CCX
451/2/3

Paper 2

1 James just bought a brand new car. The value, V dollars, of the car depreciates over time. It is given that $V = 84\,000\mathrm{e}^{kt} + 8\,500$, where t is the time in years since it was bought and k is a constant.

(a) What is the initial value of the vehicle? [1]

When $t=0$, $V = 84\,000\,e^{0} + 8\,500$ A1

$= 92\,500$.

(b) Calculate the value of k if, after 3 years, the value of the car is halved. [3]

When $t=3$, $V = \frac{1}{2}(92\,500) = 46250$

$\therefore\ 84\,000\,e^{3k} + 8\,500 = \frac{1}{2}(92\,500)$ M1

$84\,000\,e^{3k} = 37\,750$

$e^{3k} = \frac{37\,750}{84\,000}$

$3k = \ln\frac{37\,750}{84\,000}$ M1

$k = \frac{1}{3}\ln\frac{37\,750}{84\,000}$

$= -0.266\,610$

$= -0.267$ (3 s.f.) A1

(c) After having driven the car for 25 years, James decided to change to a new car. A second-hand car dealer offers to buy the old car from him for \$8 000. Without using a calculator, justify whether he should accept the offer. [2]

$V = 84\,000e^{kt} + 8\,500$

$\therefore\ V > 8\,500$ since $e^{kt} > 0$ M1

if calculate value how?

$\therefore$ The offer given by the dealer is only \$8 000 which is less than \$8 500.

$\therefore$ James should not accept the offer. A1

2 (a) Prove that $\dfrac{1-\sin x}{\cos x}+\dfrac{\cos x}{1-\sin x}=2\sec x$. [3]

$$\frac{1-\sin x}{\cos x}+\frac{\cos x}{1-\sin x}$$

$$=\frac{(1-\sin x)^2+\cos^2 x}{\cos x(1-\sin x)}$$ M1 (combine as fraction)

$$=\frac{1-2\sin x+\sin^2 x+\cos^2 x}{\cos x(1-\sin x)}$$

$$=\frac{1-2\sin x+1}{\cos x(1-\sin x)}$$ M1 (use of Pythagorean Identity)

$$=\frac{2(1-\sin x)}{\cos x(1-\sin x)}$$

$$=\frac{2}{\cos x}$$

$$=2\sec x$$ A1

(b) Reduce the equation $\frac{1-\sin x}{\cos x}+\frac{\cos x}{1-\sin x}=2-\tan^2 x$ to the form $\cos x = a$ or b, where a and b are constants and $b<0$.

(i) Find the value of a and of b. [2]

$$\frac{1-\sin x}{\cos x}+\frac{\cos x}{1-\sin x} = 2-\tan^2 x$$

$$2\sec x = 2-(\sec^2 x - 1) \quad \text{M1}$$

$$\sec^2 x + 2\sec x - 3 = 0$$

$$(\sec x + 3)(\sec x - 1) = 0$$

$$\therefore \sec x = -3 \quad \text{or} \quad \sec x = 1$$

$$\cos x = -\tfrac{1}{3} \quad \text{or} \quad \cos x = 1 \quad \text{A1}$$

$$a = 1 \qquad b = -\frac{1}{3}$$

(ii) Solve the equation $\cos x = b$ for $-\pi \le x \le 2\pi$. [2]

$$\cos x = -\tfrac{1}{3}$$

basic angle $= 1.230\,959$ M1

For $-\pi \le x \le 2\pi$,

$$x = -(\pi - 1.230\,959),\ \pi - 1.230\,959,\ \pi + 1.230\,959$$

$$= -1.91,\ 1.91,\ 4.37 \quad (3\text{ s.f.}) \quad \text{A1}$$

missing 1 value?

-4.37 (extra)?

[Turn over

3 A curve has the equation $y = 3x^2 e^{-x}$.

(a) Find the x-coordinates of the stationary points. [4]

$$y = 3x^2 e^{-x}$$

$$\frac{dy}{dx} = 6xe^{-x} + 3x^2(-e^{-x})$$

$$= 3xe^{-x}(2-x) \qquad B2$$

For stationary points, $\frac{dy}{dx} = 0$

$\therefore \quad 3xe^{-x}(2-x) = 0 \qquad$ M1

Since $e^{-x} > 0$, $\quad \therefore \quad x = 0 \quad$ or $\quad x = 2 \qquad$ A1

(b) Determine the nature of the stationary points. [3]

x	0^-	0	0^+
$\frac{dy}{dx}$	$-$	0	$+$
	\	—	/

M1

x	2^-	2	2^+
$\frac{dy}{dx}$	$+$	0	$-$
	/	—	\

$\therefore$ At $x = 0$, there is a minimum point A1

At $x = 2$, there is a maximum point. A1

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = -0.812$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 6$$

4 (i) Express $\dfrac{17+6x-5x^2}{(2x-1)(3-x)^2}$ in partial fractions. [4]

$$\frac{17+6x-5x^2}{(2x-1)(3-x)^2} = \frac{A}{2x-1} + \frac{B}{3-x} + \frac{C}{(3-x)^2} \quad \text{M1}$$

Multiplying by $(2x-1)(3-x)^2$,

$\therefore 17+6x-5x^2 = A(3-x)^2 + B(2x-1)(3-x) + C(2x-1)$

$x=3$: $5C = 17+6\times3-5\times9$

$5C = -10$

$C = -2$

$x=\frac{1}{2}$: $\frac{25}{4}A = 17+6(\frac{1}{2})-5(\frac{1}{4})$

$\frac{25}{4}A = \frac{75}{4}$

$A = 3$

x^2: $A-2B=-5 \Rightarrow 2B = A+5$

$2B = 8$

$B = 4$

M2

$$\therefore \frac{17+6x-5x^2}{(2x-1)(3-x)^2} = \frac{3}{2x-1} + \frac{4}{3-x} - \frac{2}{(3-x)^2} \quad \text{A1}$$

$-2(3-x)^{-2}$

$-2(3-x)^{-1}$

$\overline{-1(-1)}$

$+\frac{2}{3-x}$

(ii) Hence find the value of $\displaystyle\int_1^2 \frac{17+6x-5x^2}{(2x-1)(x-3)^2}\,dx$. [3]

$$\therefore \int_1^2 \frac{17+6x-5x^2}{(2x-1)(x-3)^2}\,dx = \int_1^2 \frac{3}{2x-1} + \frac{4}{3x-1} - \frac{2}{(3-x)^2}\,dx$$

$$= \left[\frac{3}{2}\ln(2x-1) + \overbrace{\frac{4}{3}\ln(3x-1)}^{\text{B1}} + \overset{\text{B1}}{\frac{2}{3-x}}\right]_1^2$$

$$=\left(\frac{3}{2}\ln 3 + \frac{4}{3}\ln 5 + 2\right) - \left(\frac{3}{2}\ln 1 + \frac{4}{3}\ln 2 + 1\right)$$

$$= \frac{3}{2}\ln 3 + \frac{4}{3}\ln\frac{5}{2} + 1$$

$$= 3.87 \quad \text{A1}$$

4ii) $\displaystyle\int_1^2 \overset{\text{B1}}{\frac{3}{2}\ln(2x-1)} - 4\ln(3-x) - \overset{\text{B1}}{\frac{2}{3-x}}\,dx$

$= \left[\frac{3}{2}\ln 3 - 4\ln(1) - 2\right] - \left[\frac{3}{2}\ln 1 - 4\ln 2 - 1\right)$

$= 3.42\ldots$ A1

[Turn over

5 A polynomial $P(x) = ax^3 + x^2 + bx - 5$ has a factor $(x + 1)$ and a remainder of 9 when divided by $(x - 2)$.

(a) Show that $a = 3$ and find the value of b. [4]

$P(x) = ax^3 + x^2 + bx - 5$

$P(-1) = 0$

$\Rightarrow a(-1) + 1 + b(-1) - 5 = 0$

$a + b = -4 \quad \cdots (1) \quad$ M1

$P(2) = 9$

$\Rightarrow a(8) + 4 + b(2) - 5 = 9$

$8a + 2b = 10 \quad \cdots (2) \quad$ M1

$(1) \times 2 \quad 2a + 2b = -8 \quad \cdots (3)$

$(2) - (3): \quad 6a = 18$

$a = 3$

$\therefore b = -7 \quad$ A2

$\therefore P(x) = 3x^3 + x^2 - 7x - 5$

(b) Hence factorise $\mathrm{P}(x)$ completely. [2]

$$P(x) = 3x^3 + x^2 - 7x - 5$$
$$= (x+1)(3x^2 - 2x - 5) \quad \text{M1}$$
$$= (x+1)(3x-5)(x+1)$$
$$= (x+1)^2(3x-5) \quad \text{A1}$$

(c) Using the result in **(b)**, solve the equation $8ax^3 - 4x^2 + 2bx + 5 = 0$. [2]

$$8ax^3 - 4x^2 + 2bx + 5 = 0$$
$$-8ax^3 + 4x^2 - 2bx - 5 = 0$$
$$a(-2x)^3 + (-2x)^2 - b(2x) - 5 = 0 \quad \text{M1}$$
$$P(-2x) = 0$$
$$\therefore (-2x+1)^2(-6x-5) = 0$$
$$x = \frac{1}{2} \text{ or } x = -\frac{5}{6} \quad \text{A1}$$

<u>OR</u> $\quad P(x) = 0 \Rightarrow x = -1 \text{ or } x = \frac{5}{3}$

$\therefore P(-2x) = 0 \Rightarrow -2x = -1 \text{ or } -2x = \frac{5}{3}$ M1

$x = +\frac{1}{2} \text{ or } x = -\frac{5}{6}$ A1

[Turn over

6 The depth of water measured from sea level around Tiny Island on a particular day, y m, is given by $y = 2 - 4\cos(30t)°$, where t is the time in hours after midnight and the function is calculated using degrees.

(a) What is the depth of the water around Tiny Island at 14 00 on that day? [2]

At 14 00, $t = 14$,

$\therefore y = 2 - 4\cos(30 \times 14)°$ M1

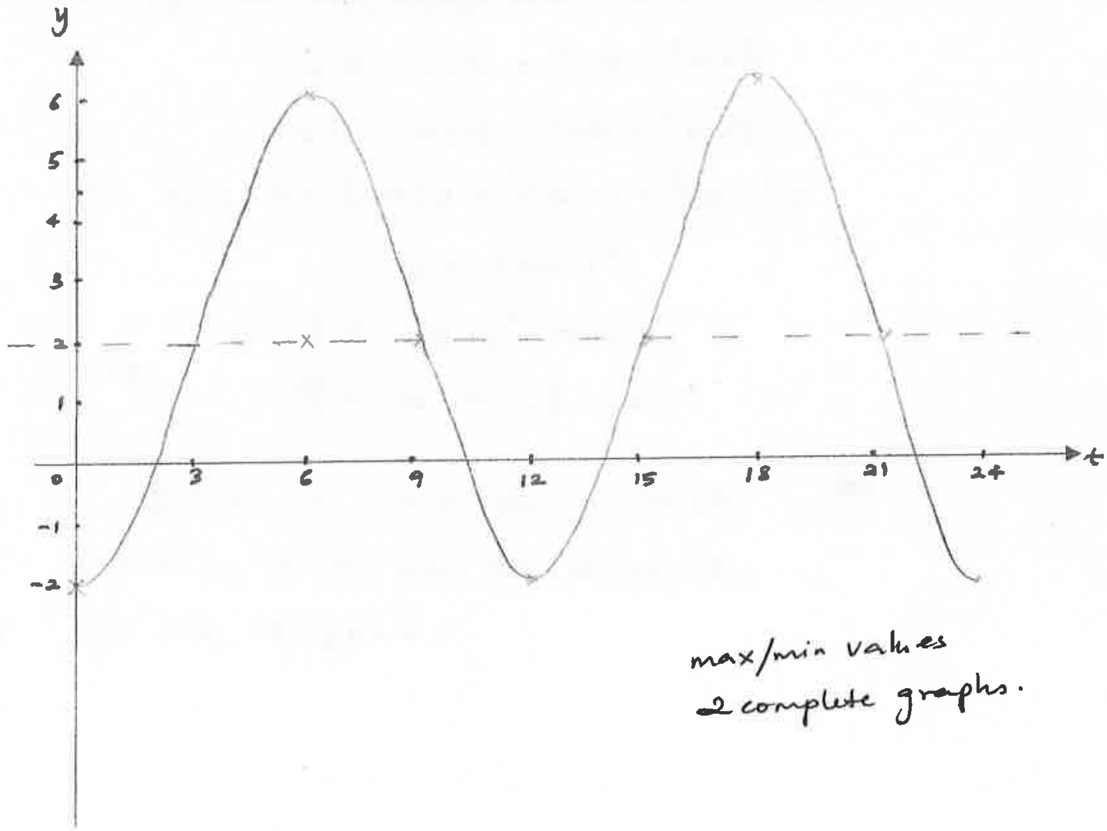
$= 2 - 4\cos 420°$

$= 2 - 4\cos 60°$

$= 2 - 4(\frac{1}{2})$

$= 0$ A1

(b) Sketch the graph of $y = 2 - 4\cos(30t)°$ where $0 \le t \le 24$. [2]



max/min values
2 complete graphs.

Tiny Island will be fully submerged underwater when the depth of the water around it reaches a depth of 4 metres.

(c) Find the duration when Tiny Island is underwater for that day. Give your answer using the 24-hour clock notation. [3]

When $y = 4$

$0 < t < \cancel{360}$ 24

$\therefore\ 2 - 4\cos(30t)^\circ = 4$

$\cos(30t)^\circ = \frac{2-4}{4}$

$= -\frac{1}{2}$

basic angle $= 60^\circ$ M1

$\therefore (30t)^\circ = 180^\circ - 60^\circ,\ 180^\circ + 60^\circ,\ 540^\circ - 60^\circ,\ 540^\circ + 60^\circ$

$= 120^\circ,\ 240^\circ,\ 480^\circ,\ 600^\circ$

$\therefore\ t = 4,\ 8,\ 16,\ 20$ A1

$\therefore$ $\cancel{0000 < t < 0400,\ 0800 < t < 1600 \text{ or } 2000 < t < 2400}$ A1

4 8 4

$0400 \leq t \leq 0800$ or $1600 \leq t \leq 2000$

7 A particle travels in a straight line from point A to point B such that t seconds after leaving point A, its velocity v ms^{-1} is given by $v = 4e^{2-\frac{t}{2}} - 3$. It comes to instantaneous rest at point B. It then travels at a constant acceleration of 5 ms^{-2} until it reaches point C, 18 seconds after leaving point A. Find

(a) the initial velocity, [1]

$v = 4e^{2-\frac{t}{2}} - 3$

When $t = 0$,

$V = 4e^{2} - 3$

$= 26.556$

$= 26.6 \text{ ms}^{-1}$ (3 s.f.) A1

(b) the time when the particle reaches B, [2]

At B, $v = 0$

$\therefore 4e^{2-\frac{t}{2}} - 3 = 0$ M1

$e^{2-\frac{t}{2}} = \frac{3}{4}$

$\therefore 2 - \frac{t}{2} = \ln\frac{3}{4}$

$\therefore t = 2\left[2 - \ln\frac{3}{4}\right]$ A1

$= 4.57536$

$= 4.58$ s (3 s.f.)

(c) the distance travelled from A to B, [3]

$$\text{Distance from A to B} = \int_0^{4.575} 4e^{2-\frac{t}{2}} - 3\, dt \quad \text{M1}$$

$$= \left[\frac{4e^{2-\frac{t}{2}}}{-\frac{1}{2}} - 3t \right]_0^{4.575} \quad \text{M1}$$

$$= -\left(8e^{2-\frac{t}{2}} + 3t \right)_0$$

$$= -\left[8e^{2-\frac{4.575}{2}} + 3(4.575) \right] + \left[8e^{2} + 0 \right]$$

$$= -19.726 + 59.112$$

$$= 39.386$$

$$= 39.4 \quad (3 \text{ s.f.}) \quad \text{A1}$$

(d) the velocity of the particle at C. [2]

At C, $a = 5 \text{ m/s}^2$.

$$v = \int 5 \cdot dt$$

$$= 5t + c_1 \quad \text{M1}$$

At B, $v = 0$.

$$\Rightarrow 5(4.575) + c_1 = 0$$

$$\Rightarrow c_1 = -22.875$$

$$\therefore \quad v = 5t - 22.875$$

At C, when $t = 18$,

$$v = 5(18) - 22.875$$

$$= 90 - 22.875 \quad \text{A1}$$

$$= 67.125 \text{ ms}^{-1}$$

[Turn over

8 **(a)** Variables x and y are related by the equation $\lg ay = \lg b^x$, where a and b are constants. Explain clearly how a and b can be calculated when a graph of $\lg y$ against x is drawn. [2]

$$\lg ay = \lg b^x$$
$$\lg a + \lg y = x \lg b$$
$$\lg y = x \lg b - \lg a \qquad \text{M1}$$

$\therefore$ When plotting $\lg y$ against x, a straight line is obtained where gradient of line $= \lg b$ and Y-intercept $= -\lg a$ } A1

(b) A solid right cone of curved surface area A cm^2 has a slant height of $(ax - b)$ cm and a base radius of $3x$ cm. The values of x and A are shown in the table.

x	0.2	0.3	0.4	0.5	0.6
A	1.90	5.65	15.30	18.90	28.27

One of the values of A is subjected to an abnormally large error.

(i) On the grid on page 15, draw a straight line graph by plotting $\frac{A}{x}$ against x. [2]

x	0.2	0.3	0.4	0.5	0.6
$\frac{A}{x}$	9.5	18.83	38.25	37.8	47.12

(ii) Hence estimate the value of each of the constants a and b. [2]

$$A = \pi r l$$
$$= \pi(3x)(ax - b)$$
$$\therefore \frac{A}{x} = 3\pi(ax - b)$$

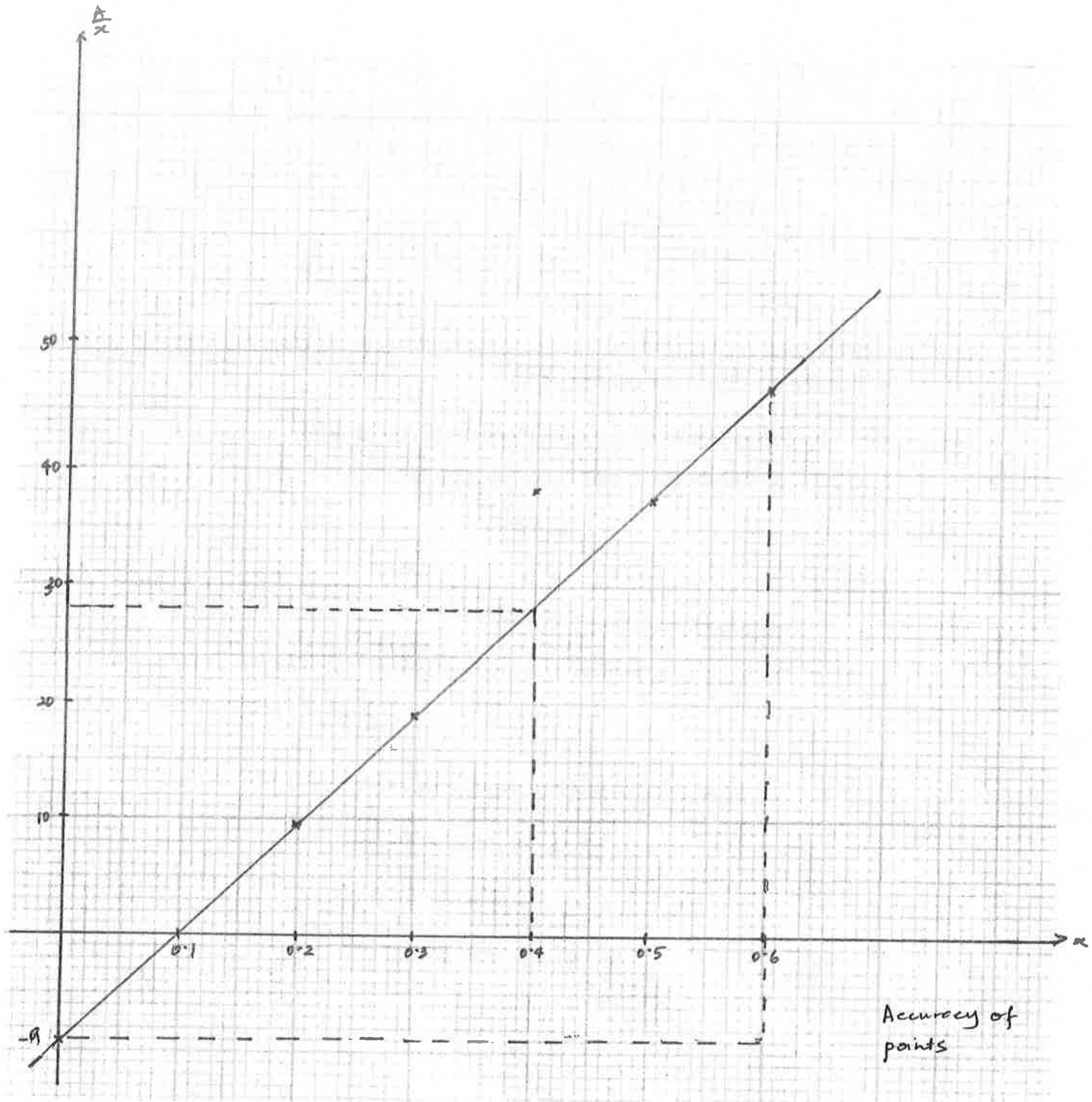
$$\therefore 3\pi a = \text{gradient}$$
$$= \frac{47 - (-9.0)}{0.6 - 0}$$
$$= 93.333$$
$$\therefore a = \frac{93.333}{3\pi} \qquad \text{A1}$$
$$= 9.90 \text{ (2 d.p.)}$$

$$-3\pi b = -9$$
$$b = \frac{9}{3\pi}$$
$$= 0.9549 \qquad \text{A1}$$
$$= 0.95 \text{ (2 d.p.)}$$

(ii) Identify the abnormal reading of A and estimate the correct value. [2]

The correct reading when $x = 0.4$ is

$$\frac{A}{x} = 28 \qquad \text{M1}$$
$$\therefore A = 28(0.4)$$
$$= 11.20 \qquad \text{A1}$$



(iii) Estimate the value of x if the curved surface area of the cone is 20 cm^2. [2]

When $A = 20$,

$\therefore 3\pi x(ax - b) = 20$

$93{\cdot}333x^2 - 9x - 20 = 0$ M1

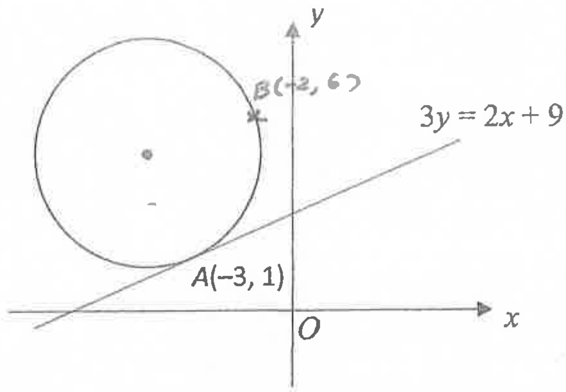
$\therefore x = \dfrac{9 \pm \sqrt{(-9)^2 - 4 \times 93{\cdot}333 \times (-20)}}{2 \times 93{\cdot}333}$

$= 0{\cdot}5136$

$= 0{\cdot}51$ (2 d.p.) A1

[Turn over

9



In the diagram, the circle with centre C passes through the point $B(-2, 6)$ and touches the line $3y = 2x + 9$ at the point $A(-3, 1)$.

(a) Find the equation of the perpendicular bisector of AB. [3]

gradient of $AB = \frac{6-1}{-2-(-3)}$

$= \frac{5}{1}$

$= 5$ M1

$\therefore$ gradient of its perpendicular bisector $= -\frac{1}{5}$

Midpoint of AB, $M = \left(\frac{-2-3}{2}, \frac{6+1}{2}\right)$

$= \left(-\frac{5}{2}, \frac{7}{2}\right)$ M1

$\therefore$ Equation of perpendicular bisector is

$y - \frac{7}{2} = -\frac{1}{5}\left(x + \frac{5}{2}\right)$

$\therefore y = -\frac{1}{5}x - \frac{1}{2} + \frac{7}{2}$

A1

$y = -\frac{1}{5}x + 3$

(b) Show that the coordinates of C is (–5, 4). [4]

C lies on the perpendicular bisector.

$\therefore \quad y = -\frac{1}{5}x + 3 \quad -(2)$

$(-3, 1) \quad y = mx + c$

$1 = -\frac{3}{2}(-3) + c$

$c = -3.5$

Eqn CA

$y = -\frac{3}{2}x - 3.5 \quad -(1)$

Let $C = (a,\ 3 - \frac{1}{5}a)$ M1

$CA \perp$ tangent

$\Rightarrow$ gradient of $CA = -\frac{3}{2}$

$\therefore \quad \dfrac{(3-\frac{1}{5}a) - 1}{a + (3)} = -\dfrac{3}{2}$ M1

$3a + 9 = -2(2 - \frac{1}{5}a)$

$3a + 9 = -4 + \frac{2}{5}a$

$\frac{13}{5}a = -13$ M1

$\therefore a = -5$

$\therefore \quad C = (-5,\ 3 - \frac{1}{5}(-5))$

$= (-5,\ 4).$ A1

(c) Find the equation of the circle. [2]

radius $= CA$

$= \sqrt{(-5+3)^2 + (4-1)^2}$

$= \sqrt{4+9}$

$= \sqrt{13}$ M1

$\therefore$ Equation of circle is

$(x+5)^2 + (y-4)^2 = 13.$ A1

[Turn over

(d) Find the number of points of intersection between the line $y = 4(x + 3)$ and the circle. Justify your answer without finding the points of intersection. [3]

$y = 4(x+3)$ ... (1)

$(x+5)^2 + (y-4)^2 = 13$ ... (2)

Sub (1) into (2),

$(x+5)^2 + (4x+12-4)^2 = 13$ M1

$x^2 + 10x + 25 + 16x^2 + 64x + 64 = 13$

$17x^2 + 74x + 76 = 0$

$D = 74^2 - 4 \times 17 \times 76$

$= +308$ M1

$\therefore$ The line intersects the circle at 2 distinct points. A1

10 (i) Show that $\frac{d}{dx}\left(\tan x \sin^2 x\right) = 2\sin^2 x + \sec^2 x - 1$. [3]

B2

$$\frac{d}{dx}(\tan x \sin^2 x) = \tan x\,(2\sin x\cos x) + \sin^2 x\,(\sec^2 x)$$

$$= \frac{\sin x}{\cos x} \times 2\sin x\cos x + \sin^2 x\left(\frac{1}{\cos^2 x}\right)$$

$$= 2\sin^2 x + \tan^2 x$$

$$= 2\sin^2 x + \sec^2 x - 1 \quad \text{A1}$$

Integrating, $\tan x \sin^2 x + c = \int (2\sin^2 x + \sec^2 x - 1)\,dx$ M1

$$\therefore \left[\tan x \sin^2 x\right]_{\frac{\pi}{3}}^{\pi} = 2\int_{\frac{\pi}{3}}^{\pi} \sin^2 x\,dx + 2\int_{\frac{\pi}{3}}^{\pi} \sec^2 x - 1\,dx$$

$$0 - \left(\sqrt{3} \times \left(\frac{\sqrt{3}}{2}\right)^2\right) = 2\int_{\frac{\pi}{3}}^{\pi} \sin^2 x\,dx + 2\left[\tan x - x\right]_{\frac{\pi}{3}}^{\pi} \quad \text{M1}$$

$$\therefore 2\int_{\frac{\pi}{3}}^{\pi} \sin^2 x\,dx = -\sqrt{3} \times \frac{3}{4} - 2\left[\tan x - x\right]_{\frac{\pi}{3}}^{\pi}$$

$$= -\frac{3\sqrt{3}}{4} - 2\left[(-\pi) - \left(\sqrt{3} - \frac{\pi}{3}\right)\right]$$

(ii) **Hence** find $\int_{\frac{\pi}{3}}^{\pi} \sin^2 x\,dx$, leaving your answer in exact form. [3]

$$\therefore \int_{\frac{\pi}{3}}^{\pi} \sin^2 x\,dx = \frac{1}{2}\left[-\frac{3\sqrt{3}}{4} + 2\pi + 2\sqrt{3} - \frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[\frac{5\sqrt{3}}{4} + \frac{\pi}{3}\right]$$

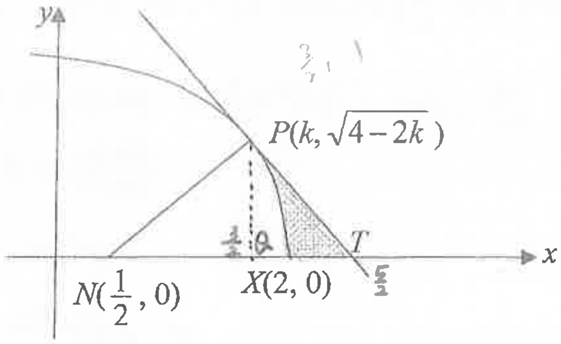
$$= \frac{1}{2}\left[\frac{15\sqrt{3} + 4\pi}{12}\right]$$

$$= \frac{15\sqrt{3} + 4\pi}{24} \quad \text{A1}$$

1006

[Turn over

11



The diagram shows part of the curve $y = \sqrt{4-2x}$ passing through the point $P\left(k, \sqrt{4-2k}\right)$, where k is a constant, $k < 2$. The curve meets the x-axis at the point $X(2, 0)$. The tangent and normal to the curve at P meet the x-axis at the points T and $N\left(\frac{1}{2}, 0\right)$ respectively.

By showing that $k = \frac{3}{2}$, find the area of the shaded region PXT. [12]

$$y = \sqrt{4-2x}$$

$$\frac{dy}{dx} = \frac{1}{2}(4-2x)^{-\frac{1}{2}}(-2)$$

$$= -\frac{1}{\sqrt{4-2x}}$$ B1

At P, $x = k$,

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{4-2k}}.$$

$\therefore$ gradient of normal at P $= \sqrt{4-2k}$ M1

Equation of normal is

$$y - \sqrt{4-2k} = \sqrt{4-2k}\,(x-k)$$ M1

5.

The normal passes through $N(\frac{1}{2}, 0)$

$$0 - \sqrt{4-2k} = \sqrt{4-2k}\,(\tfrac{1}{2} - k)$$ M1

$$-1 = \tfrac{1}{2} - k$$

$$k = \tfrac{1}{2} + 1$$

$$= \tfrac{3}{2}.$$ A1

Continuation of working space for Question 11.

$\therefore P = \left(\frac{3}{2}, \sqrt{4-2(\frac{3}{2})}\right)$
$= \left(\frac{3}{2}, 1\right).$

gradient of tangent $= -\frac{1}{1}$ M1
$= -1$

$\therefore$ Equation of tangent at P is

$y - 1 = -(x - \frac{3}{2})$. M1

$y = -x + \frac{3}{2} + 1$
$= -x + \frac{5}{2}$

At T, $y = 0$

$\therefore -x + \frac{5}{2} = 0$
$x = \frac{5}{2}$

$\therefore T = \left(\frac{5}{2}, 0\right)$ M1

7

h.

Area of $\Delta PQT = \frac{1}{2} \times 1 \times \left(\frac{5}{2} - \frac{3}{2}\right)$ M1
$= \frac{1}{2} \times 1 \times 1$
$= \frac{1}{2}$ units^2.

Area under curve from Q to X $= \int_{\frac{3}{2}}^{2} \sqrt{4-2x}\, dx$ M1

$= \left[\frac{(4-2x)^{\frac{3}{2}}}{\frac{3}{2}(-2)}\right]_{\frac{3}{2}}^{2}$ M1

$= \left[-\frac{1}{3}\sqrt{(4-2x)^3}\right]_{\frac{3}{2}}^{2}$

$= -\frac{1}{3}\left[0 - 1\right]$

$= \frac{1}{3}$

$\therefore$ Shaded area = Area of ΔPQT − Area under curve from Q to X

$= \frac{1}{2} - \frac{1}{3}$

$= \frac{1}{6}$ units^2. A1

BLANK PAGE