Candidate Name ______________________

Class	Index Number

ST ANDREW'S SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2021
SECONDARY 4 EXPRESS

ADDITIONAL MATHEMATICS 4049/02

Paper 2 (solutions)

FRIDAY 20 AUGUST 2021 2 hours 15 minutes

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

1 Simplify $\dfrac{2^{n+1}+2^{n}}{4^{\frac{1}{2}n-1}-2^{n-3}}$. [3]

$$= \frac{2(2^n)+2^n}{2^{n-2}-2^n\left(\frac{1}{8}\right)}$$

$$= \frac{2(2^n)+2^n}{2^n\left(\frac{1}{4}\right)-2^n\left(\frac{1}{8}\right)}$$

$$= \frac{2^n(2+1)}{2^n\left(\frac{1}{4}-\frac{1}{8}\right)}$$

$$= \frac{3}{1/8}$$

$$= 24$$

2 **(i)** The equation of a curve is $\frac{y}{x}=\ln 2x$. Find the coordinates of the turning point in terms of e. [3]

$\frac{y}{x}=\ln 2x$

$y=x\ln 2x$

$\frac{dy}{dx}=x\left(\frac{2}{2x}\right)+\ln 2x(1)$

$=1+\ln 2x$.

For turning point, $\frac{dy}{dx}=0$

$1+\ln 2x=0$

$\ln 2x=-1$

$2x=e^{-1}$

$x=\frac{1}{2e}$

$y=\frac{1}{2e}\ln 2\left(\frac{1}{2e}\right)$

$=\frac{1}{2e}\ln e^{-1}$

$=-\frac{1}{2e}\ln e$

$=-\frac{1}{2e}$

$\therefore$ coordinates of turning pt

$\left(\frac{1}{2e},-\frac{1}{2e}\right)$

(ii) Hence, determine the nature of this turning point. [2]

$\frac{d^2y}{dx^2}=\frac{2}{2x}$ (or $\frac{1}{x}$)

$=\frac{2}{2\left(\frac{1}{2e}\right)}$

$=2e>0$

$\therefore$ It is a minimum point.

4 The population M, of monkeys in a forest was recorded from 1960 to 2010 and the results are shown in the table below.

Year	1960	1975	1990	2010
M	5.22	15.45	58.20	380

It is given that $M = 2 + ab^t$ where t is the time measured in years from January 1955 and a and b are constants.

(i) On the grid below, draw a straight line graph to illustrate this data. [3]

$M = 2 + ab^t$

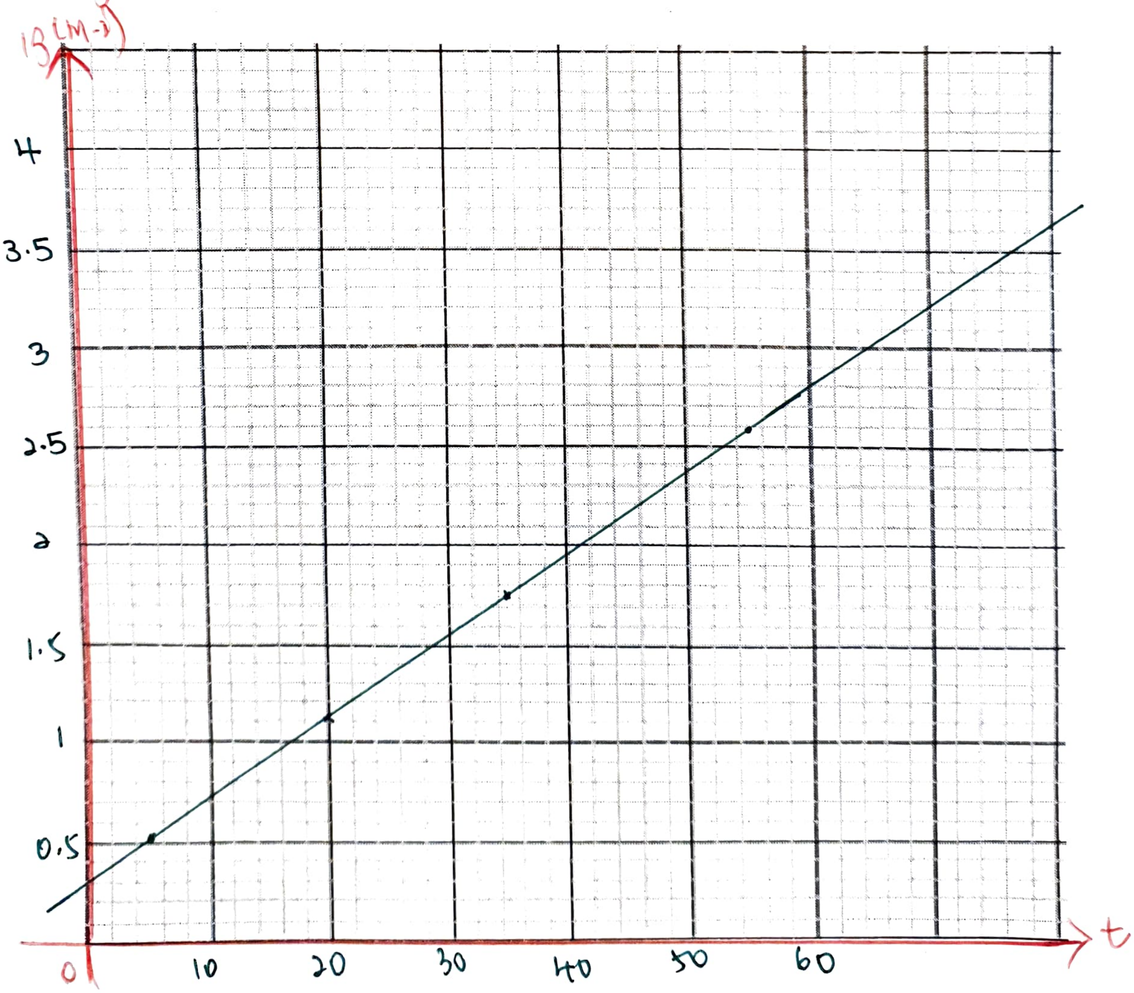
$M - 2 = ab^t$

$\lg(M-2) = \lg ab^t$

$\lg(M-2) = \lg a + \lg b^t$

$\lg(M-2) = t\lg b + \lg a$

t	5	20	35	55
$\lg(M-2)$	0.51	1.13	1.75	2.58



(ii) Use your graph to estimate

(a) the value of a and of b, [3]

$\underline{\text{gradient}} = \dfrac{2.8-0.3}{60-0}$

$= 0.0417$

$\lg b = 0.0417$, $b = 10^{0.0417}$

$= 1.10$ //

$\underline{y\text{-intercept}} = 0.3$

$\lg a = 0.3$

$a = 10^{0.3}$

$= 2$ //

(b) the year in which the population will reach 600. [2]

$\lg(600-2) = 2.78$

$t = 60$ yrs (from graph)

$1955 + 60 = 2015$ //

5 **(a)** A curve has the equation $y = \frac{1}{3}x^3 - \frac{5x^2}{2} + 12x + 3$. Determine whether y is an increasing or decreasing function of x. [4]

$y = \frac{1}{3}x^3 - \frac{5x^2}{2} + 12x + 3$

$\frac{dy}{dx} = \frac{1}{3}(3)x^2 - \frac{5}{2}(2x) + 12$

$= x^2 - 5x + 12$

$= \left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 12$

$= \left(x - \frac{5}{2}\right)^2 + 5\frac{3}{4}$

Since $\left(x - \frac{5}{2}\right)^2 \geq 0$

$\left(x - \frac{5}{2}\right)^2 + 5\frac{3}{4} > 0$

$\therefore \frac{dy}{dx} > 0$.

Hence, y is an increasing function.

(b) Show that the x-axis is not a tangent to the curve. [2]

$x^2 - 5x + 12 = 0$

$\left(x - \frac{5}{2}\right)^2 + 5\frac{3}{4} = 0$

$x - \frac{5}{2} = \pm\sqrt{-5\frac{3}{4}}$ (NA)

Since there is no x-value that satisfies the eqn.

$\therefore$ there is no real root, thus x-axis is not a tangent to the curve.

mtd 2

As $\frac{dy}{dx} > 0$ for all x, means no turning point.

$\therefore$ x-axis is not a tangent to the curve.

6 The volume of a solid right cylinder of radius r cm is $250\pi\ cm^3$.

(i) Show that the total surface area, $S\ cm^2$, of the cylinder is given by

$S = 2\pi r^2 + \frac{500\pi}{r}$. [3]

$V = \pi r^2 h$

$h = \frac{V}{\pi r^2}$

$= \frac{250\pi}{\pi r^2}$

$= \frac{250}{r^2}$

$S = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right)$

$= 2\pi r^2 + \frac{500\pi}{r}$

(ii) Calculate the value of r for which S has a stationary value. [3]

$$\frac{dS}{dr} = 4\pi r - \frac{500\pi}{r^2}$$

Let $\frac{dS}{dr} = 0$

$$4\pi r - \frac{500\pi}{r^2} = 0$$

$$4\pi r^3 - 500\pi = 0$$

$$r^3 = \frac{500\pi}{4\pi}$$

$$r^3 = \frac{500}{4}$$

$$r = 5$$

(iii) Determine whether this value of r makes S a maximum or a minimum, and find the corresponding height of the cylinder. [3]

$$\frac{d^2S}{dr^2} = 4\pi + \frac{1000\pi}{r^3} > 0$$ for all $r > 0$.

Therefore, value of $r = 5$ makes S a minimum.

$$h = \frac{250}{5^2}$$

$$= 10\text{cm}$$

7 **(a)** Prove the identity $\operatorname{cosec} A - \dfrac{\sin A}{1+\cos A} = \cot A$. [4]

LHS $\dfrac{1}{\sin A} - \dfrac{\sin A}{1+\cos A}$

$= \dfrac{1+\cos A - \sin A(\sin A)}{\sin A(1+\cos A)}$

$= \dfrac{1+\cos A - \sin^2 A}{\sin A(1+\cos A)}$

$= \dfrac{1+\cos A - (1-\cos^2 A)}{\sin A(1+\cos A)}$

$= \dfrac{\cancel{1}+\cos A - \cancel{1} + \cos^2 A}{\sin A(1+\cos A)}$

$= \dfrac{\cos A(1+\cos A)}{\sin A(1+\cos A)}$

$= \dfrac{\cos A}{\sin A}$

$= \cot A$ // (proven)

(b) Hence solve the equation $\operatorname{cosec} A - \dfrac{\sin A}{1+\cos A} = \tan 2A$ for $0° \le A \le 180°$. [3]

$\cot A = \tan 2A$

$\dfrac{1}{\tan A} = \dfrac{2\tan A}{1-\tan^2 A}$

$1 - \tan^2 A = 2\tan^2 A$

$2\tan^2 A + \tan^2 A - 1 = 0$

$3\tan^2 A = 1$

$\tan^2 A = \frac{1}{3}$

$\tan A = \pm\sqrt{\frac{1}{3}}$

basic $\cancel{A}$ = $30°$

$A = 30°, 150°$

(c) Show that there are no solutions to the equation $\operatorname{cosec} A - \dfrac{\sin A}{1+\cos A} = -\tan 2A$ for $0° \le A \le 360°$. [2]

$\dfrac{1}{\tan A} = -\tan 2A$

$\dfrac{1}{\tan A} = \dfrac{-2\tan A}{1-\tan^2 A}$

$1 - \tan^2 A = -2\tan^2 A$

$\tan^2 A = -1$ (not possible)

$\therefore$ no solution

$0 \le 2x \le 10$

$-1 \le 2x-1 \le 9$

8 **(a)** Solve $2\sin(2x-1) = -\frac{1}{2}$ where $0 \le x \le 5$. [3]

$\sin(2x-1) = -\frac{1}{4}$

S	A
T	C

basic $\measuredangle = 0.252680$

$2x-1 = -0.25261,\ \pi + 0.25268,\ 2\pi - 0.25268,\ 3\pi + 0.25268,\ 4\pi - 0.25268$

$x = 0.374,\ 2.197,\ 3.515,\ 5.3387$ (rej)

$x = 0.374,\ 2.20,\ 3.52$ (3sf)

[Turn over

(b) The height of the tides in East Coast Park is modelled by $h = 2 + 1.5\cos(kt)$, where k is a constant and t is the time after midnight. The time between high tides is 12.5 hours.

(i) What is the height of the highest tide for any given day? [1]

$2 + 1.5 = 3.5\text{m}.$

(ii) Show that $k = \dfrac{4\pi}{25}$. [2]

$\text{Period} = \dfrac{2\pi}{k}$

$12.5 = \dfrac{2\pi}{k}$

$\dfrac{25}{2} = \dfrac{2\pi}{k}$

$\therefore k = \dfrac{4\pi}{25}$ // (shown)

(iii) Find the time for which the height of the tide first reaches 2.5m. [3]

$2.5 = 2 + 1.5\cos\left(\frac{4\pi}{25}t\right)$

$0.5 = 1.5\cos\left(\frac{4\pi}{25}t\right)$

$\frac{1}{3} = \cos\left(\frac{4\pi}{25}t\right)$

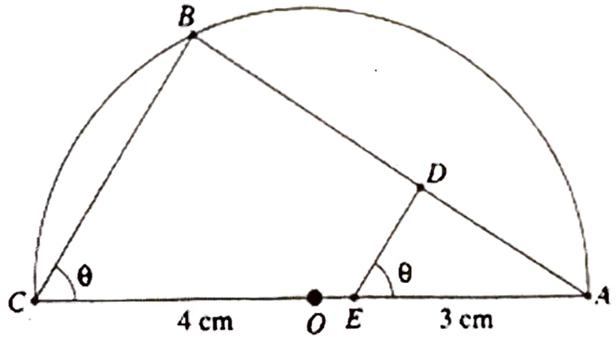
basic $\measuredangle = 1.2309$

$\frac{4\pi}{25}t = 1.2309,\ 5.0531$

$t = 2.448,\ 3.836$

Time $= 2.27$ a.m.

9 The diagram shows a triangle ABC inside a semicircle, center O and radius 4 cm. Given that $\angle ACB = \angle AED = \theta$ radians and AE = 3 cm,



(i) show that $L = BC + BD + DE = 5\sin\theta + 11\cos\theta$, [3]

$\angle ABC = \frac{\pi}{2}$ ($\angle$ in semicircle)

$\angle ABC = \angle ADE = \frac{\pi}{2}$ ($\triangle ADE$ is similar to $\triangle ABC$)

In $\triangle ABC$,

$\cos\theta = \frac{CB}{CA}$

$\cos\theta = \frac{CB}{8}$

$\therefore \boxed{CB = 8\cos\theta}$

$\sin\theta = \frac{BA}{CA}$

$\sin A = \frac{BA}{8}$

$\therefore \underline{BA = 8\sin\theta}$

In $\triangle ADE$,

$\cos\theta = \frac{DE}{AE}$

$\cos\theta = \frac{DE}{3}$

$\therefore \boxed{DE = 3\cos\theta}$

$\sin\theta = \frac{DA}{AE}$

$\sin\theta = \frac{DA}{3}$

$\therefore \underline{DA = 3\sin\theta}$

$\therefore CB + BD + DE$

$= 8\cos\theta + (BA - DA) + 3\cos\theta$

$= 8\cos\theta + 8\sin\theta - 3\sin\theta + 3\cos\theta$

$= 5\sin\theta + 11\cos\theta$.

(ii) express $5\sin\theta + 11\cos\theta$ in the form $R\cos(\theta - \alpha)$, [3]

Let $11\cos\theta + 5\sin\theta = R\cos(\theta - \alpha)$ (rewrite)

$R = \sqrt{11^2 + 5^2} = \sqrt{146}$

$\tan\alpha = \frac{5}{11}$

$\alpha = \tan^{-1}\left(\frac{5}{11}\right)$

$= 0.427$ rad

Hence $11\cos\theta + 5\sin\theta = \sqrt{146}\cos(\theta - 0.427 \text{ rad})$.

[Turn over

[2]

(iii) find the maximum value of L and the corresponding value of θ,

max value $= \sqrt{146}$ or 12.1

$\cos(\theta - 0.427) = 1$

$\theta - 0.427 = \cos^{-1} 1$

$\theta - 0.427 = 0$

$\theta = 0.427$ rad.

(iv) find the value of θ for which $L = 8$ cm. [3]

$\sqrt{146}\cos(\theta - 0.427) = 8$

$\cos(\theta - 0.427) = \frac{8}{\sqrt{146}}$

basic $\alpha = 0.84719$

$\theta - 0.427 = 0.84719$

$\theta = 1.27419$

≈ 1.27

10 In the Chingay Parade procession held at the F1 Pit building, the Beauty Pageant Float was travelling on a straight road with a velocity, v ms^{-1}, given by the equation $v = 6t - \frac{1}{2}t^2 + 5$, where t is the time after passing a fixed point A.

(a) Show that the maximum velocity is reached 6 seconds later. [3]

$a = \frac{dv}{dt} = 6 - t$

When $\frac{dv}{dt} = 0$, $6 - t = 0$
$t = 6$

$\frac{d^2v}{dt^2} = -1 \quad (<0)$

$\therefore$ max velocity is reached 6 seconds later.

Upon reaching its maximum velocity, the float started to decelerate uniformly at 2 ms^{-2}, before coming to a rest at point B to allow residents to take photographs.

(b) Find the time when the float reached B. [2]

At $t = 6$

$v = 36 - \frac{1}{2}(36) + 5$

$= 23.$

$\frac{23 - 0}{6 - t} = -2$

$23 = -12 + 2t$

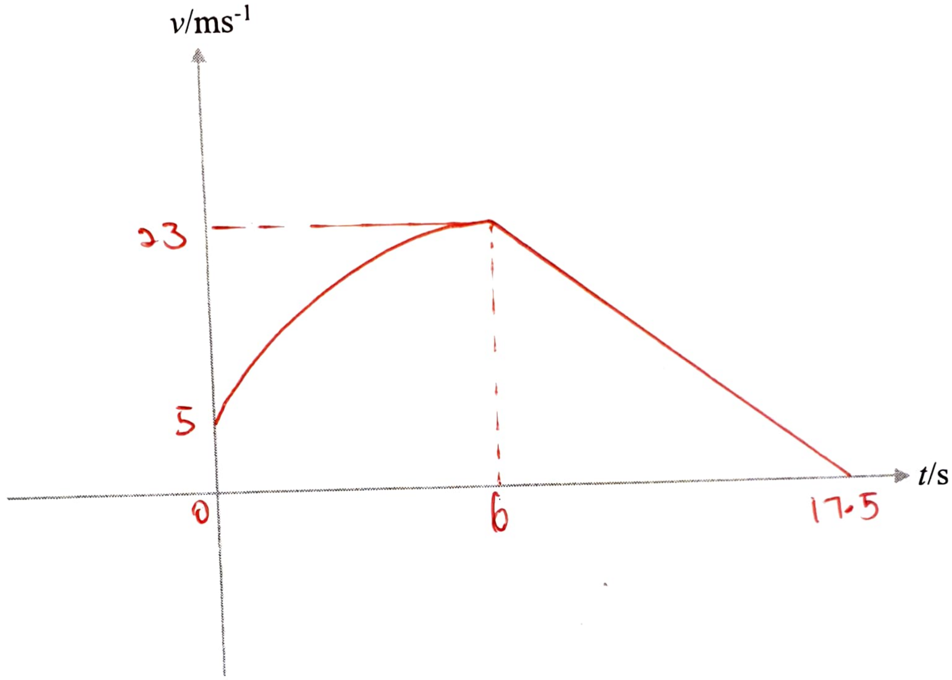
$35 = 2t$

$t = 17.5$ s

(c) Sketch the velocity-time graph from A to B. [3]

When $t = 0$, $v = 6(0) - \frac{1}{2}(0)^2 + 5$
$= 5$

When $t = 6$, $v = 6(6) - \frac{1}{2}(6)^2 + 5$
$= 23$



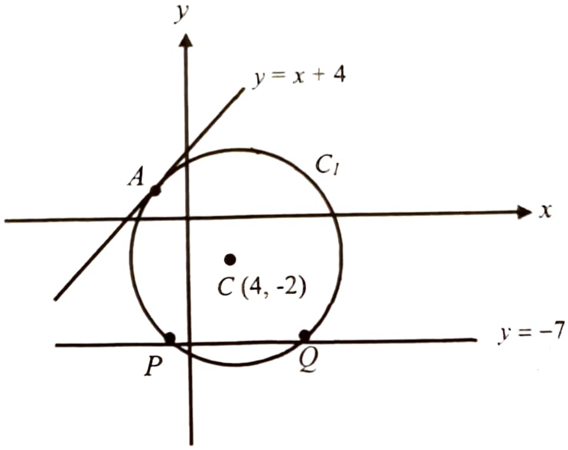
(d) Find the total distance travelled from A to B. [3]

Total Dist $= \int_0^6 (6t - \frac{1}{2}t^2 + 5)\,dt + \frac{1}{2}(17.5 - 6) \times 23$

$= \left[\frac{6t^2}{2} - \frac{t^3}{6} + 5t\right]_0^6 + 132.25$

$= 234.25\text{m}.$

11 The diagram shows circle C_1 (not drawn to scale) with the centre of the circle at $C(4, -2)$.
The line $y = 4 + x$ is a tangent to circle at the point A.



(i) Find the coordinates of A. [3]

grad of tangent at A = 1
grad of normal at A = −1
eqn of normal passes through centre,
$(y+2) = -1(x-4)$
$y = -x+2$

$x+4 = -x+2$

$2x = -2$
$x = -1$
$y = 3$
$\therefore A(-1, 3)$

(ii) Find the equation of circle C_1. [2]

radius $= \sqrt{(-1-4)^2 + [3-(-2)]^2}$
$= \sqrt{50}$ units

Eqn of circle C_1,
$(x-4)^2 + (y+2)^2 = 50$

[Turn over

(iii) Given that a line L: $y = -7$ intersects the circle C_1 at the points P and Q, find the coordinates of P and Q. [3]

$(x-4)^2 - (-7+2)^2 = 50$

$x^2 + 16 - 8x + 25 = 50$

$x^2 - 8x - 9 = 0$

$(x-9)(x+1) = 0$

$x = 9 \quad x = -1$

$P(-1,-7)$, $Q(9,-7)$

(iv) Find the shortest distance from the centre of the circle C_1 to L. [2]

Shortest dist ⇒ ⊥r bisector of P & Q!

midpt of PQ $= \left(\frac{-1+9}{2}, \frac{-7-7}{2}\right)$

$= (4,-7)$

Dist $= \sqrt{(4-4)^2 + [-2-(-7)]^2}$

$= 5$ units

or Shortest Dist

$= -2+7$

$= 5$ units.

(v) Find the coordinates of the centre of a second circle C_2 which has radius $\sqrt{89}$ units and also passes through P and Q. The centre of C_2 lies above the x-axis. [3]

x-coordinate of centre of C_2 is the same as midpt of P & Q.

$\therefore C_2 \ (4, y)$

radius $= \sqrt{89} = \sqrt{[4-(-1)]^2 + [y-(-7)]^2}$

$89 = 25 + (y+7)^2$

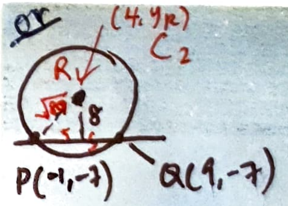
$(y+7)^2 = 64$

$y+7 = \pm 8$

$y = 1, \ -15$ (rej)

$\therefore$ centre $(4,1)$

or



RS: $\sqrt{89-5^2}$

$= \sqrt{64}$

$= 8$

$\therefore y_R + 7 = 8$

$y_R = 1$

centre $(4,1)$