(H)

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4S1
4S2/4S3

1 (a) Express $f(x) = 2x^2 + 4x - 7$ and $g(x) = 3 - 2x - x^2$ in the form $a(x-h)^2 + k$. [3]

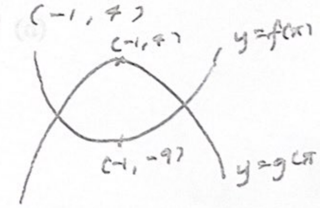
$f(x) = 2x^2 + 4x - 7$
$= 2(x^2 + 2x) - 7$
$= 2[(x+1)^2 - 1] - 7$ M1
$= 2(x+1)^2 - 2 - 7$
$= 2(x+1)^2 - 9$ A1

$g(x) = 3 - 2x - x^2$
$= -(x^2 + 2x) + 3$
$= -[(x+1)^2 - 1] + 3$
$= -(x+1)^2 + 1 + 3$
$= -(x+1)^2 + 4$ A1

(b) Hence explain if the graphs of $y = f(x)$ and $y = g(x)$ will intersect. [2]

Since for $y = f(x)$, the minimum point is $(-1, -9)$
and for $y = g(x)$, the maximum point is $(-1, 4)$

$\therefore$ The 2 graphs will intersect.



mention of max/min pt M1
answer A1

2 (a) Find the range of values of k for which

$$x^2 + 3x + k > kx + 3 \quad \text{for all real values of } x.$$

[3]

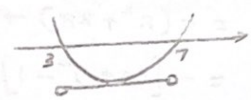
$x^2 + 3x + k > kx + 3$

$x^2 + (3-k)x + (k-3) > 0$

$\therefore \quad D < 0 \Rightarrow (3-k)^2 - 4(k-3) < 0$ M1

$(3-k)[3-k+4] < 0$

$(3-k)[7-k] < 0$ M1



$\therefore \quad 3 < k < 7$ A1

(b) Using the result in (a), justify if the curve $y = x^2 + 3x + 1$ lies entirely above the line $y = x + 3$. [2]

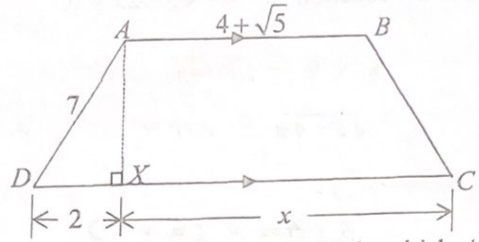
$y = x^2 + 3x + 1$

$y = x + 3$

$\therefore k = 1$ M1

The graph of $y = x^2 + 3x + 1$ does not lie entirely above the line $y = x + 3$ A1

3 (a)



The diagram shows a trapezium $ABCD$ in which $AD = 7$ cm and $AB = \left(4+\sqrt{5}\right)$ cm. AX is perpendicular to DC with $DX = 2$ cm and $XC = x$ cm.

Given that the area of the trapezium $ABCD$ is $15\left(\sqrt{5}+2\right)$ cm^2, obtain an expression for x in the form $a+b\sqrt{5}$, where a and b are integers. [3]

Area of trapezium:

$\frac{1}{2}\left[4+\sqrt{5}+a+b\sqrt{5}+2\right] \times \sqrt{7^2-2^2} = 15(\sqrt{5}+2)$ M1

$\frac{1}{2}\left[6+a+(b+1)\sqrt{5}\right]\sqrt{45} = 15(\sqrt{5}+2)$

$\left[6+a+(b+1)\sqrt{5}\right]3\sqrt{5} = 30(\sqrt{5}+2)$

$\left[6+a+(b+1)\sqrt{5}\right]\sqrt{5} = 10(\sqrt{5}+2)$

$(6+a)\sqrt{5}+5(b+1) = 10(2+\sqrt{5})$

$\therefore 6+a = 10$

$a = 4$ M1

and $5(b+1) = 20$

$b+1 = 4$

$b = 3$

$\therefore x = 4+3\sqrt{5}$ A1

[Turn over

(b) Solve the equation $x+4-\sqrt{5-\cancel{2}x}=0$. [3]

$4x$

$x + 4 - \sqrt{5-4x} = 0$

$\sqrt{5-4x} = x+4 \qquad , x \geq -4$

Squaring,

$5 - 4x = (x+4)^2$

$= x^2 + 8x + 16$ M1

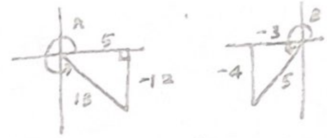
$\therefore x^2 + 12x + 11 = 0$

$(x+1)(x+11) = 0$ M1

$x = -1$ or $x = -11$

(rejected) A1

4 (a) It is given that $\cos A = \frac{5}{13}$ where A is not acute and $\sin B = -\frac{4}{5}$, where A and B lie in different quadrants. Calculate the value of $\cot A \cos (A-B)$, without using a calculator. [4]



$\cot A \cos (A-B)$

$= \frac{1}{\tan A}\left[\cos A \cos B + \sin A \sin B\right]$

$= \frac{1}{\left(\frac{-12}{5}\right)}\left[\left(\frac{5}{13}\right)\left(-\frac{3}{5}\right) + \left(-\frac{12}{13}\right)\left(-\frac{4}{5}\right)\right]$ B2

$= -\frac{5}{12}\left[-\frac{15}{65} + \frac{48}{65}\right]$ B1

$= -\frac{5}{12} \times \frac{33}{65}$

$= -\frac{11}{52}$ A1

(b) Without using a calculator, find the value of θ such that $\cos 2\theta = \sin 50°$, where $-90° \leq \theta \leq 90°$. [3]

$\cos 2\theta = \sin 50°$

$= \cos 40°$

$\therefore$ basic angle $= 40°$ M1

$\therefore 2\theta = -40°, 40°$ M1

$\theta = -20°, 20°$ A1

5 (a) The cubic polynomial $P(x)$ is such that the coefficient of x^3 is 3 and the roots of the equation $P(x) = 0$ are -2, 3 and k. Given that $P(x)$ has a remainder of 42 when divided by $(x+1)$, find

(i) the value of k, [3]

Let $P(x) = 3(x+2)(x-3)(x-k)$ M1

$P(-1) = 42$

$\Rightarrow 3(-1+2)(-1-3)(-1-k) = 42$ M1

$12(1+k) = 42$

$1+k = \frac{42}{12}$

$= \frac{7}{2}$

$k = \frac{5}{2}$ A1

p7? (ii) the remainder when $P(x)$ is divided by x. [1]

remainder $= P(0)$ ✓ M1

$= 3(2)(-3)(-k)$

$= 18k$

$= 18\left(\frac{5}{2}\right)$

$= 45$ ~~A1~~ M1

(b) Factorise $16x^3 - 2(x-1)^3$ completely. [3]

$16x^3 - 2(x-1)^3$

$= 2\left[8x^3 - (x-1)^3\right]$ M1

$= 2\left[(2x)^3 - (x-1)^3\right]$

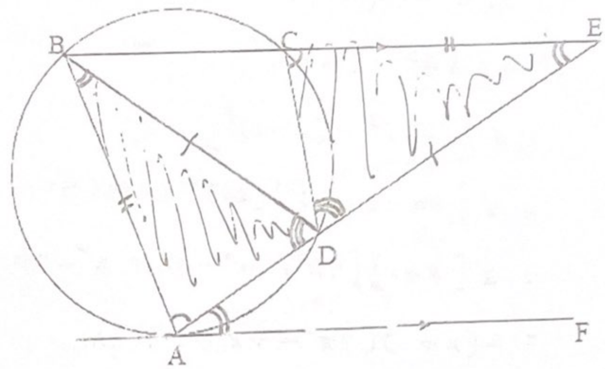
$= 2\left[2x - (x-1)\right]\left[(2x)^2 + 2x(x-1) + (x-1)^2\right]$ M1

$= 2\left[x+1\right]\left[4x^2 + 2x^2 - 2x + x^2 - 2x + 1\right]$

$= 2(x+1)(7x^2 - 4x + 1)$ A1

[Turn over

6 In the diagram, AF is a tangent to the circle at A. ADE and BCE are straight lines. AF is parallel to BE and $AB = CE$.



Prove that

(i) angle ABD = angle CED, [2]

$\angle ABD = \angle DAF$ (alternate segment Theorem) M1

$= \angle CED$ (alternate angles) M1

(ii) triangle ABD is congruent to triangle CED, [3]

From (i), $\angle ABD = \angle CED$

$AB = CE$ (given) } M1

$\angle BAD = 180° - \angle BCD$ (angles in opposite segments)

$= 180° - (180° - \angle ECD)$ (angles on a straight line)

$= \angle ECD$

$\therefore \triangle ABD \equiv \triangle CED$ (ASA) A1

(iii) BD bisects angle ABC. [3]

From (ii), $BD = DE$ M1

$\therefore \triangle BDE$ is isosceles.

i.e. $\angle DBE = \angle CED$

$= \angle ABD$ (from (i)) M1

$\therefore$ BD bisects $\angle ABC$. A1

7 (a) Given that $y = \ln\left(\dfrac{2x+1}{x-3}\right)^2$, find the range of values for which y is an increasing function. [5]

$$y = \ln\left(\frac{2x+1}{x-3}\right)^2$$

$$= 2\ln\left(\frac{2x+1}{x-3}\right)$$

$$= 2\left[\ln(2x+1) - \ln(x-3)\right] \quad \text{M1}$$

$$\frac{dy}{dx} = 2\left(\frac{2}{2x+1} - \frac{1}{x-3}\right) \quad \text{B1}$$

$$= 2\left[\frac{2(x-3)-(2x+1)}{(2x+1)(x-3)}\right]$$

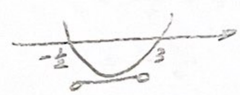
$$= 2\left[-\frac{7}{(2x+1)(x-3)}\right] \quad \text{M1}$$

$$= -\frac{14}{(2x+1)(x-3)},$$

For y to be an increasing function, $\frac{dy}{dx} > 0$

$$\therefore -\frac{14}{(2x+1)(x-3)} > 0$$

$$\Rightarrow (2x+1)(x-3) < 0 \quad \text{M1}$$



A1

$$\therefore -\tfrac{1}{2} < x < 3$$

(b) Find the angle at which the tangent to the curve at point where $x = 1$ makes with the positive x-axis. [2]

When $x = 1$, $\frac{dy}{dx} = -\frac{14}{3(-2)}$

$$= \frac{7}{3}$$

Let θ be the angle between the tangent and positive x-axis.

$$\therefore \tan\theta = \frac{7}{3} \quad \text{M1}$$

$$\theta = \tan^{-1}\frac{7}{3}$$

$$\approx 66.8^\circ. \quad \text{A1}$$

[Turn over

8 (a) Find the term independent of x in the expansion of $\left(2x^2 - \dfrac{1}{3x^3}\right)^{10}$. [3]

$\left(2x^2 - \frac{1}{3x^3}\right)^{10}$

$T_{r+1} = \binom{10}{r}(2x^2)^{10-r}\left(-\frac{1}{3x^3}\right)^r$

power of $x = 2(10-r) + (-3r)$

$= 20 - 5r$

For term independent of x, power of $x = 0$

$\therefore 20 - 5r = 0$

$r = 4$ M1

$\therefore T_5 = \binom{10}{4}(2x^2)^6\left(-\frac{1}{3x^3}\right)^4$ M1

$= 210\,(64x^{12})\left(\frac{1}{81x^{12}}\right)$

$= \frac{4480}{27}$ or 165.926

$= 165\frac{25}{27}$ A1

(b) The first two terms in the expansion of $(1-3x)\left(1+\frac{x}{2}\right)^n$ in ascending powers of x are $1-px^2$, where p is a constant. Find the values of n and p. [5]

$$(1-3x)\left(1+\frac{x}{2}\right)^n$$

$$= (1-3x)\left[1+\binom{n}{1}\left(\frac{x}{2}\right)+\binom{n}{2}\left(\frac{x}{2}\right)^2+\cdots\right] \quad B1$$

$$= (1-3x)\left[1+\frac{n}{2}x+\frac{n(n-1)}{2}\left(\frac{x^2}{4}\right)+\cdots\right]$$

$$= (1-3x)\left[1+\frac{n}{2}x+\frac{n(n-1)}{8}x^2+\cdots\right]$$

Coeff of $x = 0 \Rightarrow \frac{n}{2}-3=0$ M1

$$\frac{n}{2}=3$$

$$n=6 \quad A1$$

Coeff of x^2 : $-p = \frac{n(n-1)}{8} - 3\left(\frac{n}{2}\right)$ M1

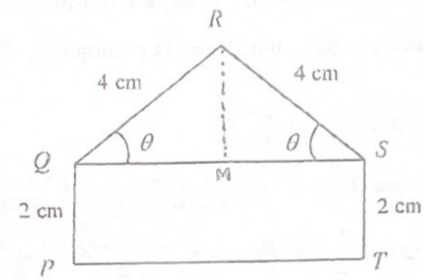
$$= \frac{6\times5}{8} - \frac{3\times6}{2}$$

$$= \frac{15}{4} - 9$$

$$= -\frac{21}{4}$$

$$\therefore p = \frac{21}{4} \quad A1$$

9 In the diagram, $PQRST$ is a piece of cardboard. $PQST$ is a rectangle with $PQ = 2$ cm and QRS is an isosceles triangle with $QR = RS = 4$ cm. $\angle RSQ = \angle RQS = \theta$ radians.



$\sin\theta = \frac{RM}{4}$

$RM = 4\sin\theta$ M1

$\cos\theta = \frac{QM}{4}$

$QM = 4\cos\theta$

(i) Show that the area, A cm^2, of the cardboard is given by $A = 8\sin 2\theta + 16\cos\theta$. [3]

Area of cardboard, $A = \frac{1}{2}(4\sin\theta)(2\times 4\cos\theta) + (8\cos\theta)\times 2$ M1

$\therefore A = 16\sin\theta\cos\theta + 16\cos\theta$

$= 8\sin 2\theta + 16\cos\theta$ A1

(ii) Given that θ can vary, find the stationary value of A. giving your answers in exact value. [5]

$\frac{dA}{d\theta} = 16\cos 2\theta - 16\sin\theta$ M1

For stationary value of A, $\frac{dA}{d\theta} = 0$

$\therefore 16\cos 2\theta - 16\sin\theta = 0$ M1

$16(1 - 2\sin^2\theta) - 16\sin\theta = 0$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$ M1

$\therefore \sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ (rejected)

$\therefore \theta = \frac{\pi}{6}$ A1

Stationary value of $A = 8\sin\frac{\pi}{3} + 16\cos\frac{\pi}{6}$

$= 8\times\frac{\sqrt{3}}{2} + 16\times\frac{\sqrt{3}}{2}$

$= 12\sqrt{3}$ A1

(iii) Determine if this value of A is a maximum or a minimum. [2]

$$\frac{d^2A}{d\theta^2} = 16(-2\sin 2\theta) - 16\cos\theta$$

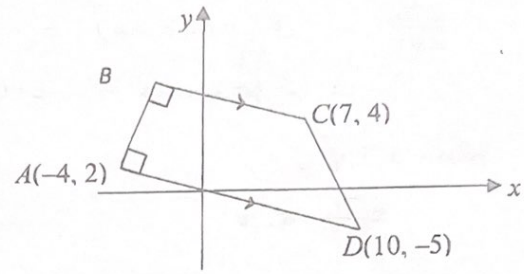
$$= -\cancel{86}\,32\sin 2\theta - 16\cos\theta \quad \text{M1}$$

When $\theta = \frac{\pi}{6}$,

$$\frac{d^2A}{d\theta^2} < 0$$

$\therefore$ A is a maximum. A1

10 The diagram shows a trapezium $ABCD$ and the coordinates of A, C and D are $(-4, 2)$, $(7, 4)$ and $(10, -5)$ respectively. AD is parallel to BC and perpendicular to AB.



(a) Show that the coordinates of B are $(-1, 8)$. [4]

gradient of AD $= \frac{2-(-5)}{-4-10}$

$= \frac{7}{-14}$

$= -\frac{1}{2}$

$\therefore$ gradient of AB $= 2$ since $AB \perp AD$.

$\therefore$ Equation of AB is

$y - 2 = 2(x+4)$ M1

$y = 2x + 8 + 2$

$y = 2x + 10$ ... (1)

Equation of BC is

$y - 4 = -\frac{1}{2}(x-7)$ M1

$y = -\frac{1}{2}x + \frac{7}{2} + 4$

$y = -\frac{1}{2}x + \frac{15}{2}$ ... (2)

(1) = (2): $2x + 10 = -\frac{1}{2}x + \frac{15}{2}$ M1

$\frac{5}{2}x = -\frac{5}{2}$

$x = -1$

Sub into (1): $y = -2 + 10$

$= 8$

A1

$\therefore$ $B = (-1, 8)$

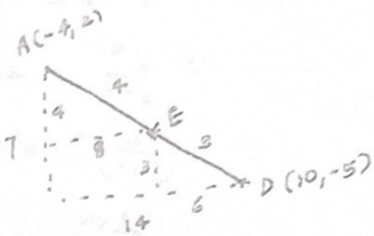
(b) The point E lies on the line AD such that $\dfrac{\text{area of triangle } ABE}{\text{area of triangle } ABD} = \dfrac{4}{7}$.

Find the coordinates of E. [2]

$$\frac{\text{Area of } \triangle ABE}{\text{Area of } \triangle ABD} = \frac{4}{7}$$

$$\frac{\frac{1}{2} \times AB \times AE}{\frac{1}{2} \times AB \times AD} = \frac{4}{7}$$

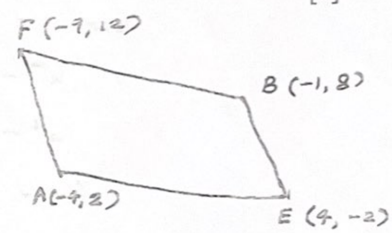
$$\frac{AE}{AD} = \frac{4}{7} \quad \text{M1}$$



$\therefore E = (-4+8,\ 2-4)$

$= (4, -2)$. A1

(c) The point F with coordinates $(-9, 12)$ lies on the line CB produced. Show that $AFBE$ is a parallelogram. [2]



Midpoint of F $= \left(\frac{-9+4}{2}, \frac{12-2}{2}\right)$

$= \left(-\frac{5}{2}, 5\right)$

Midpoint of A $= \left(\frac{-4-1}{2}, \frac{2+8}{2}\right)$

$= \left(-\frac{5}{2}, 5\right)$

Since the diagonals bisect each other, then AFBE is a parallelogram.

OR grad of FB $= \frac{8-12}{-1+9} = -\frac{4}{8} = -\frac{1}{2}$

grad of AE $= \frac{2-(-2)}{-4-4} = \frac{4}{-8} = -\frac{1}{2}$ $\Big] \therefore FB // AE$ M2

grad of FA $= \frac{12-2}{-9+4} = \frac{10}{-5} = -2$

grad of BE $= \frac{8+2}{-1-4} = \frac{10}{-5} = -2$ $\Big\} \therefore FA // BE$

$\therefore$ AFBE is a parallelogram.

[Turn over

11
~~10~~ Solve the following equations

(a) $2^{2x+1} = 6^{x-1}$, [3]

$$2^{2x+1} = 6^{x-1}$$

$$2^{2x} \times 2 = \frac{6^x}{6}$$

$$\therefore \frac{6^x}{2^{2x}} = 2 \times 6$$

$$\left(\frac{6}{2^2}\right)^x = 12 \quad \text{M1}$$

$$\left(\frac{3}{2}\right)^x = 12$$

$$x = \log_{\frac{3}{2}} 12 \quad \text{M1}$$

$$= \frac{\lg 12}{\lg \frac{3}{2}}$$

$$= 6.13 \quad (3 \text{ s.f.}) \quad \text{A1}$$

(b) $3^{2x} - 3^x = 2\left(3^x + 3^2\right)$, [3]

Let $u = 3^x$.

$$\therefore u^2 - u = 2(u + 9) \quad \text{M1}$$

$$u^2 - 3u - 18 = 0$$

$$(u - 6)(u + 3) = 0$$

$$u = 6 \quad \text{or} \quad u = -3$$

$$\therefore 3^x = 6 \quad \text{or} \quad 3^x = -3 \quad \text{M1}$$

(no solution)

$$x = \frac{\lg 6}{\lg 3}$$

$$= 1.63 \quad (3 \text{ s.f.}) \quad \text{A1}$$

$2^{\frac{1}{2}} = (\sqrt{4})^{\frac{1}{2}}$
$= (4^{\frac{1}{2}})^{\frac{1}{2}}$
$= 4^{\frac{1}{4}}$

(c) $\log_4 x^2 - \log_{\sqrt{2}}(3-x) = \log_3 \frac{1}{9}$. [4]

$$\log_4 x^2 - \log_{\sqrt{2}}(3-x) = \log_3 \frac{1}{9}$$

$$\frac{\log_2 x^2}{\log_2 4} - \frac{\log_2 (3-x)}{\log_2 \sqrt{2}} = -2$$

$$\frac{2\log_2 x}{2} - \frac{\log_2 (3-x)}{\frac{1}{2}} = -2 \qquad \text{M1}$$

$$\log_2 x - 2\log_2 (3-x) = -2$$

$$\log_2 \frac{x}{(3-x)^2} = -2$$

$$\frac{x}{(3-x)^2} = 2^{-2} = \frac{1}{4} \qquad \text{M1}$$

$$4x = (3-x)^2$$

$$\therefore x^2 - 6x + 9 = 4x$$

$$x^2 - 10x + 9 = 0$$

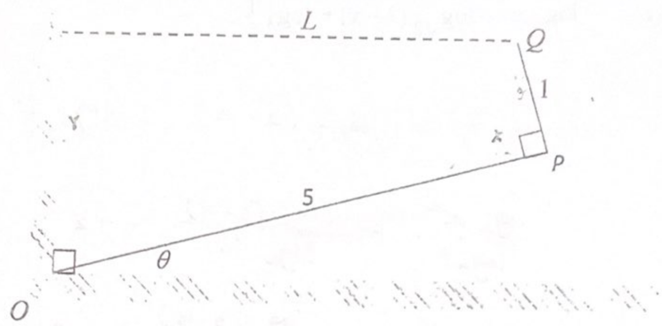
$$(x-9)(x-1) = 0 \qquad \text{M1}$$

$$x = 9 \quad \text{or} \quad x = 1$$

But $x < 3$,

$\therefore x = 1$ A1

12



A L-shaped structure, OPQ, can be rotated about O. $OP = 5$ m, $PQ = 1$ m and OP makes an acute angle θ with the ground. Given that the horizontal distance from Q to the wall is L m,

(a) show that $L = 5\cos\theta - \sin\theta$, [2]

$\cos\theta = \frac{PY}{5} \Rightarrow PY = 5\cos\theta$

$\sin\theta = \frac{PX}{1} \Rightarrow PX = \sin\theta$ M1

$\therefore L = YX$

$= PY - PX$

$= 5\cos\theta - \sin\theta$ A1

(b) express L in the form $R\cos(\theta + \alpha)$, [3]

$\therefore L = R\cos(\theta + \alpha)$

$\therefore R = \sqrt{5^2 + 1^2}$

$= \sqrt{26}$ M1

$\tan\alpha = \frac{1}{5}$

$\Rightarrow \alpha = 11.309$ M1 $\alpha = 0.19739$ rad.

$\therefore L = \sqrt{26}\cos(\theta + 11.3^\circ)$ A1

(c) state the minimum value of L and find the corresponding value of θ, [2]

minimum $L = 0$ A1

when $\cos(\theta + 11.3°) = 0$

$\theta + 11.3° = 90°$

$\theta = 90° - 11.3°$

$= 78.7°$ A1

$\theta = 1.3734$

(d) find the value of θ when $L = 4$, [2]

$L = 4 \Rightarrow \sqrt{26}\cos(\theta + 11.309°) = 4$

$\cos(\theta + 11.309°) = \frac{4}{\sqrt{26}}$

basic angle $= \cos^{-1}\left(\frac{4}{\sqrt{26}}\right)$

$= 38.3288°$. M1 0.6689

$\therefore \theta + 11.309° = 38.3288°$

$\theta = 27.0°$ A1 $\theta = \underline{0.472}$

(e) explain why the maximum value of L is not R. [1]

maximum L occurs when $\theta = 0°$. M1

$\therefore$ maximum $L = 5$ m $\neq R$

OR If max $L = R$, then $\cos(\theta + 11.3°) = 1$

$\therefore \theta + 11.3° = 0°$ or $360°$

$\theta = -11.3°$ or $348.7°$

But θ is acute.

$\therefore$ max $L \neq R$.