Name : ________________________________

Class : _________

TEMASEK JUNIOR COLLEGE

Unit 5: Projectile Motion

Learning Outcomes

(a) describe and use the concept of weight as the force experience by a mass in a gravitational field.

(b) describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

(c) derive, from the definition of work done by a force, the equation $\Delta E_p = mg\Delta h$ for gravitational potential energy changes in a uniform gravitational field (e.g. near the Earth's surface).

(d) recall and use the equation $\Delta E_p = mg\Delta h$ to solve problems.

(e) describe qualitatively, with reference to forces and energy, the motion of bodies falling in a uniform gravitational field with air resistance, including the phenomenon of terminal velocity.

5.1 Weight *LO(a)*

Weight is the force experience by a mass in a gravitational field.

In unit 3, we saw that all objects released near the surface of the Earth fall with the same acceleration (the acceleration of free fall) if air resistance can be neglected. The force causing this acceleration is the gravitational attraction of the Earth on the object. The gravitational force which acts on an object is called the weight of the object. We can apply Newton's second law to the weight. For a body of mass m falling with the acceleration of free fall g, the weight W is given by

$$W = mg$$

The SI unit of force is the newton (N). This is also the unit of weight.

Because weight is a force and force is a vector, we ought to be aware of the direction of the weight of an object. It is towards the centre of the Earth.

mass	**weight**
a measure of the inertia of a body	a measure of the gravitational force on a body
SI unit: kg	SI unit: N
same regardless of location	varies with gravitational field strength at each location

Example 1

A boy throws a ball vertically upwards. It rises to a maximum height, where it is momentarily at rest, and falls back to his hands.

Which of the following gives the acceleration of the ball at various stages in its motion?
Take vertically upwards as positive. Neglect air resistance.

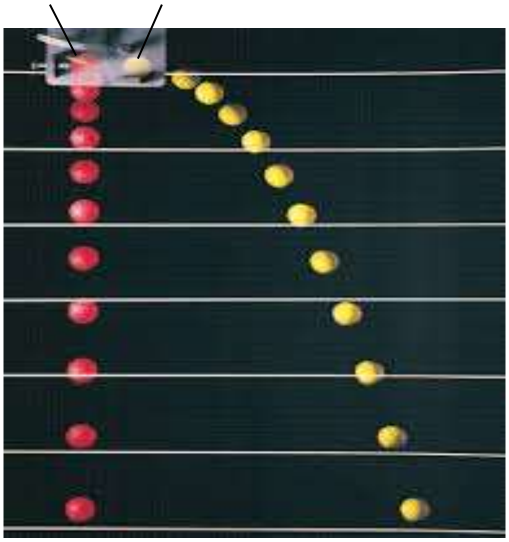
	rising	at maximum height	falling
A	-9.81 m s^{-2}	0	$+9.81$ m s^{-2}
B	-9.81 m s^{-2}	-9.81 m s^{-2}	-9.81 m s^{-2}
C	$+9.81$ m s^{-2}	$+9.81$ m s^{-2}	$+9.81$ m s^{-2}
D	$+9.81$ m s^{-2}	0	-9.81 m s^{-2}

5.2 Projectile Motion *LO(b)*

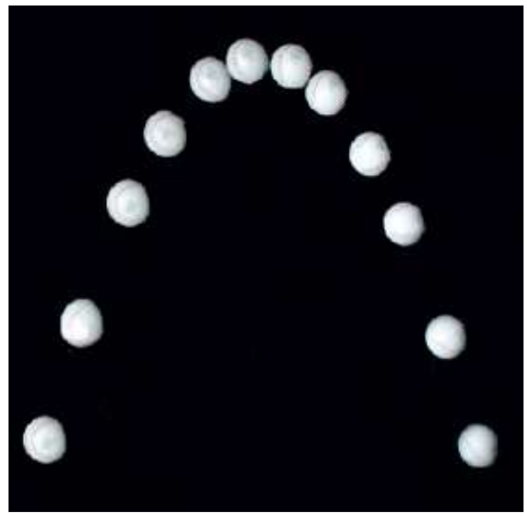
So far we have been dealing with motion along a straight line; that is, one-dimensional motion. We will now look at the motion of particles moving in paths in two dimensions.

The stroboscopic photographs below depicts three balls A, B and C released simultaneously.

Ball A Ball B



Ball C



Ball A is dropped **freely** while ball B is **projected horizontally**. Ball C is **projected at an angle** to the horizontal. Notice that in equal time intervals balls A and B fall the same vertical distance. This is because both balls undergo the same vertical acceleration due to gravity. Balls A and B also reach the ground at the same time!

The paths or trajectory of balls B and C are **parabolic**.

Ball A is an example of a one-dimensional linear accelerated motion whereas balls B and C are said to undergo *projectile motion*, which is a two-dimensional motion under a constant force.

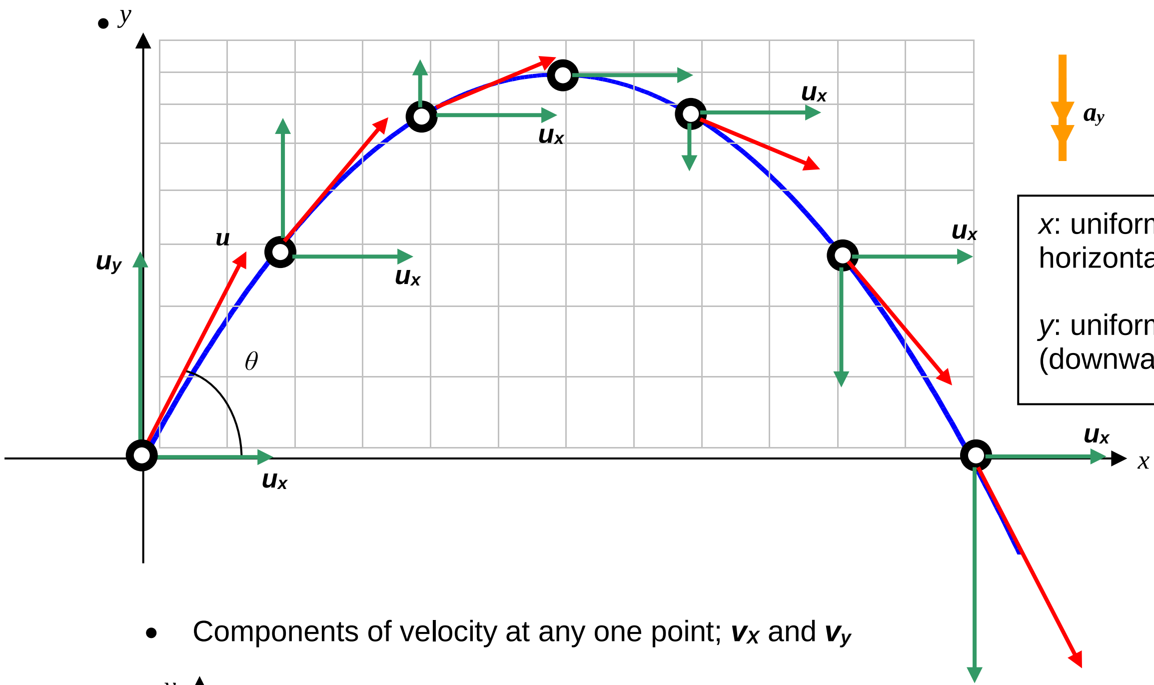
A **projectile motion** can be considered as a combination of **two independent motions.**

1. **Horizontal motion** with ***uniform velocity*** (since there is no force in horizontal direction).

2. **Vertical motion** with ***uniform acceleration*** (due to force of gravity).

Horizontal and vertical motion of a projectile

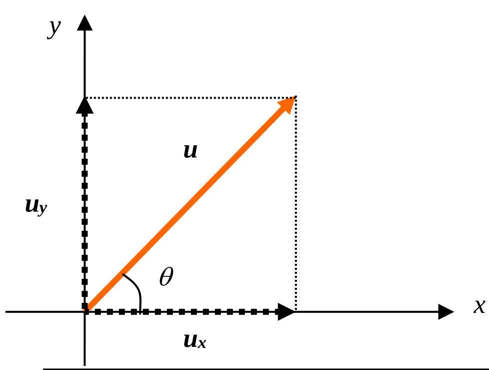
- Projectile motion involves **motion in the *x* and *y* directions simultaneously**.
- For non-linear motion, the **direction of velocity** at each point of its trajectory is **tangent** to the trajectory at that point.
- At each point, the velocity of the projectile can be **resolved into its horizontal and vertical components** respectively.

- 

x: uniform velocity in the horizontal (*x* direction)

y: uniform acceleration (downwards) due to gravity

- Components of velocity at any one point; $\boldsymbol{v_x}$ and $\boldsymbol{v_y}$



At time $t = 0$, when
velocity = u, the <u>**magnitudes**</u> of the components are given by:

$\boldsymbol{u_x = u \cos \theta}$
$\boldsymbol{u_y = u \sin \theta}$

If substituting the components of velocity into equation of motion, you have to apply the sign (- or +) for the initial velocity according to the sign convention and direction of velocity.

- If both the components are known, both the magnitude and direction of the vector can be found.

Magnitude of vector: $u = \sqrt{u_x^2 + u_y^2}$
Direction of vector: $\tan\theta = \dfrac{u_y}{u_x}$ in this way, the angle of projection can be found.

5.2.1 Analysis of projectile motion

Key terms used in projectile motion

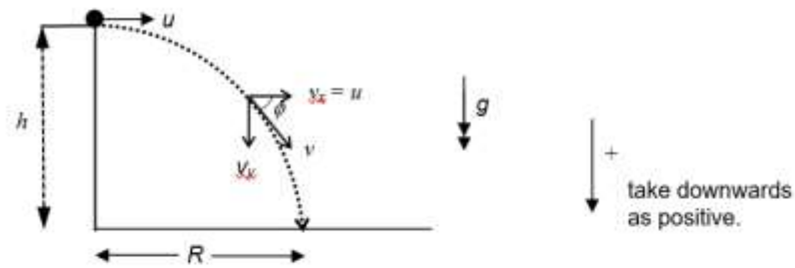
- **Trajectory –** the path taken by a projectile
- **Range, *R* –** the distance on the plane between the point of projection and the point of impact
- **Angle of projection –** the angle between the direction of projection and the horizontal plane through the point of projection
- **Time of flight –** time taken from the point of projection to the point of impact

Strategies used in solving problems involving projectile motion *without* air resistance

1. If diagram of path is not given, sketch one, identifying start and end point.
2. Resolve given velocity into its *x* and *y* components.
3. Apply the relevant equations of motion separately to the horizontal and vertical motion**.** (Remember to take sign conventions into consideration!)

Horizontal direction, *x*	**Vertical direction, *y***
$u_x = ?$	$u_y = ?$
$\mathbf{v_x = u_x}$ $\quad$ $\mathbf{(a_x = 0)}$	$v_y = u_y + a_y t$
$s_x = u_x t$ $\quad$ $\mathbf{(a_x = 0)}$	$s_y = u_y t + \frac{1}{2} a_y t^2$
	$v_y^2 = u_y^2 + 2 a_y s_y$

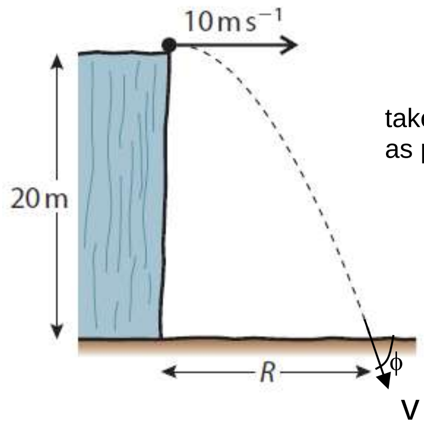
Projectile launched horizontally



	Horizontal direction	Vertical direction
Initial velocity *(Starting point is the highest point)*	$u_x = u$ The horizontal component of the object's velocity remains unchanged throughout its flight.	$u_y = 0$ The vertical component of the object's velocity increases as the object falls towards ground.
Sign convention*	right positive → +	downward positive ↓ +
Acceleration	$a_x = 0$	$a_y = 9.81\ \text{m s}^{-2}$ (taking downwards as positive)
Component of velocity at any given time, *t*.	$v_x = u_x = u$	$v_y = u_y + a_y t$ $= gt$ *since* $u_y = 0$
Time of flight *T*	Consider the vertical motion; using $\boldsymbol{s_y = u_y t + ½\, a_y t^2}$ and taking the downward direction as positive, $h = 0 + ½\, gT^2$ $T = \sqrt{\dfrac{2h}{g}}$	
Displacement	horizontal range = s_x Using $\boldsymbol{s_x = u_x t + ½\, a_x t^2}$ since $\boldsymbol{a_x} = 0$, $\boldsymbol{s_x = u_x t = ut}$ Range, $R = uT = u\sqrt{\dfrac{2h}{g}}$	vertical displacement = s_y Using $\boldsymbol{s_y = u_y t + ½\, a_y t^2}$ since $\boldsymbol{u_y} = 0$, $\boldsymbol{a_y} = g$ $\boldsymbol{s_y = ½\, gt^2}$
	By combining the two equations, we have $y = \dfrac{g(x/u)^2}{2} = \dfrac{g}{2u^2}x^2$. This implies $y \propto x^2$. Thus the trajectory of a projectile is a **parabola**.	

Example 2

A 50 g ball is thrown horizontally from a cliff top with a horizontal speed of 10 m s^{-1}.



take downwards as positive.

x	y
$u_x = 10$ ms^{-1}	$u_y = 0$
$v_x = ?$	$v_y = ?$
$s_x = R = ?$	$s_y = 20$ m
$a_x = 0$	$a_y = 9.81$ ms^{-2}
Time of flight, $T = ?$	

Apply equation of motions separately on the variables in *x* and *y* direction.

If the cliff is 20 m high, calculate

(a) the magnitude and direction of the velocity of the ball when it strikes the sea,

Solution:

Horizontal component of velocity:

Vertical component of velocity:

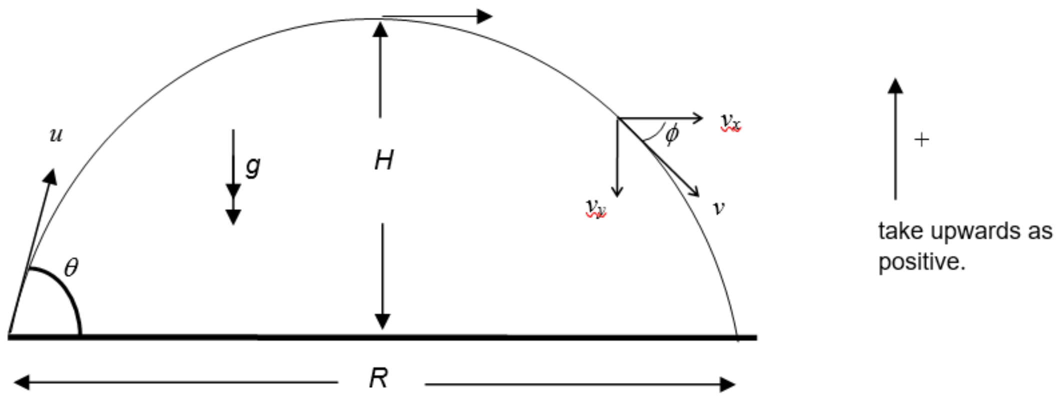
$\therefore$ Final velocity:

The ball strikes the sea at __________ m s^{-1} at angle ___________ below the horizontal.

(b) the range of the ball.

Solution:

Projectile launched at an angle to the horizontal



	Horizontal direction	**Vertical direction**
Initial velocity	$u_x = u \cos\theta$	$u_y = u \sin\theta$
Sign convention*	right positive +	upward positive +
Acceleration	$a_x = 0$ *no acceleration in x direction*	$a_y = -9.81$ m s^{-2} *(acceleration due to gravity is downwards)*
Component of velocity at any given time, *t*.	$v_x = u_x = u \cos\theta$	$v_y = u_y + a_y t = u \sin\theta - gt$
Time of flight *T*	<u>Time of flight, *T (use information in the y direction)*</u> Consider the vertical motion. Let *t* be the time taken to reach maximum height. Using $\boldsymbol{v_y = u_y + a_y t}$ to find *t.* At the highest point of the trajectory, $\boldsymbol{v_y} = 0$, $t = t_{max\text{-}height}$ $0 = u \sin\theta + (-g)\, t_{max\text{-}height}$ $t_{max\text{-}height} = \frac{u \sin\theta}{g}$ ***Without air resistance***, the time to go up is equal to the time to come down, so **time of flight, $\boldsymbol{T = 2 \times t_{max\text{-}height}}$** so $T = 2t = \frac{2u \sin\theta}{g}$

<table>
<tr>
<td>Displacement</td>
<td>horizontal displacement = s_x

• To find range, $s_x = R$
Using $\boldsymbol{s_x = u_x t + \frac{1}{2} a_x t^2}$
since $\boldsymbol{a_x} = 0$,
$\boldsymbol{s_x = u_x t} = (u \cos\theta)T$
$= \dfrac{u^2 \sin 2\theta}{g}$
Note that R is maximum when $\sin 2\theta = 1$ i.e. when $\theta = 45^o$.</td>
<td>vertical displacement = s_y

• To find maximum height, $s_y = H$
Using $\boldsymbol{v_y^{\,2} = u_y^{\,2} + 2as_y}$
at the highest point of trajectory,
$\boldsymbol{v_y} = 0$</td>
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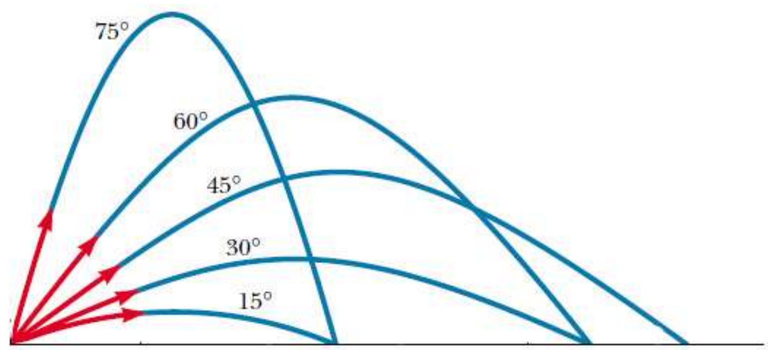
<td></td>
<td colspan="2">

<u>Some features of projectile motion:</u>
(i) The largest Range occurs at angle of projection $\theta = 45^o$.
(ii) The largest maximum Height occurs at angle of projection $\theta = 90^o$.</td>
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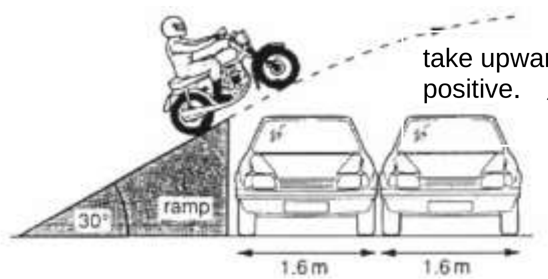
Example 3

A stuntman on a motorcycle plans to ride up a ramp in order to jump over a number of cars as illustrated in the figure.



x	y
$u_x = 14 \cos 30^o$	$u_y = 14 \sin 30^o$
$v_x = ?$	$v_y = ?$
$s_x = R = ?$	$s_y = ?$
$a_x = 0$	$a_y = -9.81\ ms^{-2}$
Time of flight, $T = ?$	

Apply equation of motions separately on the variables in x and y direction.

The speed of the motorcycle as it leaves the ramp is 14 m s^{-1}.

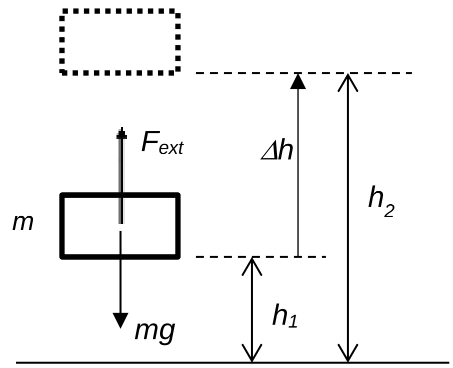
(a) Neglecting air resistance, determine the time interval between leaving the end of the ramp and reaching maximum height.

(b) The cars are each of width 1.6 m and the same height as the ramp. Estimate the maximum number of cars which the motorcycle can jump over.

<u>Solution:</u>

5.3 Gravitational Potential Energy in a Uniform Field *LO(c)(d)*

Consider an object raised up at a constant speed by an applied force F_{ext} from height h_1 to h_2. At constant speed, acceleration and net force is zero, thus F_{ext} must be equal and opposite to the weight *mg* of the object.



Work done by the applied force F_{ext} is

$$
\begin{aligned}
W &= F_{ext}\,\Delta h \\
&= mg\,(h_2 - h_1) \\
&= mgh_2 - mgh_1 \\
&= \text{Final potential energy - Initial potential energy} \\
&= \Delta E_p \text{ or } \Delta U
\end{aligned}
$$

Thus, an object raised over height Δh above the earth's surface has a change in gravitational potential energy

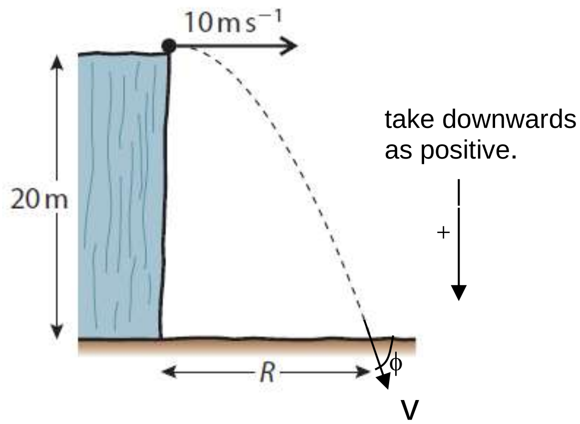
$$\boxed{\Delta E_p = mg\Delta h}$$

<u>Note :</u>

- This change in gravitational potential energy is a change in the potential energy of the object-Earth system.
- $\Delta E_p = mg\Delta h$ is valid only for objects near the surface of the Earth, where the gravitational acceleration g is approximately constant.
- The change in potential energy can be either negative or positive. For instance, we can set the initial position to be a table top. If a body is moved to a position below a tabletop, the upward force F_{ext}, whose magnitude is *mg*, acts opposite to the downward change in displacement to give a negative work done since $\Delta h = h_{final} - h_{initial}$ will be negative.

Example 4

A 50 g ball is thrown horizontally from a cliff top with a horizontal speed of 10 m s^{-1}.



(a) If the cliff is 20 m high, calculate the change in gravitational potential energy of the ball when it strikes the sea.
(b) Hence determine the speed of the ball when it strikes the sea.

<u>Solution</u>

(a) loss in gravitational potential energy = $mg\Delta h$ = 0.050 (9.81)(20) = 9.8 J

(b) Loss in GPE = Gain in KE

$$9.8 = \frac{1}{2}(0.050)v^2 - \frac{1}{2}(0.050)(10)^2$$

$$v = 22 \text{ m s}^{-1}$$

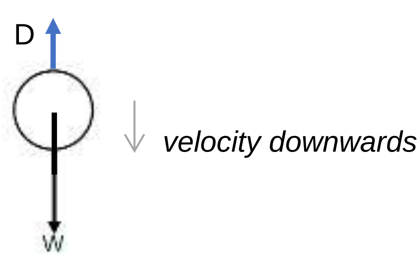
5.4 Effect of Air Resistance & Terminal Velocity *LO(e)*

5.4.1 Effects of Air Resistance on falling body

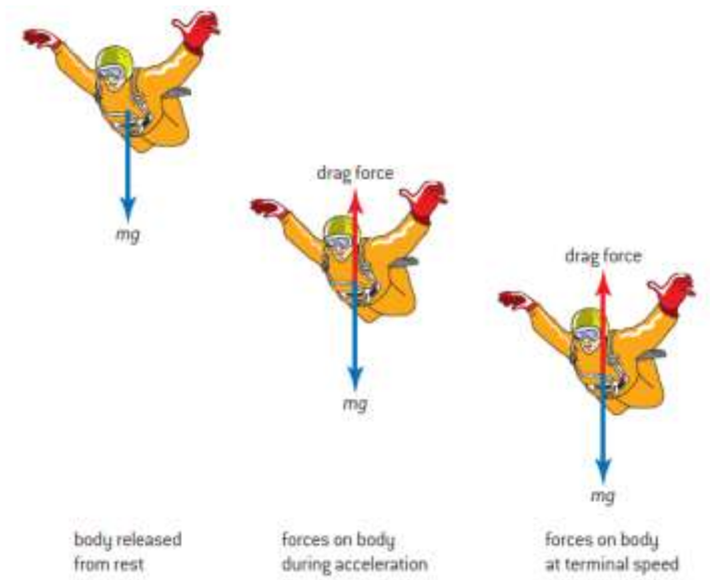
In the presence of air resistance, a body moving through air is subjected to two external forces: its weight, *W*, and air resistance or drag force, *D.* Drag force always opposes motion and acts in a direction opposite to the velocity of the body.

Object falling down with drag force: F_{net} = W - D

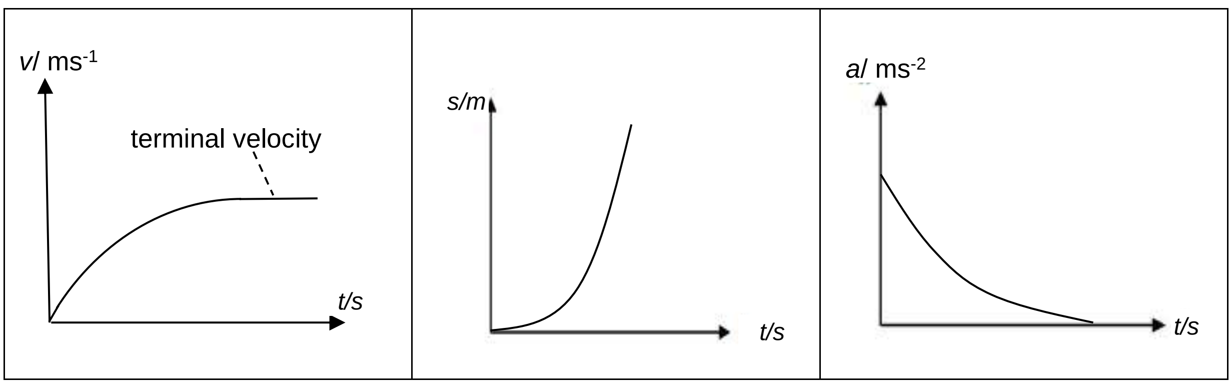
ma = W - D



- At low speeds with no turbulence, we say that magnitude of drag force, D, is proportional to speed; $D \propto v$.
- At higher speeds with turbulence, magnitude of drag force is proportional to square of speed; $D \propto v^2$. □



- Initially, speed is zero so drag force is zero. Body accelerates due to weight.
- As speed increases, drag force increases. Speed increases at a decreasing rate.
- Eventually, drag force equals to weight. Net force is zero, acceleration is zero and its velocity will remain constant. This maximum velocity is called terminal velocity.



Example 5

A student throws a ball vertically upwards with an initial speed.

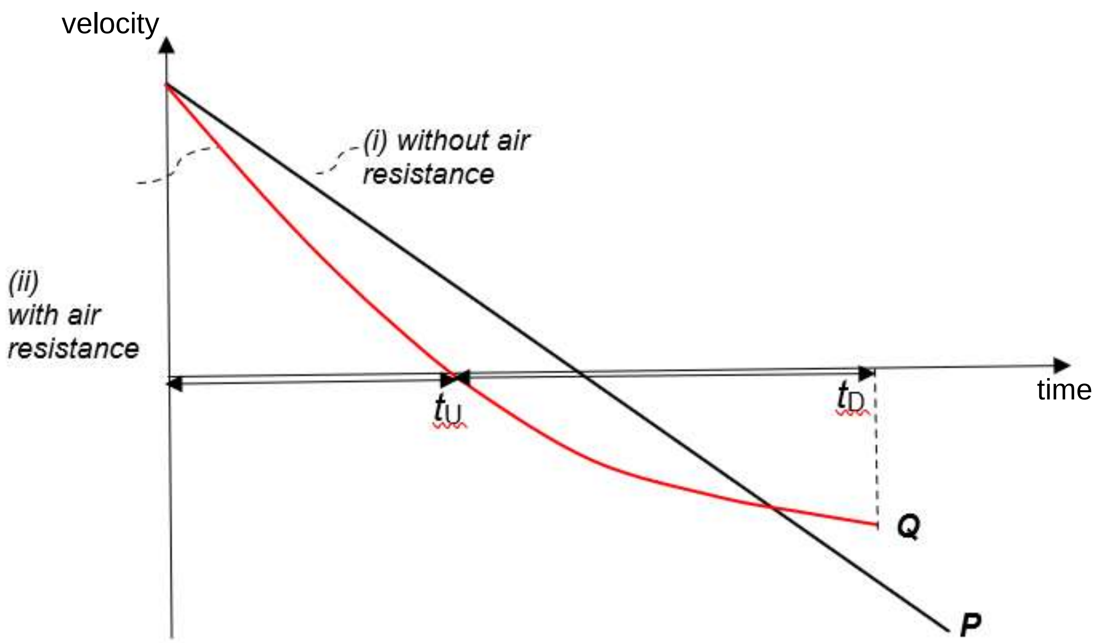
(a) Taking upwards as positive, sketch on the same axis the velocity-time graphs for the motion of the ball until it returns to the student's hand, for the case when

1. air resistance is negligible. Label the graph P,
2. air resistance is present. Label the graph Q.

(b) Explain the shape of the graph Q you have drawn in terms of forces and energy.

Solution:

Taking upward direction as positive



(c) Explanation of shape of graph Q:

- As the object moves upwards, the time of flight when there is air resistance is shorter.

Using Newton's Second Law

$F_{net} = W + D$

$ma_{up} = mg + D$

$$a_{up} = g + \frac{D}{m} > g$$

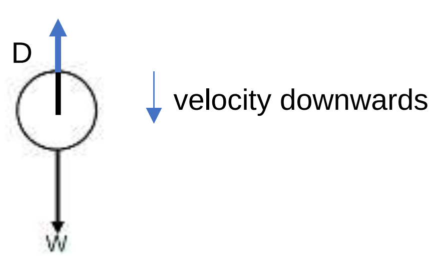
As both drag force and weight are acting downwards, net force is greater than weight and thus deceleration of the ball is greater than *g*. This results in gradient of graph Q being steeper than graph P. Velocity of the object decreases to zero at a faster rate. The maximum height is lower and is reached in a shorter time.

- As the object moves downwards, the time of flight when there is air resistance is longer.

$F_{net} = W - D$

$ma_{down} = mg - D$

$$a_{down} = g - \frac{D}{m} < g$$



Drag force acts opposite to weight. Net force is smaller than weight and object's acceleration is less than *g*. This results in gradient of graph Q being less steep than graph P. Velocity of the object increases from zero at a slower rate. It will take a longer time for the ball to travel the same distance on the way down than on the way up i.e. $t_D > t_U$

With air resistance, there is work done by drag force. The maximum height reached is lower as less of the kinetic energy store will be transferred to gravitational potential energy store.

Since a lower maximum height is reached, and there is work done by drag force on the way down, the ball returns to student's hand with a lower kinetic energy store, and hence a smaller speed than the initial speed. This also means that the average upward speed is greater than the average downward speed.

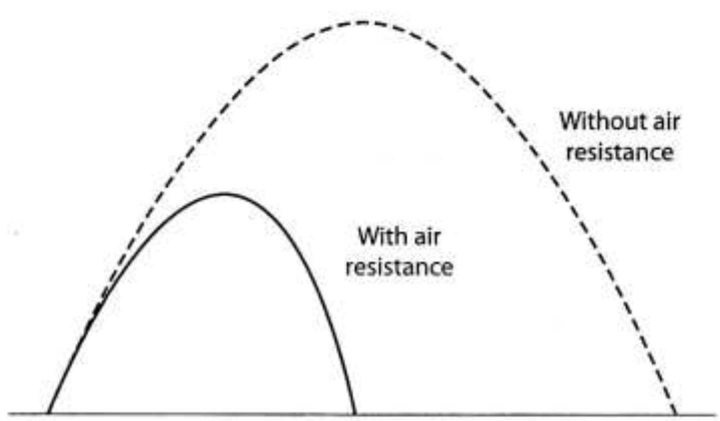
Since the distance up and down is the same, and time = distance/average speed, **time to go up is therefore less than time to come down**, i.e. $t_u < t_D$.

5.4.2 Projectile Motion with air resistance

In the real world, projectiles are subject to air resistance. The projectile experiences a **vertical drag** and a **horizontal drag** that oppose its vertical and horizontal motion respectively.

As it travels up, both drag force and weight act downwards and thus the net downward force is larger. By Newton's Second Law, the downward acceleration is therefore larger than 9.81 m s^{-2}. This causes the body to reach a smaller altitude and to reach this maximum height sooner ($\boldsymbol{v_y}$ reduces to zero in a shorter time).
The effect of the horizontal drag will shorten the range of the projectile and the effect of the vertical drag will be to reduce the maximum height reached by the projectile.

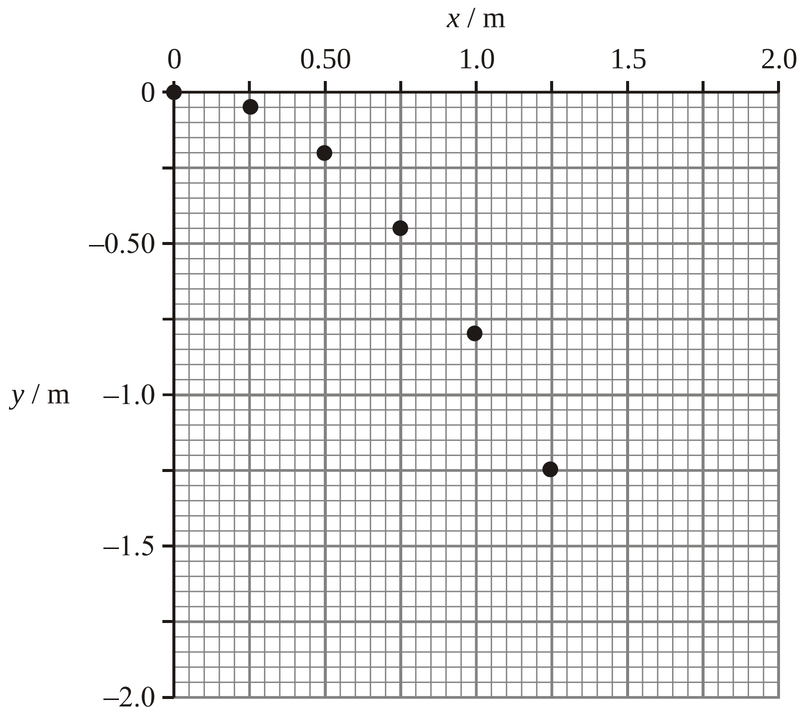
The actual path of a ball in the presence of air resistance is likely to be as shown below.



Features of the new path:
1. Both the height and range are smaller.
2. The path is no longer a parabola – the way down is steeper than the way up.
3. The final speed is less than the initial speed.

Unit 5: Projectile Motion Tutorial Questions

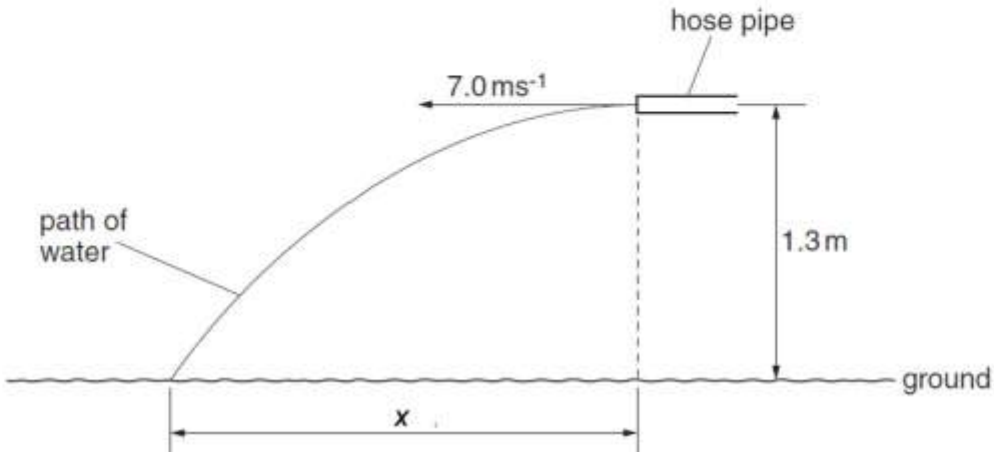
1 A marble is projected horizontally from the edge of a wall 1.8 m high with an initial speed *u*.
A series of flash photographs are taken of the marble as shown below.
The time interval between each image of the marble is 0.10 s.



(a) Explain how the figure shows that air resistance is negligible.
(b) Use the figure to determine the initial horizontal speed *u* of the ball.
(c) Use the figure to estimate the acceleration due to gravity of the ball.
(d) On the photograph, draw a suitable line to determine the horizontal range R from the base of the wall to the point where the marble hits the ground. Explain your working.

[2.5 m s^{-1}, 10 m s^{-2}, R = 1.5 m]

2 Figure shows the path of water from a hose pipe.



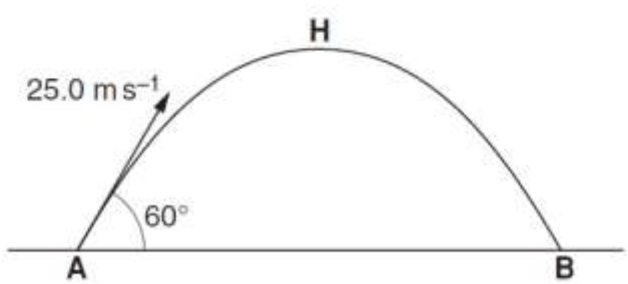
Water is projected horizontally from the hose pipe at a speed of 7.0 m s^{-1}. The hose is 1.3 m above the ground.

(a) Determine the horizontal distance x at which the water strikes the ground.

(b) Determine the speed and angle at which the water strikes the ground.

[3.6 m, 8.6 m s^{-1} and 36° below horizontal]

3 Figure shows the path of a ball that is thrown from point A to point B. The ball reaches maximum height at point H.



The ball is thrown at an initial velocity at 25.0 m s^{-1} at 60° to the horizontal. The horizontal distance from A to B is *x*. Assume that there is no air resistance.

(a) Calculate the vertical and horizontal components of the ball's initial velocity.

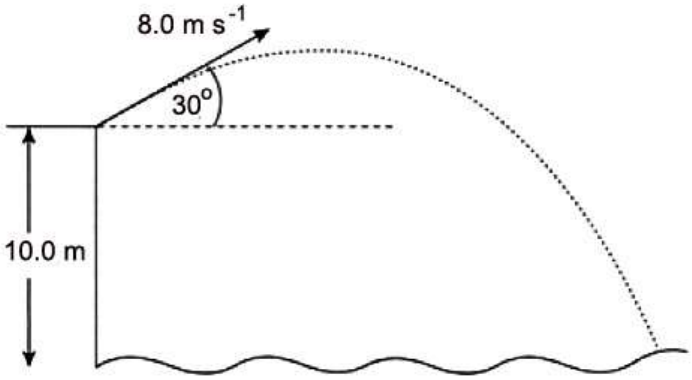
(b) State the velocity of the ball at point H.

(c) Calculate the time to travel to point H.

(d) Calculate the horizontal distance from A to B.

[u_y = 21.7 m s^{-1}, u_x = 12.5 m s^{-1}, 12.5 m s^{-1}, 2.21 s, 55.3 m]

4 An object is thrown upwards at an angle of 30° to the horizontal from the top of a 10.0 m vertical cliff and lands in the sea. The initial velocity of the object is 8.0 m s^{-1}.



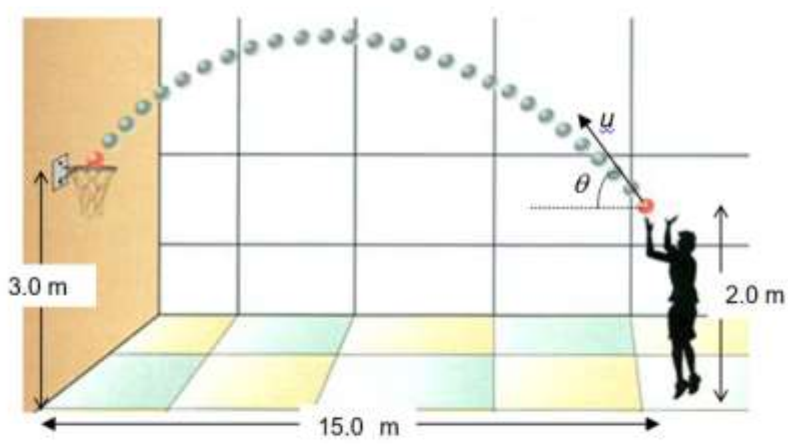
(a) Calculate the time taken
- (i) for the object to reach the highest point in its flight
- (ii) for the object to hit the sea

(b) Calculate the displacement
- (i) vertically, at the highest point in its flight
- (ii) horizontally when it hits the sea

(c) Sketch the path of the trajectory if air resistance is not negligible.

[0.41 s, 1.89 s, 0.82 m, 13.1 m]

5 With 2.0 seconds on the clock left, a TJC basketball player, at a distance 15.0 m away from the 3.0 m high hoop, throws the ball at a height of 2.0 m.



Determine the
(a) initial speed u,
(b) angle of projection θ of the ball
such that it just enters the hoop as the referee blows the final whistle.

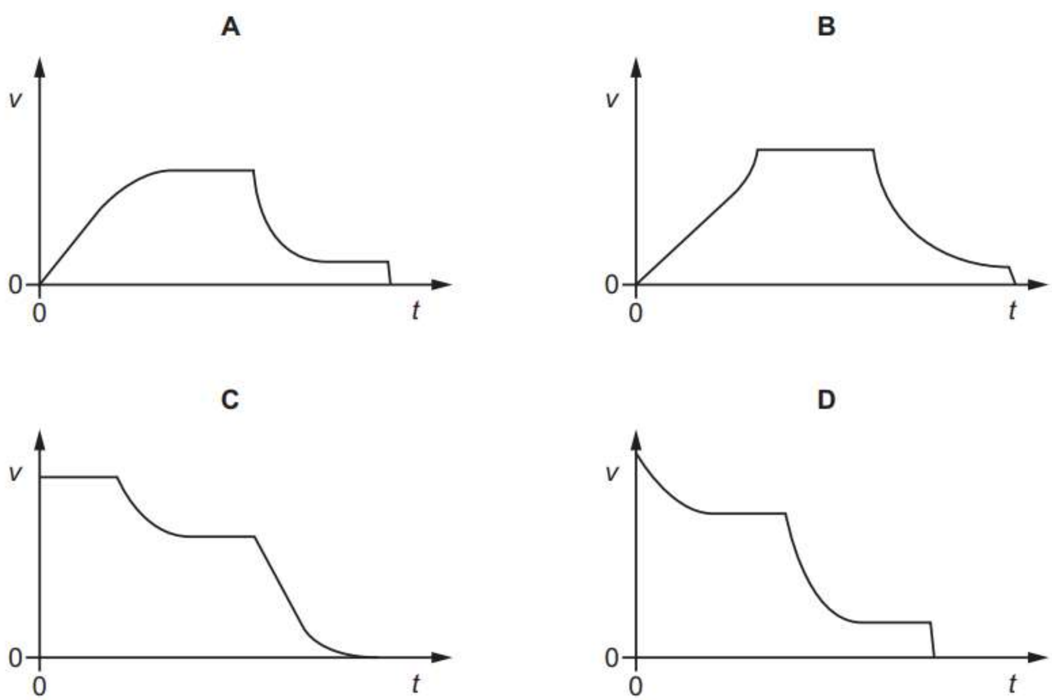
[12.7 m s^{-1}, 54°]

6 A 50 g ball is thrown from a window with an initial velocity of 8.0 ms^{-1} at an angle of 30° above the horizontal. Using energy methods, determine (a) the kinetic energy of the ball at the top of its flight and (b) its speed when it is 3.0 m below the window. Does the answer to (b) depend on the mass of the ball or the initial angle? [1.2 J, 11 m s^{-1}]

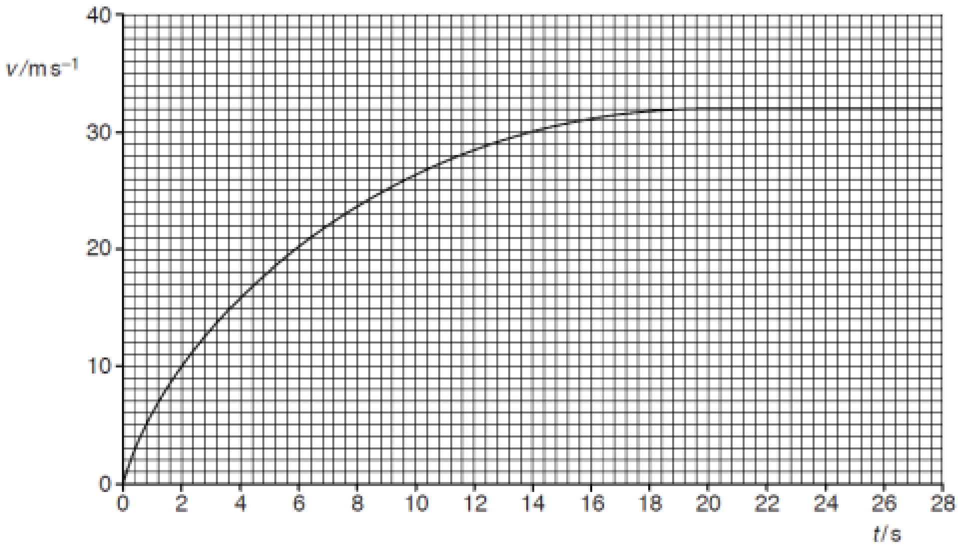
Air resistance

7 A parachutist falls vertically from rest at time t = 0 from a hot-air balloon. She falls for some distance before opening her parachute.

Which graph best shows the variation with time t of the speed v of the parachutist?



8 A sky-diver jumps from a high-altitude balloon.
He experiences a drag force proportional to his speed. The graph shows how his vertical speed *v* varies with time *t*.



(a) Explain briefly, using forces acting, why the acceleration of the sky-diver
(i) is 9.81 m s^{-2} at the start of the jump,
(ii) decreases with time.

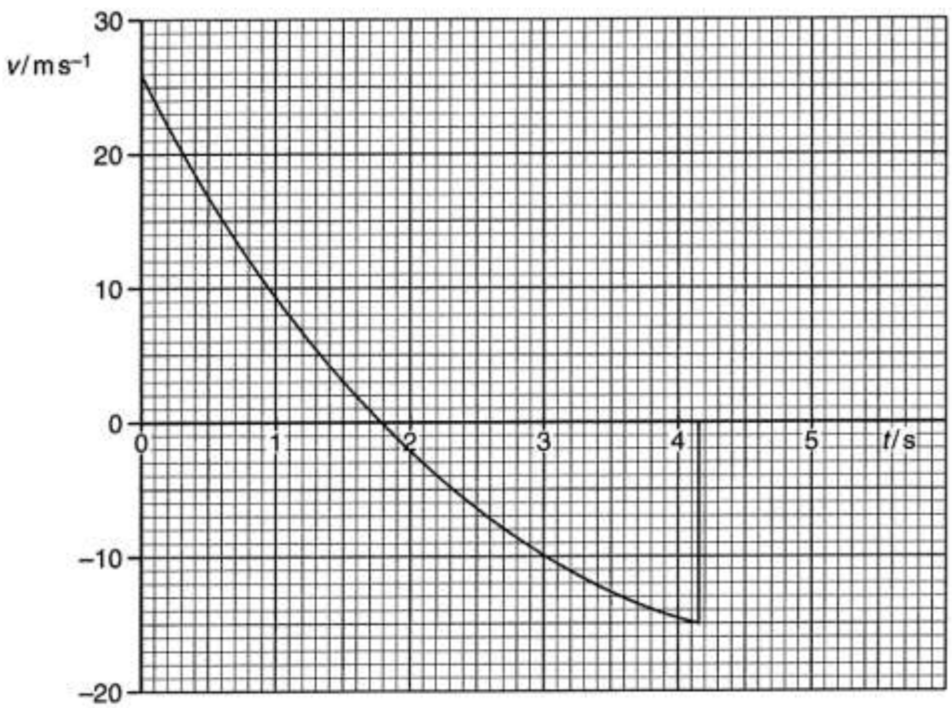
(b) Use the figure to determine
(i) the magnitude of the acceleration of the sky-diver at time t = 6.0 s.
(ii) the distance the sky-diver has fallen at time t = 6.0 s.
(iii) the terminal velocity of the diver.

(c) Sketch the corresponding variation with time of the acceleration of the sky-diver.

[1.9 m s^{-2}, 70 m, 32 m s^{-1}]

9 [N03/III/1 part]

(a) The graph below shows the variation with time t of the velocity v of a ball from the moment it is thrown with a velocity of 26 m s^{-1} vertically upwards.



(i) State the time at which the ball reaches its maximum height. [1]

(ii) State the feature of a velocity-time graph that enables the acceleration to be determined. [1]

(iii) Just after the ball leaves the thrower's hand, it has a downward acceleration of approximately 20 m s^{-2} . Explain how this is possible. [2]

(iv) State the time at which acceleration is g. Explain why the acceleration has this value only at this particular time. [2]

(v) Sketch an acceleration-time graph for the motion (in air). Show the value of g on the acceleration axis. [3]

(b) Explain why, for all real vertical throws, the time taken to reach maximum height must be shorter than the time taken to return to the starting point. [2]

(c) The ball in (a) starts with kinetic energy of 54 J.

(i) Calculate the mass of the ball.

(ii) Describe qualitatively how the amount of kinetic energy changes during the motion. [4]

[1.80 s, 1.80 s, 0.16 kg]