1}} - 14e$$

$$8e\left(e^{x}\right) = \frac{15}{e\left(e^{x}\right)} - 14e$$

Let $y = e^{x}$.

$$8ey = \frac{15}{ey} - 14e$$

$$8y^2 + 14y - 15 = 0$$

$$(4y-3)(2y+5) = 0$$

$$y = \frac{3}{4} \quad \text{or} \quad y = -\frac{5}{2} \text{ (rej.)}$$

$$e^{x} = \frac{3}{4}$$

$$x = \ln\frac{3}{4} \approx -0.288$$

(b) Solve the equation $3\sin 2\theta = \cos\theta$ for $-\pi \le \theta \le \pi$. [4]

$$3\sin 2\theta = \cos\theta$$

$$6\sin\theta\cos\theta = \cos\theta$$

$$6\sin\theta\cos\theta - \cos\theta = 0$$

$$\cos\theta(6\sin\theta - 1) = 0$$

$$6\sin\theta - 1 = 0 \quad \text{or} \quad \cos\theta = 0$$

$$\sin\theta = \frac{1}{6} \qquad \theta = -\frac{\pi}{2}, \frac{\pi}{2}$$

$$\theta \approx 0.167,\ 2.79$$

Note: Many students cancelled out $\cos\theta$ and thus missed out one equation with 2 answers.

5 **(a)** Solve the equation $\log_5 x+1=2\log_x 5$. [4]

$$\log_5 x+1=2\log_x 5$$
$$=2\left(\frac{1}{\log_5 x}\right)$$
Let $y=\log_5 x$:
$$y+1=\frac{2}{y}$$
$$y^2+y-2=0$$
$$(y-1)(y+2)=0$$
$$y=1 \quad \text{or} \quad y=-2$$
$$\log_5 x=1 \quad \text{or} \quad \log_5 x=-2$$
$$x=5 \quad \text{or} \quad x=\frac{1}{25}$$

(b) Solve the equation $\log_9(2+x)=\log_{16}4-\log_{\frac{1}{3}}\sqrt{1-2x}$. [5]

$$\log_9(2+x)=\log_{16}4-\log_{\frac{1}{3}}\sqrt{1-2x}$$
$$\frac{\log_3(2+x)}{\log_3 9}=\frac{\log_2 4}{\log_2 16}-\frac{\log_3\sqrt{1-2x}}{\log_3\frac{1}{3}}$$
$$\frac{\log_3(2+x)}{2}=\frac{2}{4}-\frac{\frac{1}{2}\log_3(1-2x)}{-1}$$
$$\frac{\log_3(2+x)}{2}=\frac{1}{2}+\frac{1}{2}\log_3(1-2x)$$
$$\log_3(2+x)=1+\log_3(1-2x)$$
$$\log_3\frac{2+x}{1-2x}=1$$
$$\frac{2+x}{1-2x}=3$$
$$2+x=3-6x$$
$$7x=1$$
$$x=\frac{1}{7}$$

6 **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left[x\sqrt{3x-1}\right]=\frac{Ax+B}{\sqrt{3x-1}}$, where A and B are constants. [4]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x\sqrt{3x-1}\right] &= \sqrt{3x-1}+x\bullet\frac{1}{2}(3x-1)^{-\frac{1}{2}}\bullet 3 \\
&= \sqrt{3x-1}+\frac{3}{2}x(3x-1)^{-\frac{1}{2}} \\
&= (3x-1)^{-\frac{1}{2}}\left[(3x-1)+\frac{3}{2}x\right] \\
&= (3x-1)^{-\frac{1}{2}}\left[\frac{9}{2}x-1\right] \\
&= \frac{\frac{9}{2}x-1}{\sqrt{3x-1}} \quad \text{or} \quad \frac{9x-2}{2\sqrt{3x-1}}
\end{aligned}$$

(ii) Hence or otherwise, find $\int_1^2 \frac{9x-2}{\sqrt{3x-1}}\,\mathrm{d}x$, giving your answer correct to 3 significant figures. [4]

$$\begin{aligned}
\int \frac{9x-2}{\sqrt{3x-1}}\ \mathrm{d}x &= 2\int \frac{9x-2}{2\sqrt{3x-1}}\ \mathrm{d}x \\
&= 2x\sqrt{3x-1}+c
\end{aligned}$$

$$\begin{aligned}
\int_1^2 \frac{9x-2}{2\sqrt{3x-1}}\ \mathrm{d}x &= \left[2x\sqrt{3x-1}\right]_1^2 \\
&= 2(2)\sqrt{3(2)-1}-2(1)\sqrt{3(1)-1} \\
&= 4\sqrt{5}-2\sqrt{2} \\
&\approx 6.12
\end{aligned}$$

7 The function f is defined for $x \neq -\frac{1}{3}$ and is such that $f''(x) = 4e^{2x} + \frac{9}{(3x+1)^2}$.

Given that $f'(0) = -1$ and $f(0) = 2$, find an expression for $f(x)$. [7]

$$f'(x) = \int 4e^{2x} + \frac{9}{(3x+1)^2}\, dx$$

$$= 2e^{2x} + \frac{9(3x+1)^{-1}}{(-1)(3)} + c$$

$$= 2e^{2x} - \frac{3}{3x+1} + c$$

$$f'(0) = -1$$

$$2 - 3 + c = -1$$

$$c = 0$$

$$f'(x) = -2e^{2x} - \frac{3}{3x+1}$$

$$f(x) = \int 2e^{2x} - \frac{3}{3x+1}\, dx$$

$$= e^{2x} - \ln(3x+1) + d$$

$$f(0) = 2$$

$$1 - \ln(1) + d = 2$$

$$d = 1$$

$$f(x) = e^{2x} - \ln(3x+1) + 1$$

8 **(a)** The equation of a curve is $y = \frac{1}{\sqrt{10x-1}} + \frac{1}{3}x$, for $x \neq \frac{1}{10}$. Find the equation of the normal to the curve at the point $\left(1, \frac{2}{3}\right)$. [4]

$$y = \frac{1}{\sqrt{10x-1}} + \frac{1}{3}x$$
$$= (10x-1)^{-\frac{1}{2}} + \frac{1}{3}x$$

$$\frac{dy}{dx} = -\frac{1}{2}(10x-1)^{-\frac{3}{2}}(10) + \frac{1}{3}$$
$$= -\frac{5}{(10x-1)^{\frac{3}{2}}} + \frac{1}{3}$$

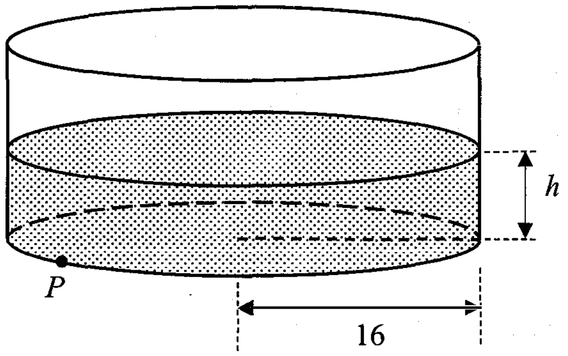
$$\left.\frac{dy}{dx}\right|_{x=1} = -\frac{5}{(10-1)^{\frac{3}{2}}} + \frac{1}{3}$$
$$= \frac{4}{27}$$

Gradient of normal at $P = -\frac{27}{4}$

Eqn of normal at P:

$$y - \frac{2}{3} = -\frac{27}{4}(x-1)$$
$$y = -\frac{27}{4}x + \frac{89}{12} \qquad \text{or} \qquad 12y = 81x + 89$$

(b) The diagram shows a cylindrical water tank with base radius 16 cm.



Water is flowing into the tank at a constant rate of $4.8\pi k$ cm^3/min, where k is a constant. At time t minutes, the depth of the water in the tank is h metres. There is a hole at point P at the bottom of the tank and a cork is used to stop water leakage. When the cork is unplugged, water will leak from the hole at a rate of 6.4π cm^3/min. Find an expression for the rate of change of h in terms of k after the cork is unplugged.

[4]

$$\frac{dV_{\text{In}}}{dt} = 4.8\pi k, \qquad \frac{dV_{\text{out}}}{dt} = 6.4\pi$$

$$\frac{dV}{dh} = 256\pi$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{256\pi} \times (4.8\pi k - 6.4\pi)$$

$$\frac{dh}{dt} = \frac{3k}{160} - \frac{1}{40}$$

or $$\frac{dh}{dt} = \frac{3k-4}{160}$$

9 **(i)** The expansion of $(1+px)^n$, where $n > 0$, by the binomial theorem is $1+36x+66p^2x^2+kx^3+\ldots$. Find the value of n, of p and of k. [5]

$$(1+px)^n = 1+\binom{n}{1}(px)+\binom{n}{2}(px)^2+\binom{n}{3}(px)^3+\ldots$$
$$= 1+npx+\binom{n}{2}p^2x^2+\binom{n}{3}p^3x^3+\ldots$$

Comparing x^2:

$$\binom{n}{2} = 66$$
$$\frac{n(n-1)}{2} = 66$$
$$n^2-n-132=0$$
$$(n-12)(n+11)=0$$
$$n=12 \quad \text{or} \quad n=-11 \quad \text{(rej.)}$$

Comparing x:

$$np=36$$
$$12p=36$$
$$p=3$$

Comparing x^3:

$$k=\binom{n}{3}p^3$$
$$=\binom{12}{3}(3)^3$$
$$=5940$$

(ii) Find the term independent of x in the expansion of $(3+x)\left(2x-\frac{1}{x}\right)^8$. [4]

For $\left(2x-\frac{1}{x}\right)^8$:

$$T_{r+1}=\binom{8}{r}(2x)^{8-r}\left(-\frac{1}{x}\right)^r$$
$$=\binom{8}{r}2^{8-r}(-1)^r x^{8-2r}$$

Let $8-2r=0 \Rightarrow r=4$

Term independent of $x=\binom{8}{4}2^4(-1)^4$
$=1120$

For $(3+x)\left(2x-\frac{1}{x}\right)^8$:

Term independent of $x=3(1120)$

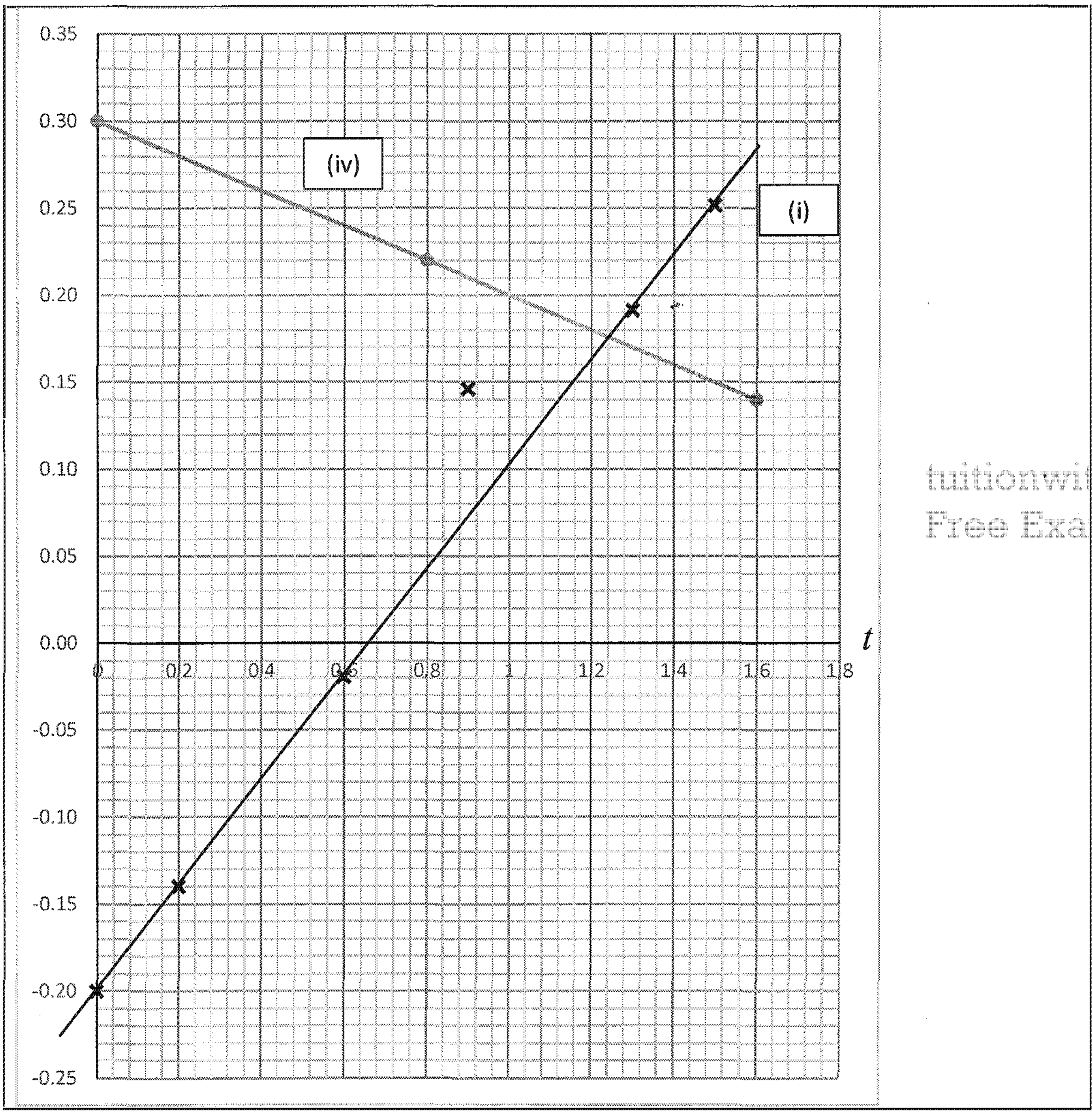
$=3360$

10 The table shows, to 2 decimal places, the population, P, of a certain kind of bacteria in a test-tube, measured at hourly intervals, t.

t	0.2	0.6	0.9	1.3	1.5
P	0.72	0.96	1.40	1.55	1.78

(i) On the grid on the next page, plot lg P against t and draw a straight line graph. [3]

t	0.2	0.6	0.9	1.3	1.5
lg P	-0.14	-0.02	0.15	0.19	0.25



(ii) Determine which value of P, in the table above, is the incorrect reading.
Use the graph to estimate a value of P to replace the incorrect recording of P. [2]

Incorrect reading of P = 1.40
From graph, when $t = 0.9$, $\lg P = 0.075$, $P = 10^{0.075} \approx 1.19$

(iii) Find the gradient of your straight line and hence express P in the form $\frac{A^t}{10^{2k}}$, where A and k are constants. [4]

$$(0,-0.2), \quad (1.2,0.16)$$

$$\text{Gradient} = \frac{-0.2-0.16}{0-1.2} \approx 0.3000$$

$$(\lg P)-\text{intercept} = -0.2$$

$$\lg P = 0.3000t - 0.2$$

$$P = 10^{0.3000t-0.2}$$

$$= \frac{\left(10^{0.3}\right)^t}{10^{2(0.1)}} \quad \text{or} \quad \frac{(2)^t}{10^{2(0.1)}}$$

(iv) On the same diagram, draw the line representing $P^{10} = 10^{3-t}$ and hence estimate the value of t for which $\left(\frac{A^t}{10^{2k}}\right)^{10} = 10^{3-t}$. [2]

$$P^{10} = 10^{3-t}$$

$$10\lg P = 3 - t$$

$$\lg P = -\frac{1}{10}t + \frac{3}{10}$$

Draw the straight line: $\lg P = -\frac{1}{10}t + \frac{3}{10}$

From graph, $t \approx 1.24$

END OF PAPER