Solutions

1 Given the function $y=\sqrt{\frac{1}{x}-2}$, where $x<\frac{1}{2}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and show that y has no turning points. [4]

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2\sqrt{\frac{1}{x}-2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}-2\right)$$

$$=\frac{1}{2\sqrt{\frac{1}{x}-2}}\left(-\frac{1}{x^2}\right)$$

$$=-\frac{1}{2x^{\frac{3}{2}}\sqrt{-2x+1}}$$

Fraction over a fraction is not complete.

When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $-\frac{1}{2x^{\frac{3}{2}}\sqrt{-2x+1}}=0$ has no solution. $\therefore$ y has no turning points.

2 **(i)** Express $\dfrac{-4x^3+10x^2+x-2}{x^2(2-x)}$ in partial fractions. [5]

$$\begin{array}{r} 4 \\ -x^3+2x^2 \overline{\smash{\big)}\, -4x^3+10x^2+x-2} \\ \underline{-(-4x^3+8x^2)} \\ 2x^2+x-2 \end{array}$$

$$\frac{-4x^3+10x^2+x-2}{x^2(2-x)} = 4 + \frac{2x^2+x-2}{x^2(2-x)}$$

$$\frac{2x^2+x-2}{x^2(2-x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(2-x)}$$

$$2x^2+x-2 = Ax(2-x)+B(2-x)+Cx^2$$

Sub $x=0$,

$-2=2B$

$B=-1$

Sub $x=2$,

$8=4C$

$C=2$

Sub $x=1$,

$1=A-1+2$

$A=0$

$$\frac{-4x^3+10x^2+x-2}{x^2(2-x)} = 4 + \frac{2}{(2-x)} - \frac{1}{x^2}$$

(ii) Hence find $\int \frac{-4x^3+10x^2+x-2}{x^2(2-x)}\,\mathrm{d}x$. [3]

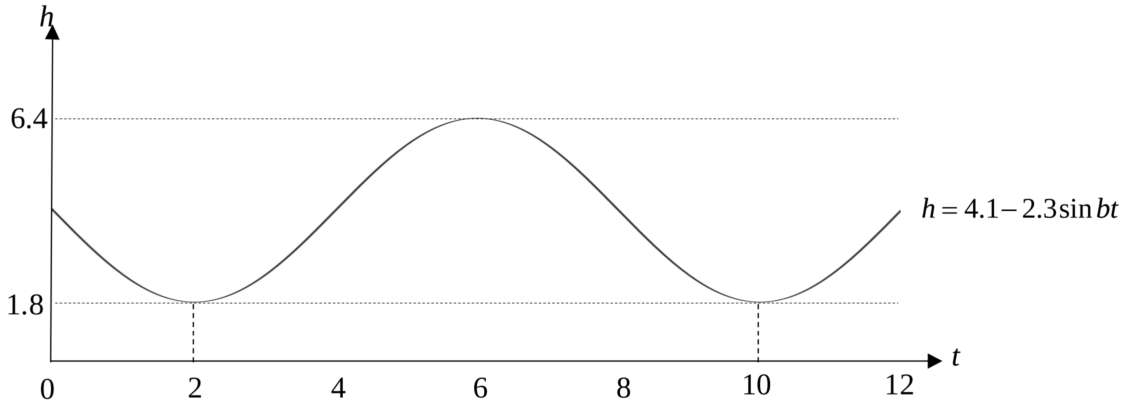
$$\int \frac{-4x^3+10x^2+x-2}{x^2(2-x)}\,\mathrm{d}x$$

$$= \int 4+\frac{2}{2-x}-\frac{1}{x^2}\,dx$$

$$= 4x-2\ln(2-x)+\frac{1}{x}+c,$$ where c is an arbitrary constant.

Deduct mark if did not put + c

3 The diagram shows the graph of $h = 4.1 - 2.3\sin bt$, for $0 \le t \le 12$.



(i) State the value of *b*. [1]

$$b = \frac{2\pi}{8} = \frac{\pi}{4}$$

(ii) The height, *h* m, of the tidal wave at a harbour *t* hours after 4am is given by $h = 4.1 - 2.3\sin bt$. A yacht at the harbour can only leave and return to the harbour when the height at the wave is at least 2 metres. Calculate the range of time in hours and minutes for the yacht to leave and return in the next 10 hours. [4]

$$4.1 - 2.3\sin\left(\frac{\pi}{4}t\right) \ge 2$$

Must be in rad mode.

$$2.3\sin\left(\frac{\pi}{4}t\right) \le 2.1$$

Solve for *t*,

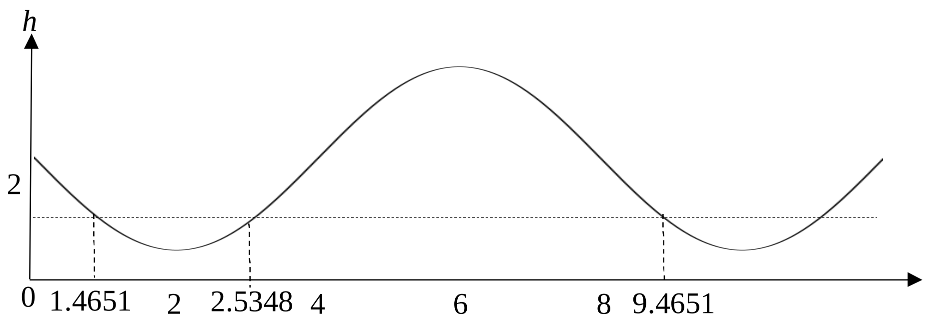
$$\sin\left(\frac{\pi}{4}t\right) = \frac{21}{23}$$

$$\alpha = 1.1507$$

$$\frac{\pi}{4}t = 1.1507,\ 1.9909,\ 7.4339$$

$$t = 1.4651,\ 2.5348,\ 9.4651$$

1.4651 h = 1 hour 27.9 min
2.5348 h = 2 hour 32.1 min
9.4651 h = 9 hour 27.9 min



Yacht to sail out between 4 am to 5.27 am and return between 6.33 am to 1.27 pm.

4 A curve is defined by $y = \dfrac{x}{\sqrt{x^2 - 2}}$.

Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and obtain the exact x – coordinates of the point(s) on the curve at which the gradient is $-\dfrac{1}{4}$. [5]

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x)\sqrt{x^2-2} - \frac{d}{dx}\left(\sqrt{x^2-2}\right)x}{\left(\sqrt{x^2-2}\right)^2}$$

$$= \frac{1\cdot\sqrt{x^2-2} - \frac{x}{\sqrt{x^2-2}}x}{\left(\sqrt{x^2-2}\right)^2}$$

$$= -\frac{2}{\sqrt{\left(x^2-2\right)^3}}$$

Fraction over a fraction is incomplete.

When gradient is $-\dfrac{1}{4}$,

$$-\frac{2}{\sqrt{\left(x^2-2\right)^3}} = -\frac{1}{4}$$

$$\sqrt{\left(x^2-2\right)^3} = 8$$

$$\left(x^2-2\right)^{\frac{3}{2}} = 8$$

$$\left(x^2-2\right) = 4$$

$$x^2 = 6$$

$$x = \pm\sqrt{6}$$

5 The line $9y = x - 31 - a$ is a normal to curve $y = -ax^3 + 3x - 5$ at the point $(-1, -4)$.

(i) Find a. [3]

Method 1

Sub $(-1, -4)$ into either line or curve:

$9(-4) = (-1) - 31 - a$

$-36 = -32 - a$

$a = 4$

Method 2

$y = -ax^3 + 3x - 5$

$\frac{dy}{dx} = -3ax^2 + 3$

$9y = x - 31 - a$

$y = \frac{1}{9}x - \frac{31+a}{9}$

At $(-1, -4)$,

since grad(normal) $\times$ grad(curve) = -1

Gradient of curve $= -9$

$-3a(-1)^2 + 3 = -9$

$-a + 1 = -3$

$a = 4$

(ii) With the value of a, show that the normal at $(-1, -4)$ will not cut the curve again. [5]

When $a = 4$,

$-4x^3 + 3x - 5 = \frac{1}{9}(x - 35)$

$-36x^3 + 27x - 45 = x - 35$

$-36x^3 + 26x - 10 = 0$

$18x^3 - 13x + 5 = 0$

At $(-1, -4)$, $(x+1)$ is a factor of $18x^3 - 13x + 5$.

$18x^3 - 13x + 5 = (x+1)(18x^2 - bx + 5)$

Compare coefficients of x^2,

$0 = 18 - b$

$b = 18$

Discriminant of $18x^2 - 18x + 5$

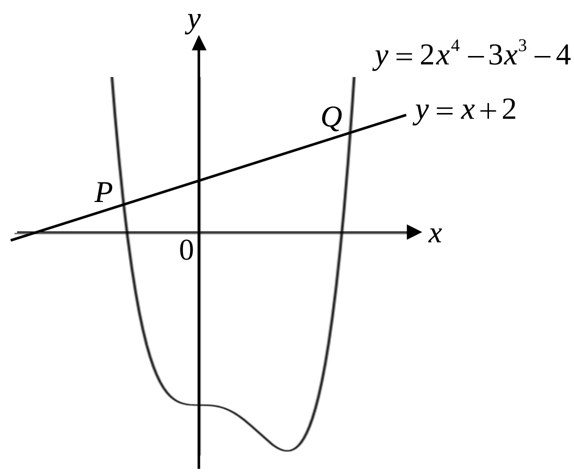
$= (-18)^2 - 4(18)(5)$

$= -36 < 0$

No real roots for $18x^2 - 18x + 5$.

Normal at $(-1, -4)$ will not cut the curve again

6 **(a)** The diagram shows the curve $y = 2x^4 - 3x^3 - 4$ and the line $y = x + 2$. The curve and the line intersects at points $P(-1,1)$ and $Q(2,4)$.



Find the area bounded by the line $y = x + 2$ and the curve $y = 2x^4 - 3x^3 - 4$. [3]

$$\int_{-1}^{2} (x+2) - \left(2x^4 - 3x^3 - 4\right) \mathrm{d}x$$

$$= \left[-\frac{2x^5}{5} + \frac{3x^4}{4} + \frac{x^2}{2} + 6x \right]_{-1}^{2}$$

$$= \left(-\frac{2(2)^5}{5} + \frac{3(2)^4}{4} + \frac{(2)^2}{2} + 6(2) \right) - \left(-\frac{2(-1)^5}{5} + \frac{3(-1)^4}{4} + \frac{(-1)^2}{2} + 6(-1) \right)$$

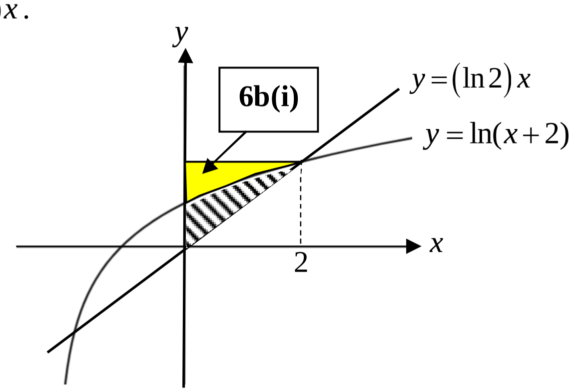
$$= \frac{351}{20} \text{ or } 17.55 \text{ units}^2$$

(b) **(i)** Find the exact value of $\int_{\ln 2}^{\ln 4} e^x - 2 \, dx$. [2]

$$\int_{\ln 2}^{\ln 4} e^x - 2 \, dx$$
$$= \left[e^x - 2x\right]_{\ln 2}^{\ln 4}$$
$$= (4 - 2\ln 4) - (2 - 2\ln 2)$$
$$= 2 - \ln 4 \quad (\text{or} \quad 2 - 2\ln 2)$$

Exact!

(ii) The diagram below shows part of the curve of $y = \ln(x+2)$ and the line $y = (\ln 2)x$.



Use your result in **6b(i)** to find the area of the shaded region. [3]

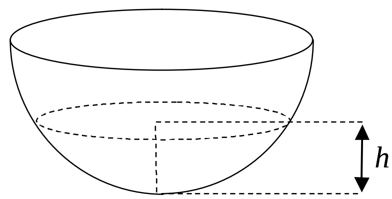
Method 1
(wrt y-axis)
Area of triangle
$\frac{1}{2} \times 2 \times 2\ln 2$
$= 2\ln 2$
area of the shaded region
$= 2\ln 2 - (2 - 2\ln 2)$
$= 4\ln 2 - 2$

Method 2
$y = \ln(x+2)$
$e^y = x + 2$
$x = e^y - 2$
area of the shaded region
$= \frac{1}{2} \times \ln 4 \times 2 - \int_{\ln 2}^{\ln 4} (e^y - 2) \, dy$
$= \ln 4 - (2 - \ln 4)$
$= 2\ln 4 - 2$ or $4\ln 2 - 2$ units^2

7 The diagram shows a hollow hemispherical bowl. The bowl is initially empty. At time $t=0$, water flows into the bowl. When the height of the water is at h m, the volume of water, V m^3, is given by

$$V=\frac{1}{12}\pi h^2(3-4h),\ 0\le h\le 0.25.$$

Water flows into the bowl at a constant rate of $\frac{\pi}{600}$ m^3/s.



(i) Find the rate of change of height when the height of the water is at 15 cm. [3]

$$V=\frac{1}{12}\pi h^2(3-4h)$$
$$=\frac{1}{12}\pi\left(3h^2-4h^3\right)$$
$$\frac{dV}{dh}=\frac{1}{12}\pi\left(6h-12h^2\right)$$

When $h=0.15$,

$$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$$
$$\frac{\pi}{600}=\frac{1}{12}\pi\left(6(0.15)-12(0.15)^2\right)\times\frac{dh}{dt}$$
$$\frac{\pi}{600}=\frac{21\pi}{400}\times\frac{dh}{dt}$$
$$\frac{dh}{dt}=\frac{2}{63}\text{ m/s}$$

(ii) Find the time taken for the bowl to be filled. [2]

When $h=0.25$,

$$V=\frac{1}{2}\left(\frac{4}{3}\pi\times 0.25^3\right)$$
$$=\frac{\pi}{96}\text{ m}^3$$

time to fill the bowl = $\frac{\pi}{96}\div\frac{\pi}{600}$

=6.25 s

8 Solve the equation $\log_3(x+1)-\dfrac{8}{\log_{27}(x+1)}=2$. [5]

$$\log_3(x+1)-\frac{8}{\left(\dfrac{\log_3(x+1)}{\log_3 27}\right)}=2$$

$$\log_3(x+1)-\frac{8}{\left(\dfrac{\log_3(x+1)}{3}\right)}=2$$

$$\log_3(x+1)-\frac{3(8)}{\log_3(x+1)}=2$$

$$\left[\log_3(x+1)\right]^2-24=2\log_3(x+1)$$

$$\left[\log_3(x+1)\right]^2-2\log_3(x+1)-24=0$$

Let $\log_3(x+1)=u$

$u^2-2u-24=0$

$(u-6)(u+4)=0$

$u=6$ or $u=-4$

$\log_3(x+1)=6$ or $\log_3(x+1)=-4$

$(x+1)=3^6$ $\quad$ $(x+1)=3^{-4}$

$x=728$ $\quad$ $x=-\dfrac{80}{81}$

9 **(i)** Find the exact value of k such that $\cos\left(x+\frac{\pi}{3}\right)-\cos\left(x-\frac{\pi}{3}\right)=k\sin x$. [3]

$$\cos\left(x+\frac{\pi}{3}\right)-\cos\left(x-\frac{\pi}{3}\right)$$
$$=\frac{1}{2}\cos(x)-\frac{\sqrt{3}}{2}\sin(x)-\left(\frac{1}{2}\cos(x)+\frac{\sqrt{3}}{2}\sin(x)\right)$$
$$=-\sqrt{3}\sin(x)$$
$$\therefore k=-\sqrt{3}$$

(ii) Hence determine a suitable value of x and find the value of $\cos\left(\frac{7\pi}{6}\right)-\cos\left(\frac{\pi}{2}\right)$. [2]

$$x+\frac{\pi}{3}=\frac{7\pi}{6} \quad ; \quad x-\frac{\pi}{3}=\frac{\pi}{2}$$
$$x=\frac{5\pi}{6} \qquad\quad x=\frac{5\pi}{6}$$

Recognise $x=\frac{5\pi}{6}$.

$$\cos\left(\frac{7\pi}{6}\right)-\cos\left(\frac{\pi}{2}\right)=-\sqrt{3}\sin\frac{5\pi}{6}$$
$$=-\frac{\sqrt{3}}{2}$$

10 A particular curve for which $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sin 2x - x + 1$ passes through the point $H(0,3)$. The gradient of the curve at H is 1.5. Find the equation of the curve. [5]

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sin 2x - x + 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\cos(2x) - \frac{x^2}{2} + x + c$$

At $H(0,3)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1.5$

$$-\frac{1}{2}\cos(0) + c = 1.5$$

$$-\frac{1}{2} + c = 1.5$$

$$c = 2$$

$$y = -\frac{1}{4}\sin(2x) - \frac{x^3}{6} + \frac{x^2}{2} + 2x + d$$

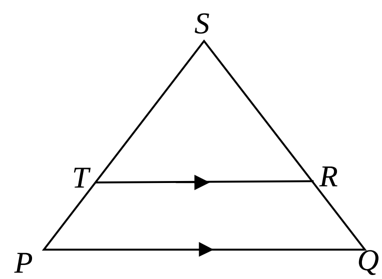
At $H(0,3)$,

$$-\frac{1}{4}\sin(0) + d = 3$$

$$d = 3$$

$$y = -\frac{1}{4}\sin(2x) - \frac{x^3}{6} + \frac{x^2}{2} + 2x + 3$$

11



The diagram shows two triangles *TRS* and *PQS*. *TR* is parallel to *PQ* and *TR* = $4-\sqrt{8}$ cm and *PQ* = $4+\sqrt{32}$ cm

Find the ratio of $\dfrac{\text{area of triangle } TRS}{\text{area of triangle } PQS}$ in the form $\dfrac{a+b\sqrt{2}}{2}$, where a and b are constants. [4]

Recognise that △TRS and △PQS are similar triangles.

$$\frac{\Delta TRS}{\Delta PQS}=\left(\frac{4-\sqrt{8}}{4+\sqrt{32}}\right)^2$$

$$=\left(\frac{4-2\sqrt{2}}{4+4\sqrt{2}}\right)^2$$

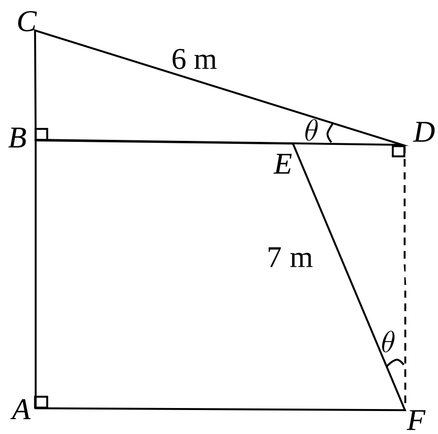
$$=\frac{24-16\sqrt{2}}{48+32\sqrt{2}}$$

$$=\frac{3-2\sqrt{2}}{2(3+2\sqrt{2})}\times\frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$

$$=\frac{9-12\sqrt{2}+8}{2(1)}$$

$$=\frac{17-12\sqrt{2}}{2}$$

12



The diagram shows a park such that lengths *CD* and *EF* are 6 m and 7 m respectively and *AF* = *BD*. The acute angle *CDB* and *EFD* is θ radians. A fence is built along *CBEF*.

(i) Show that the length of fence, *P* m, is given by $P = 7 + m\cos\theta + n\sin\theta$ where *m* and *n* are integers to be found. [2]

$CB = 6\sin\theta$

$BE = 6\cos\theta - 7\sin\theta$

$P = 6\sin\theta + (6\cos\theta - 7\sin\theta) + 7$

$\quad = 7 + 6\cos\theta - \sin\theta$

$m = 6$; $n = -1$

(ii) Express *P* in the form $7 + R\cos(\theta + \alpha)$, where $R > 0$ and α is an acute angle. [4]

$R = \sqrt{6^2 + 1^2} = \sqrt{37}$

$\alpha = \tan^{-1}\dfrac{1}{6} = 0.16515$

(iii) Given that 10 m of wire was used for fencing, find the value of θ. [3]

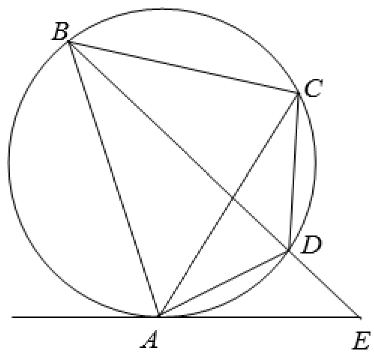
$$7+\sqrt{37}\cos(\theta+0.16515)=10$$

$$\cos(\theta+0.16515)=\frac{3}{\sqrt{37}}$$

$$\theta+0.16515= 1.05504$$

$$\theta=0.890 \text{ (3s.f.)}$$

13



In the diagram above, a line from point E intersects a circle at point D and B. The line AE is tangent to the circle at the point A. Line BE bisects angle ABC.

Prove that

(i) Triangle DAC is isosceles. [4]

$\angle DAE = \angle ACD = \angle ABD$ (alt $\angle$ segment)

Or

$\angle ABD = \angle ACD$ ($\angle$ in the same segment)

$\angle ABD = \angle DBC$ (BE bisects ABC)

$\angle DBC = \angle DAC$ ($\angle$ in the segment)

$\therefore \angle DAC = \angle ACD$

$\angle DAC$ and $\angle ACD$ form the base angle of isosceles ΔDAC .

(ii) Triangle DAE is similar to Triangle ABE. [2]

$\angle DAE = \angle ABE$ (from (i))
$\therefore \angle DEA = \angle AEB$ (common)
ΔDAE is similar to ΔABE (AA).

(iii) $AE \times AB = BE \times CD$. [2]

Since ΔDAE is similar to ΔABE,

$$\frac{AE}{BE} = \frac{AD}{AB}$$

From (i) $AD = CD$

$\therefore AE \times AB = BE \times CD$

14 A particle starts from rest travels in a straight line so that, t seconds after passing a fixed point O, its velocity v m/s, is given by $v = e^{-0.5t} - \frac{1}{4}$. The particle comes to instantaneous rest at A. Find

(i) the acceleration of the particle at A, [4]

At A, $v = 0$.

$e^{-0.5t} - \frac{1}{4} = 0$

$e^{-0.5t} = \frac{1}{4}$

$-0.5t = \ln\frac{1}{4}$

$t = 2\ln 4$ or 2.77259

$\frac{dv}{dt} = -0.5e^{-0.5t}$

At $t = 2\ln 4$, $\frac{dv}{dt} = -0.5e^{-0.5(2\ln 4)} = -\frac{1}{8}\ m/s^2$

(ii) Explain why the total distance travelled by the particle in the interval $t = 0$ to $t = 5$ is not obtained by finding the value of s when $t = 5$. [2]

When $t = 5$, s gives the displacement of the particle from O to when $t = 5$. It does not take into account the total distance the particle makes from due to the reverse/change of direction at $t = 2\ln 4$.

(iii) the total distance travelled by the particle between $t = 0$ and $t = 5$. [5]

$$
\begin{aligned}
s &= \int v \, \mathrm{d}t \\
&= \int e^{-0.5t} - \frac{1}{4} \, \mathrm{d}t \\
&= \frac{e^{-0.5t}}{-0.5} - \frac{1}{4}t + c \\
&= -2e^{-0.5t} - \frac{1}{4}t + c
\end{aligned}
$$

When $t = 0$, $s = 0$

$$
\begin{aligned}
-2e^{-0.5(0)} - \frac{1}{4}(0) + c &= 0 \\
-2 + c &= 0 \\
c &= 2
\end{aligned}
$$

$$s = -2e^{-0.5t} - \frac{1}{4}t + 2$$

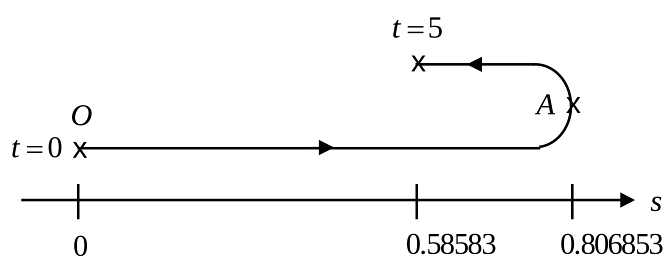
When $t = 2\ln 4$, $s = -\frac{1}{2} - \frac{1}{2}\ln 4 + 2 = 0.806853$

When $t = 5$, $s = -2e^{-0.5(5)} - \frac{1}{4}(5) + 2 = 0.58583$

Total distance travelled from $t = 0$ to $t = 5$

$= 0.806853 + (0.806853 - 0.58583)$

$= 1.03$ m



End of paper