TEMASEK JUNIOR COLLEGE

Unit 8: Newton's Law of Gravitation

Learning Outcomes

Candidates should be able to:

(a) recall and use Newton's law of gravitation in the form $F = \dfrac{Gm_1m_2}{r^2}$.

(b) derive, from Newton's law of gravitation and the definition of gravitational field strength, the field strength due to a point mass, $g = \dfrac{GM}{r^2}$.

(c) recall and use $g = \dfrac{GM}{r^2}$ for the gravitational field strength due to a point mass to solve problems.

(d) show an understanding that near the surface of the Earth, gravitational field strength is approximately constant and is equal to the acceleration of free fall.

(e) define the gravitational potential at a point as the work done per unit mass by an external force in bringing a small test mass from infinity to the point.

(f) solve problems using the equation $\phi = -\dfrac{GM}{r}$ for the gravitational potential in the field due to a point mass.

(g) show an understanding that the gravitational potential energy of a system of two point masses is $U_G = -\dfrac{\mathrm{GMm}}{\mathrm{r}}$

(h) recall that gravitational field strength at a point is equal to the negative potential gradient at that point and use this to solve problems.

(i) analyse problems related to escape velocity by considering energy stores and transfers.

(j) analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes.

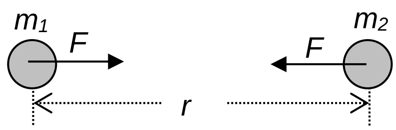
(j) show an understanding of geostationary orbits and their application.

1. NEWTON'S LAW OF GRAVITATION LO(a)

You may have heard the legend that Newton was struck on the head by a falling apple while napping under a tree. This alleged accident supposedly prompted him to imagine that perhaps all objects in the Universe were attracted to each other in the same way the apple was attracted to the Earth. It was in 1687 that Newton first published his work on the **law of gravitation**.

Newton's law of gravitation states that two point masses attract each other with a force that is proportional to the product of their masses and inversely proportional to the square of their separation.

Gravitational force $\boxed{F = \dfrac{Gm_1m_2}{r^2}}$



where m_1 and m_2 are the two point masses and r the distance between them. The constant of proportionality G is called the universal gravitational constant. $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.

<u>Note</u>:

1. The expression is applicable ONLY between point masses or spheres of uniform density.

2. For spheres, r is the distance between their centres and NOT surface to surface.

3. The gravitational force is directed along the line joining their centres.

4. The gravitational forces acting on the two masses form an *action-reaction* pair.

5. By the usual convention in Physics, an attractive force should have a negative sign (i.e. $F = -\dfrac{Gm_1m_2}{r^2}$). But since gravitational repulsion does not exist, the negative sign is omitted.

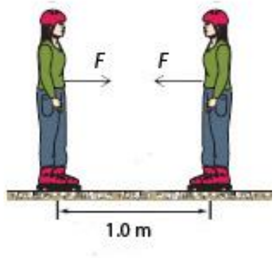
6. Newton's law of gravitation is an example of an *inverse square law* of force. That is, the force varies inversely with the square of the distance between the two masses.

7. The value of G (= 6.67×10^{-11} N m^2 kg^{-2}) is very small. Thus, although gravitational force of attraction exists between all bodies, it is a **weak** force and is only significant when very *massive* objects are involved.

Example 1: Find the gravitational force which

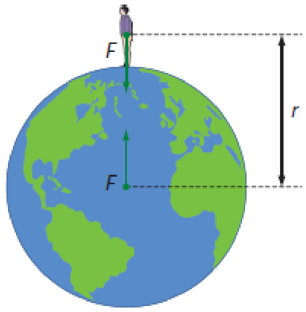
(a) a person of mass 60 kg exerts on another person of the same mass separated by a distance of 1.0 m.

Solution:



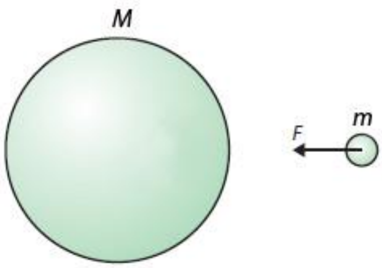
(b) the Earth of mass 6.0×10^{24} kg exerts on a man of mass 60 kg standing at its surface. The Earth has a radius of 6.4×10^{6} m.

Solution:



2. THE GRAVITATIONAL FIELD LO(b)

Any mass M sets up a "field of force" or *gravitational field* around it. This field is invisible but it can be detected by placing a small point mass m nearby. This small mass m, also called a test mass, experiences an attractive gravitational force F.



The gravitational field is *a* region of space where a mass experiences a gravitation force. **The gravitational field strength at a point is defined as the gravitational force acting per unit mass at the point.**

Gravitational field strength g = gravitational force per unit mass = $\dfrac{F}{m}$

Unit of g: N kg^{-1}. It is a vector quantity. *The direction of the gravitational field strength is the same as the direction of the force that acts on a small mass placed in the field*. Therefore g points in the direction towards the mass M.

Exercise: *Show, using base units, that N kg^{-1} is equivalent to m s^{-2}.*
Solution:

$$\begin{aligned} \text{N kg}^{-1} &= \text{kg m s}^{-2}\ \text{kg}^{-1} \\ &= \text{m s}^{-2} \end{aligned}$$

At any point, the gravitational field strength is equal to the acceleration due to gravity.

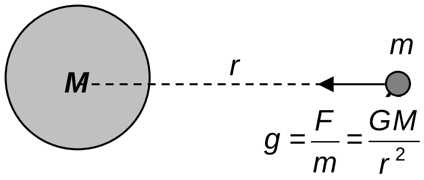
2.1 Derivation of gravitational field strength *g* due to a point mass *M* *LO(b),(c)*

From Newton's law of gravitation, the attractive gravitational force on a point mass *m* caused by another point mass *M*, with a distance *r* between their centres, is given by

$$F = \frac{GMm}{r^2}$$

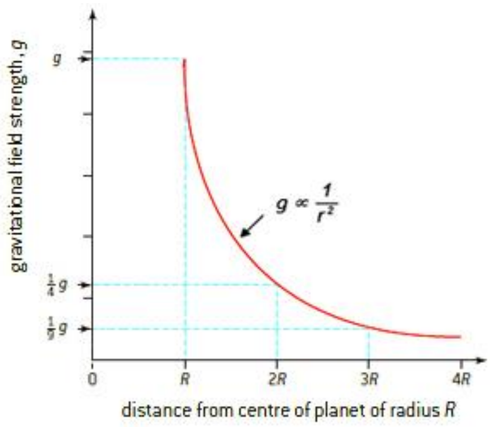
By definition, the gravitational field strength due to the mass *M* at a distance of *r* from its centre is thus given by

$$\boxed{g = \frac{F}{m} = \frac{GM}{r^2}}$$



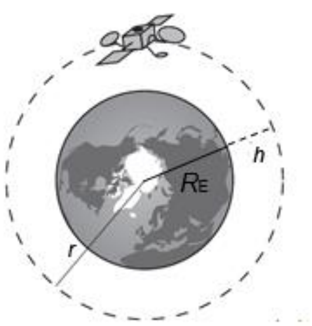
Since $g \propto \frac{1}{r^2}$, the field strength obeys the inverse square law with distance.

Thus the gravitational field is sometimes known as an *inverse square law field.*

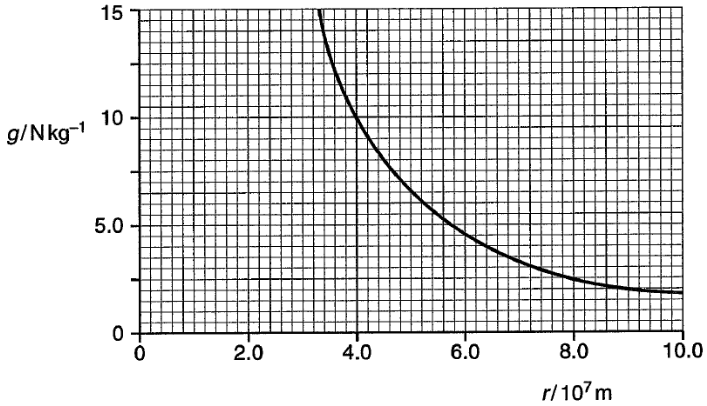


Example 2:
The International Space Station (ISS) operates at an altitude *h* of 350 km. What is the gravitational field strength at its orbit?
Given: Mass of Earth $M_E = 5.98 \times 10^{24}$ kg, Radius of Earth R_E = 6370 km.

Solution:



Example 3:
Figure shows how the gravitational field strength g varies with distance r from the centre of a planet of radius 2.7 x 10^7 m. Show that the field strength obeys an inverse square law.



Solution:
Choose any 3 points on the curve and show that gr^2 = constant

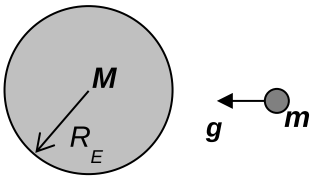
$r/10^7$ m	g/Nkg^{-1}	$gr^2/10^7$ Nkg^{-1}m^2
4.0	10.0	160
6.0	4.5	160
8.0	2.5	160

Since gr^2 = constant,

$$\Rightarrow \quad g = \frac{1}{r^2}$$

2.2 Variation of *g* with distance *r* from the Earth's centre

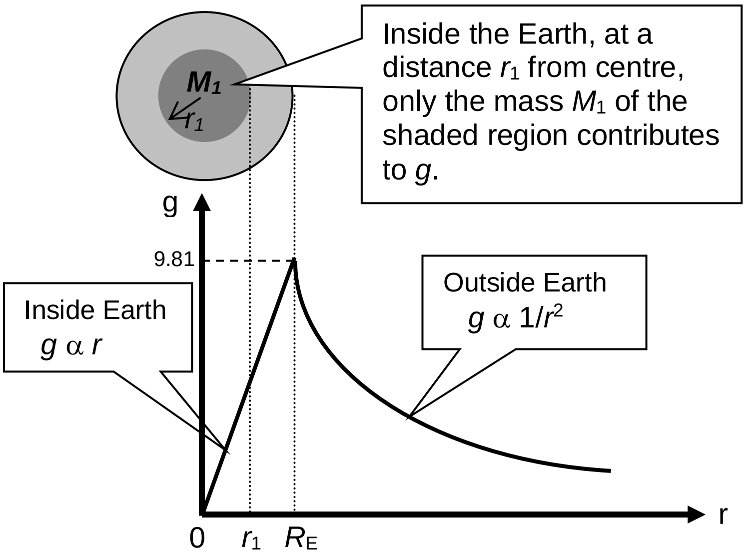
Outside the Earth
Outside the Earth, assuming that it is spherical, the Earth behaves as a *point mass*, with all the mass of the sphere concentrated at its centre. Any small mass m outside the Earth is attracted by the total mass M of the Earth.



Hence the gravitational field strength of the Earth for $r > R_E$ is given by equation

$$g = \frac{GM}{r^2} \quad \Rightarrow \quad g \propto \frac{1}{r^2}$$

Inside the Earth (assuming uniform density ρ)
Inside the Earth, a small mass m at any distance r_1, is attracted by the mass M_1 of the shaded region of radius r_1.



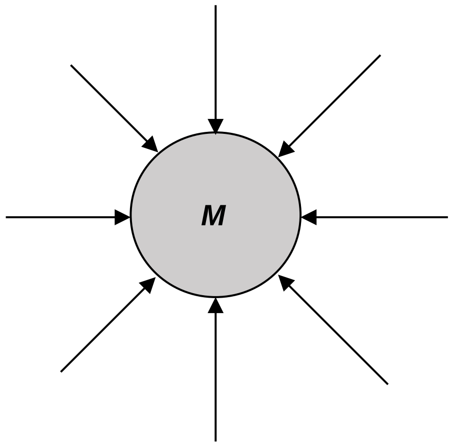
$$g = G\frac{M_1}{r_1^2} = G\frac{\rho V_1}{r_1^2}$$

$$= G\frac{\rho \frac{4}{3}\pi r_1^3}{r_1^2} = \frac{4}{3}G\rho\pi r_1$$

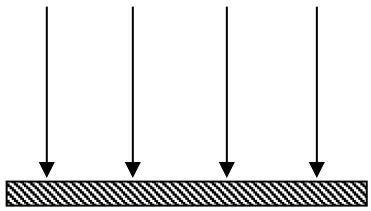
$$\Rightarrow g \propto r$$

Graph of variation of g from inside Earth to outside Earth

2.3 Gravitational Field Lines LO(d)



radial field



uniform field

(near surface of Earth)

Gravitational field lines are useful for visualizing the gravitational field due a mass M in any region of space. These are imaginary lines which *show the direction of the gravitational force acting on a unit mass*.

Since gravitational force is attractive, the field lines point radially inward towards the centre of a point mass or a spherical mass. The stronger the gravitational field strength, the closer/denser are the field lines.

Near the surface of the Earth, the field strength is approximately constant and hence the field is assumed to be uniform. For a uniform field, the lines are parallel and equally spaced.

2.4 True weight and Apparent weight

The *true weight* of a body is equal to the gravitational force acting on it.

true weight = mass × gravitational field strength

A body is truly weightless in the **absence** of a gravitational field, e.g. at an infinite distance from the Earth and other astronomical bodies.

The *apparent weight* of a body is equal to the force that the body exerts on its support. It is therefore **equal to the Normal Contact Force *N*** that acts on the body due to the support. It will not be equal to the true weight if the body is *accelerating.*

For example,

(i) the apparent weight of a man standing inside an elevator, which is accelerating upward, is greater than his true weight [since $N = mg + ma$],

(ii) the apparent weight of a man in a lift that snaps is zero since it is *falling freely* under gravity [since $N = mg - ma = 0$ if $a = g$].

2.5 Factors affecting *g* measured on Earth

1. The non-spherical shape of the Earth

The Earth is ellipsoidal i.e. it is flattened at the Poles and bulges at the Equator.

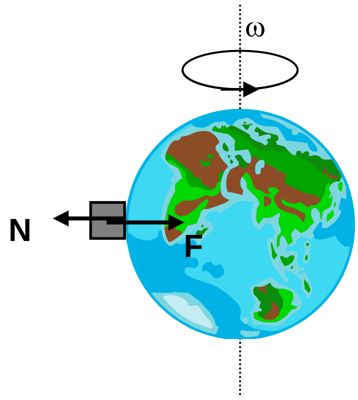
Since $g \propto 1/r^2$, the value of *g* is greater at the poles than at the Equator.

2. The density of the Earth

The density of the Earth is non-uniform due to unequal deposits of minerals in the crust causing small variations in *g*. Variations in density results in variations of *g* and this is useful in oil prospecting.

3. The rotation of the Earth

The Earth spins on its axis with a period T = 24 hours. Any object on the Earth's surface undergoes circular motion with the same angular velocity $\omega = \dfrac{2\pi}{T}$ as the Earth. Each object experiences a centripetal acceleration.



Consider an object of mass *m* at the Equator of radius *R*.
Two forces gravitational force F and normal contact force N acts on the object.
By Newton's second law,

net force on object = centripetal force

$$F - N = mR\omega^2$$

where F is the gravitational force acting on object,
N is the normal force acting on object,
R is the radius of the Earth,

Substitute gravitational force $F = \dfrac{GMm}{R^2} = mg$ (true weight) into the above equation,

$$mg - N = mR\omega^2$$

$$\Rightarrow \quad N = mg - mR\omega^2$$
$$= m(g - R\omega^2)$$

The above equation shows that apparent weight N is less than true weight mg.

Let $N = mg' = m(g - R\omega^2)$

$\Rightarrow$ apparent acceleration due to gravity, $g' = g - R\omega^2$

Thus due to the rotation of the Earth, the apparent acceleration of free fall **g' at the equator is less than g.**

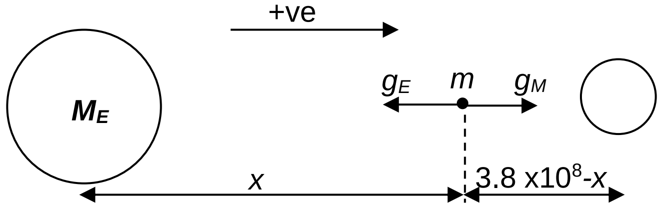
$$g' = g - R\omega^2 = 9.81 - 6400 \times 10^3 \times \left(\frac{2\pi}{24\times60\times60}\right)^2 = 9.78 \text{ m s}^{-2}$$

2.6 Resultant Gravitational Field Strength

LO(c)

If two or more masses are present, then the **resultant gravitational field strength at any point is the vector sum of the individual gravitational field strengths**.

Take the example of a space capsule of mass *m* travelling between the Earth and the Moon. The space capsule experiences a resultant gravitational field due to the Earth and the Moon.



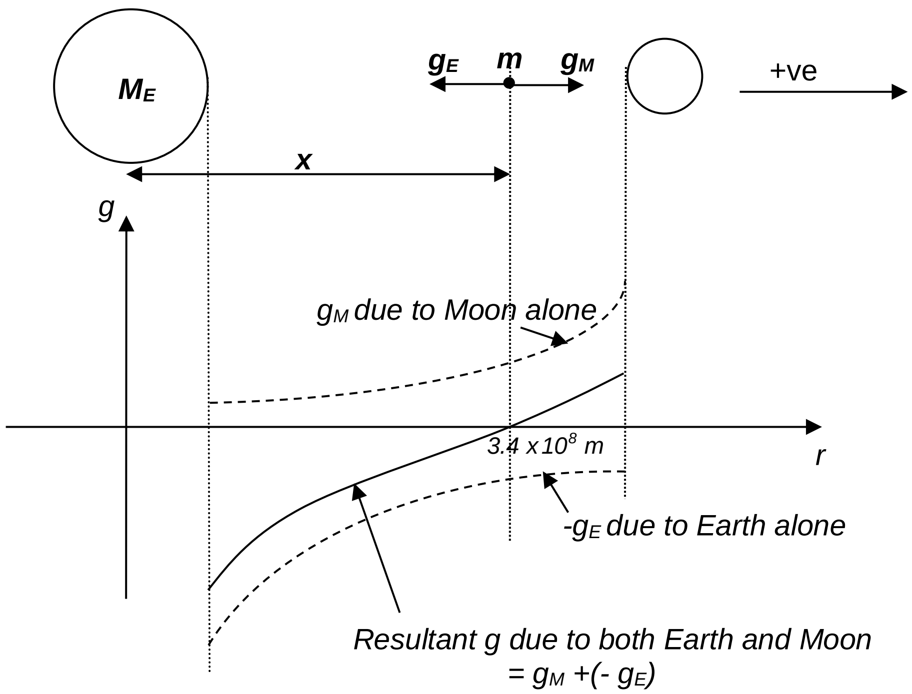
Given:
$M_E = 6.0 \times 10^{24}$ kg,
$M_M = 7.4 \times 10^{22}$ kg,
Distance between the centres of the Earth and the Moon = 3.8×10^8 m.

At any distance x, the resultant gravitational field strength is the vector addition of the individual field strength graphs due to Earth and Moon.

$\rightarrow+$ Resultant field strength $g = g_M + (-g_E) = g_M - g_E$

$$= \frac{GM_M}{(3.8 \times 10^8 - x)^2} - \frac{GM_E}{x^2}$$

The variation of the resultant field strength *g* with distance *x* is given below.



Resultant g due to both Earth and Moon

$$= g_M + (-g_E)$$

$$= \frac{GM_M}{(3.8 \times 10^8 - x)^2} - \frac{GM_E}{x^2}$$

Example 4: Determine the position of the zero gravity point between the Earth and the Moon.

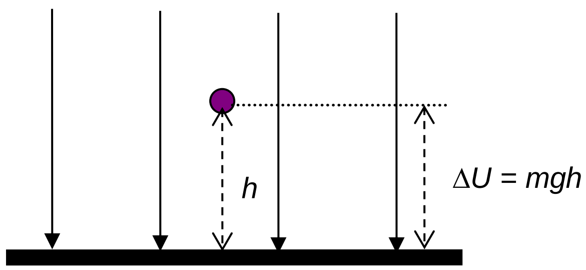
Solution:
At a particular position called the **zero gravity point**, the resultant gravitational field acting on the space capsule is equal to zero,

$$g = g_M - g_E = 0 \Rightarrow g_E = g_M$$

Hence at distance $x = 3.4 \times 10^8$ m, the space capsule experience **zero gravity**.

3. GRAVITATIONAL POTENTIAL AND GRAVITATIONAL POTENTIAL ENERGY

Consider an object near the Earth's surface. We can assume ***g* to be constant** for short distances above the Earth's surface.



When it moves a vertical distance h, then the change in the gravitational potential energy can be considered as

$$\Delta U = mgh$$

Note that this formula $\Delta U = mgh$ is only applicable if height h << radius of the Earth, as g is approx. constant for small heights above the Earth's surface.

For objects situated close to the surface of the Earth compared to its radius (e.g. apple on tree, rock on hill) the equation $\Delta U = mgh$ is usually valid.

However for larger distances (e.g. satellites in orbit, the moon), this equation is no longer valid, as g decreases with $\frac{1}{r^2}$.

3. 1 Definitions of Gravitational Potential Energy *U* and Gravitational Potential ϕ

LO(e), (g)

Gravitational potential energy *U* at a point is defined as the work done by an external agent in bringing a body from infinity to that point (without any change in its kinetic energy).

$$U = W$$

The equation for gravitational potential energy is given by $\boxed{U = -\dfrac{GMm}{r}}$

It is a scalar quantity and has units of J.

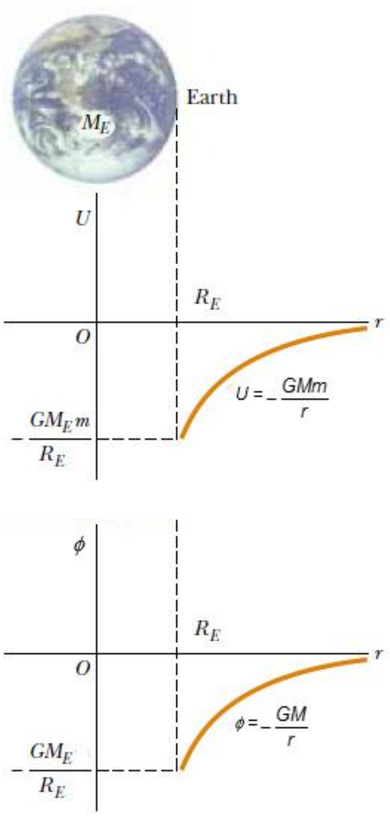
Gravitational potential ϕ at a point is defined as the work done per unit mass by an external agent in bringing a body from infinity to that point (without any change in its kinetic energy).

$$\phi = \frac{W}{m} = \frac{U}{m}$$

The equation for gravitational potential is given by $\boxed{\phi = -\dfrac{GM}{r}}$

It is a scalar quantity and has units of J kg^{-1}.

A sketch of the variation of *U and* ϕ vs *r* outside the Earth are as shown below.



Note:

1. Both ϕ and *U* varies with $-\dfrac{1}{r}$.
2. At infinity, *both* ϕ and *U* are zero. (highest value)
3. As an object moves towards the surface of Earth, both ϕ and *U* decrease and become more negative.

Note: at distance *r* far away from the Earth or any planet, we can no longer use *U = mgh as g is not constant.* We have to use this equation $U = -\dfrac{GMm}{r}$ to calculate gravitational potential energy!

3.2 Relationship between *F, U, g* and ϕ LO(h)

In summary,

1. The ***gravitational potential energy*** at a point is given by $U = -\dfrac{GMm}{r}$.
2. The ***gravitational potential*** at a point is given by $\phi = -\dfrac{GM}{r}$.
3. Gravitational force $F = -\dfrac{dU}{dr}$, where $\dfrac{dU}{dr}$ is known as the **potential energy gradient**.
4. Gravitational field strength $g = -\dfrac{d\phi}{dr}$, where $\dfrac{d\phi}{dr}$ is known as the **potential gradient**.

5. The **magnitude of the gravitational field strength is equal to the potential gradient at the point** and the negative sign indicates that ***g*** **points in the direction of decreasing** $\boldsymbol{\phi}$**.**

The relationships between the quantities are summarised in the following table.

	F	$\boldsymbol{\phi}$
g	$g = \dfrac{F}{m}$	$g = -\dfrac{d\phi}{dr}$
U	$F = -\dfrac{dU}{dr}$	$\phi = \dfrac{U}{m}$

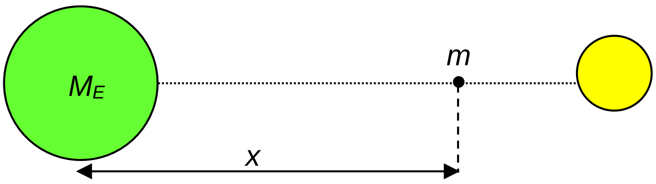
Question: Why are ϕ and *U negative values?*
2 reasons:
1. The gravitational potential at infinity is chosen to be **zero**. (i.e. when at $r = \infty$, $\phi = 0$).
2. Work done by the external agent from infinity to any point is **negative**. (since the force by the external agent is in the opposite direction to the displacement of mass *m* as it moves from infinity towards the point.) Hence this negative work done is converted as negative potential energy at the point.

3.3 Resultant Gravitational Potential LO(f)

Since gravitation potential is a scalar quantity, the **resultant gravitational potential at the point is given by scalar addition of the potentials due to two or more masses present.**

<u>Example 5:</u>
For the space capsule question in Section 2.6,
(a) Calculate the gravitation potential at the point at which it experiences zero gravitational field strength.
(b) Sketch the variation of the resultant gravitational potential with distance from the Earth to the Moon.
(Mass of the Earth = 6.0×10^{24} kg, mass of the Moon = 7.4×10^{22} kg, distance between the centres of the Earth and the Moon = 3.8×10^{8} m.)
(c) Use the resultant gravitational curve to determine the minimum energy required to send the 1200 kg space capsule from the Earth to the Moon.

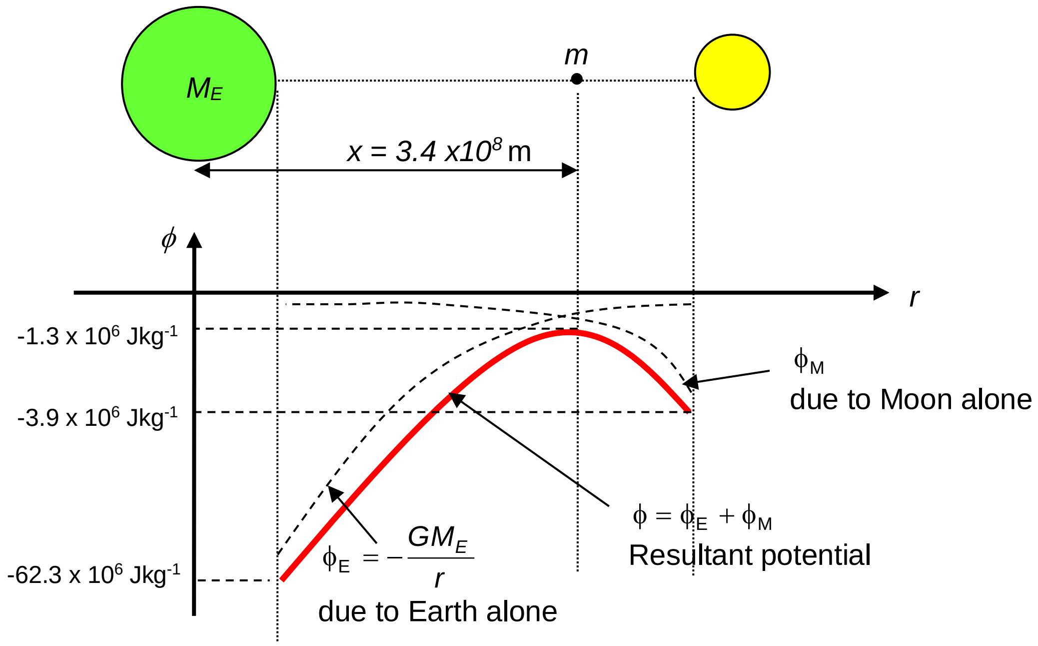
Solution:
(a)



The resultant potential at position x is given by

$$\phi = \phi_E + \phi_M \qquad \text{(Scalar addition of the potentials due to Earth and Moon)}$$

(b) The resultant potential curve is the scalar addition of the potential curves due to Earth and Moon.



(c) The minimum energy to send a space capsule from Earth to Moon is given by the work done to 'climb' to the highest potential point.

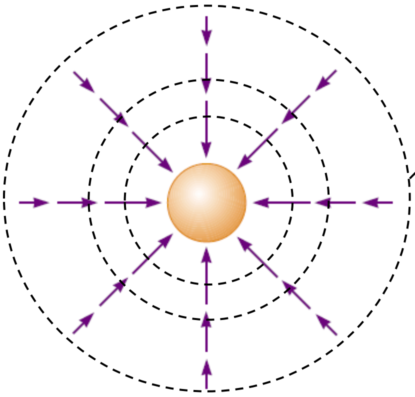
$$\begin{aligned} W &= \Delta U \\ &= m\Delta\phi \\ &= 1200\,[-1.3 \times 10^6 - (-62.3 \times 10^6)] \\ &= 7.32 \times 10^{10}\text{ J} \end{aligned}$$

3.4 Equipotential Surfaces

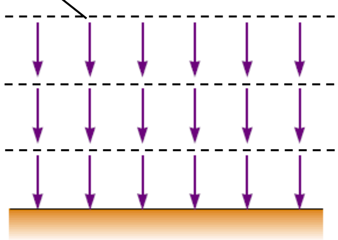
Equipotential surfaces are surfaces on which the potential is the same at all points on the surface.

No work is needed to move a mass along an equipotential surface.

Equipotential surfaces can be represented on a diagram by drawing equipotential contours (dotted lines). Note that the **equipotentials are perpendicular to the gravitational field lines** at all points.



The equipotential surfaces round the Earth are imaginary spherical shells centred at the Earth's centre.

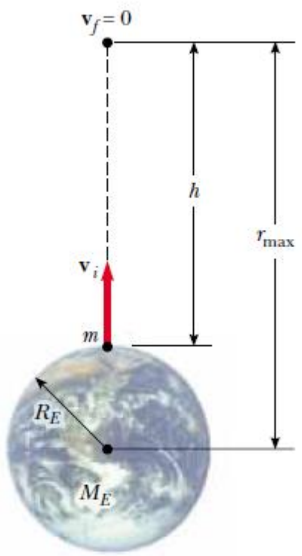


Near the Earth's surface, the field is approximately uniform and the equipotentials are parallel and evenly spaced.

3.5 Escape Speed

LO(i)

Escape speed is the minimum speed at which a body can be projected from the surface of a planet so that it reaches an infinite distance from the planet. (i.e. it escapes completely from the gravitational pull of the planet.)



Take for example a mass m projected from the surface of the Earth, the minimum escape speed is such that the initial kinetic energy of mass m is sufficient to be converted to potential energy of the mass at infinity.

Loss in KE = Gain in GPE

Initial KE – Final KE = Final GPE – Initial GPE

$$\tfrac{1}{2} m v_i^2 - 0 = 0 - \left(-\frac{GM_E m}{R_E}\right)$$

Escape speed $v = \sqrt{\dfrac{2GM_E}{R_E}}$ ------(1)

Since $g = \dfrac{GM_E}{R_E^{\,2}}$, escape speed $v = \sqrt{2gR_E}$ -----(2)

Example 6:
Calculate the escape speed from the Earth for a 5 000 kg spacecraft, and determine the kinetic energy it must have at the Earth's surface in order to move infinitely far away from the Earth.

Solution:

Escape speed $v = \sqrt{\dfrac{2GM_E}{R_E}}$

$$= \sqrt{\frac{2x6.67x10^{-11}x5.98x10^{24}}{6.37x10^{6}}} = 1.12 \times 10^4 \text{ m s}^{-1}$$

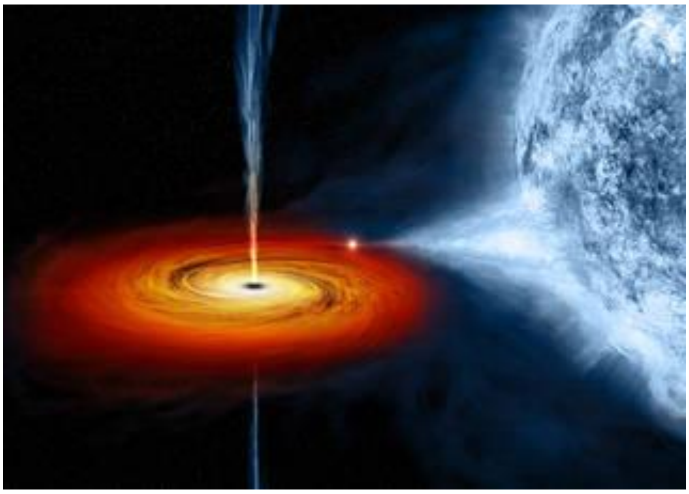
Kinetic Energy required $= \frac{1}{2} mv^2$
$= \frac{1}{2} \times 5000 \times (1.12 \times 10^4)^2$
$= 3.14 \times 10^{11}$ J

Existence of black holes

What is a black hole? How is it formed? What is the escape speed of a black hole?

A black hole is a region in space where the pulling force of gravity is so strong that light is not able to escape.

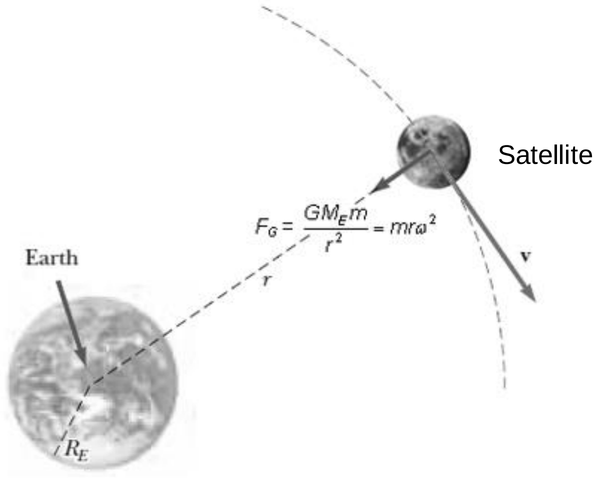
A massive star may collapse or shrink under its own gravitational forces. If the star has a mass greater than three times the mass of the Sun, the collapse may continue until the star becomes a very small object in space, commonly referred to as a black hole. When an object such as a spacecraft comes close to a black hole, it experiences an extremely strong gravitational force and is sucked and trapped inside it forever.



The escape speed for a black hole is very high, **greater than the speed of light *c***, because even visible light radiation from any object within the black hole cannot escape.

4 PLANETARY AND SATELLITE MOTION LO(j)

The orbit of the Moon or any satellite round the Earth or of a planet round the Sun can be assumed to be circular. **The centripetal force necessary for the circular motion is provided by the gravitational force of attraction.**



For the satellite in circular motion about any planet of mass M, the gravitational force provides the centripetal force.

Gravitational Force = Centripetal Force

$$\frac{GMm}{r^2} = mr\omega^2$$

$$\frac{GM}{r^2} = r\left(\frac{2\pi}{T}\right)^2 \quad \text{where } T \text{ is the period of rotation}$$

$$T^2 = \frac{4\pi^2}{GM} r^3$$ (Note $T^2 \propto r^3$, *which is Kepler's 3rd Law*)

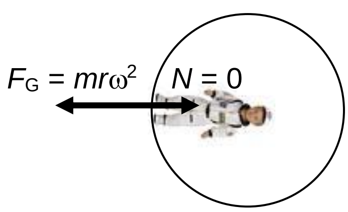
Period $T = 2\pi\sqrt{\frac{r^3}{GM}}$

Using $\frac{GMm}{r^2} = \frac{mv^2}{r}$,

We can derive the orbital speed $v = \sqrt{\frac{GM}{r}}$

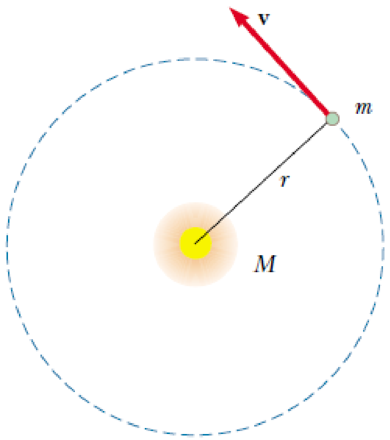
Question: Explain briefly why an astronaut in a satellite orbiting the Earth experiences weightlessness.

Explanation: The gravitational force acting on both the satellite and the astronaut by the Earth **fully provides the centripetal force** needed for circular motion. As such, the astronaut experiences zero normal contact force and hence he experiences weightlessness.



Astronaut inside a satellite

4.1 Energy Considerations in Planetary and Satellite Motion



Since $\dfrac{GMm}{r^2} = \dfrac{mv^2}{r}$

$\Rightarrow mv^2 = \dfrac{GMm}{r}$

Kinetic Energy of satellite $T = \frac{1}{2} mv^2 = \dfrac{GMm}{2r}$

Potential Energy of satellite $U = -\dfrac{GMm}{r}$

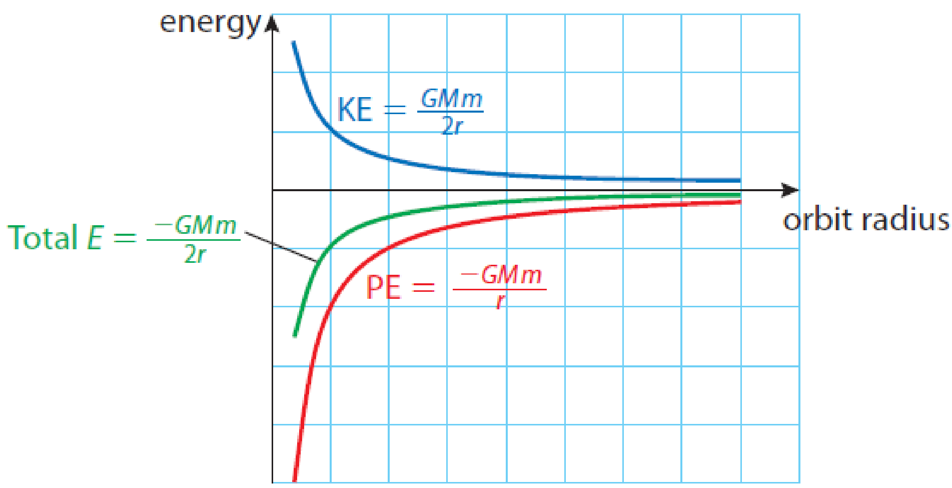
Total Energy of satellite $E = T + U$

$= \dfrac{GMm}{2r} + (-\dfrac{GMm}{r})$

$= -\dfrac{GMm}{2r}$

The total energy E is negative, which means that the satellite is in a *bound* orbit around the planet. External work done is required to separate the satellite from the planet.

The graphs of KE, PE and Total Energy of a satellite is given below.



4.2 Geostationary satellites

LO(k)

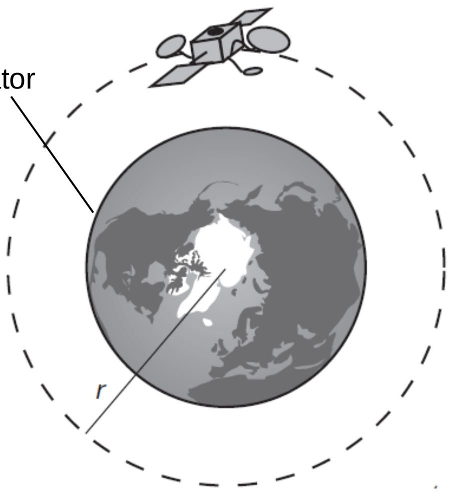
Artificial satellites have a variety of uses. Many are used for making observations of the Earth's surface for commercial, environmental, meteorological and military purposes. Others are used for astronomical observations, benefiting greatly from being above the Earth's atmosphere. Still others are used for navigation, telecommunications and broadcasting.

Many of these satellites are placed in what is called **geostationary** orbit.

Features of a geostationary orbit:

1. It is an equatorial orbit, i.e. the orbit includes the equator.
2. It has a period of 24 h, i.e. *same period as the Earth.*
3. It moves in the same direction of rotation as the Earth, i.e. rotates from west to east.

As such, a satellite in geostationary orbit is *always above the same point on the Equator*. Thus a geostationary orbit is sometimes known as a 'parking orbit'.



For a satellite in geostationary orbit,
Gravitational Force provides the Centripetal Force

$$\frac{GMm}{r^2} = mr\omega^2$$

$$\frac{GM}{r^2} = r\left(\frac{2\pi}{T}\right)^2$$

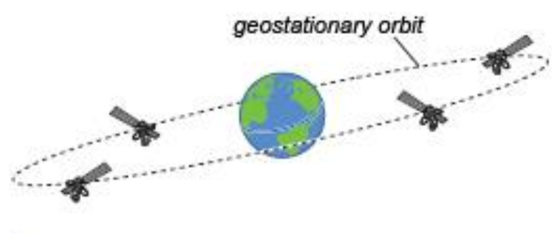
$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$$

Substitute G = 6.67×10^{-11} N m^2 kg^{-2},
Mass of Earth $M = 6.0 \times 10^{24}$ kg and
Period $T = 24 \times 60 \times 60 = 8.64 \times 10^4$ s,

Radius of geostationary orbit
r = $\underline{4.2 \times 10^7 \text{ m}}$

Geostationary Satellites in Communication

The idea of a 'parking orbit' was first suggested in 1945 by the engineer and science fiction writer Arthur C. Clarke. There are over 300 satellites in such orbits. They are used for telecommunications (transmitting telephone messages around the world) and for satellite television transmission. A base station on Earth sends the TV signal up to the satellite, where it is amplified and broadcast back to the ground. Satellite receiver dishes are a familiar sight in a neighbourhood; they all point towards the same point in the sky. Because the satellite is in a geostationary orbit, the dish can be fixed. Geostationary satellites have a lifetime of ten years. They need a fuel supply to maintain them in the correct orbit, and to keep them pointing correctly towards the Earth. Eventually they run out of fuel and need to be replaced.



Video of geostationary satellite:

Video of geostationary versus polar satellite

Gravitational Field Tutorial

(Take $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, Radius of Earth R_E = 6400 km, g = 9.81 m s^{-2})
Solutions to questions 9, 13, 15 and 17 are provided.

Gravitational Force and Field Strength

1 A spherical planet has diameter 1.2×10^4 km. The gravitational field strength at the surface of the planet is 8.6 N kg^{-1}. The planet may be assumed to be isolated in space and to have its mass concentrated at its centre. Calculate the mass of the planet.

[4.6 x 10^{24} kg]

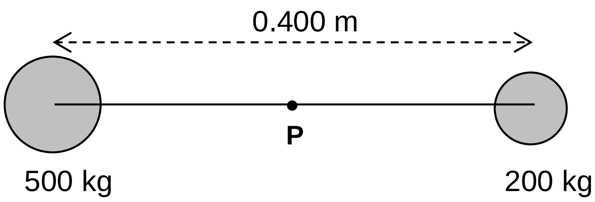
2 Information related to the Earth and the Moon is given below.

$\frac{\text{Radius of Earth}}{\text{Radius of Moon}} = 3.7$, $\frac{\text{Mass of Earth}}{\text{Mass of Moon}} = 81$

Gravitational field strength due to the Earth at its surface = 9.8 N kg^{-1}
Using these data, calculate the gravitational field strength due to the Moon at its surface.

[1.7 N kg^{-1}]

3 A 500 kg object and a 200 kg object are separated by 0.400 m.

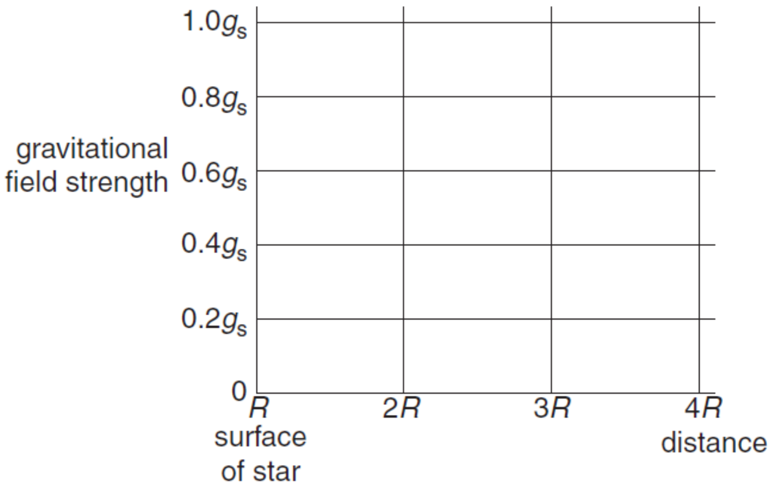


(a) Find the resultant gravitational field strength at a point P midway between them
(b) Find the force exerted on a 50 kg object placed at P.

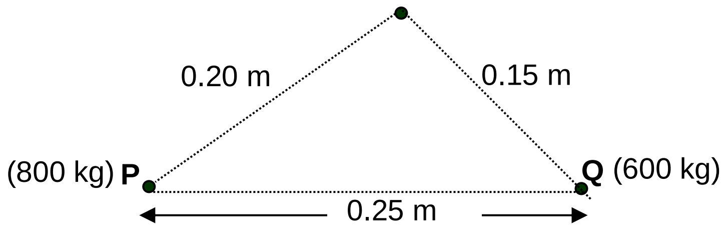
(c) At what position (other than an infinite distance) can the 50 kg object be placed so as to experience a zero net force?

[5.0 x 10^{-7} Nkg^{-1} towards the 500 kg object, 2.50 x 10^{-5} N, 0.245 m from the 500 kg object]

4 An isolated star has radius R. The mass of the star may be considered to be a point mass at the centre of the star. The gravitational field strength at the surface of the star is g_s. Sketch a graph to show the variation of the gravitational field strength of the star with distance from its centre. You should consider distances in the range R to $4R$.



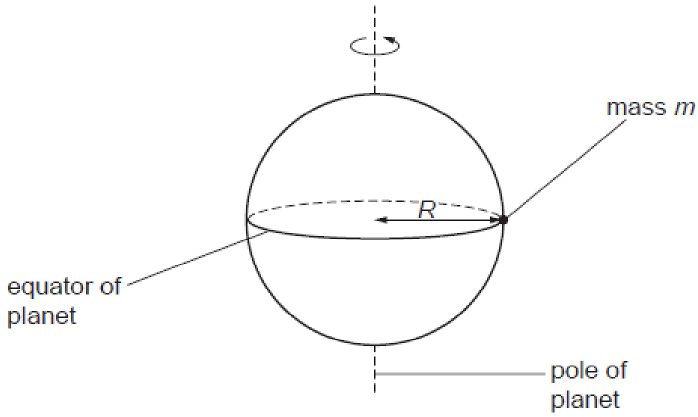
5 Masses P of 800 kg and Q of 600 kg are separated by 0.25 m as shown in the diagram below.



(a) Determine the magnitude of the resultant gravitational field strength due to both bodies at a point X, which is 0.20 m from P and 0.15 m from Q.

(b) If a mass of 100 kg is placed at X, determine the magnitude of the gravitational force it will experience. [2.2×10^{-6} N kg^{-1}, 2.2×10^{-4} N]

6 A spherical planet has mass M and radius R. The planet may be assumed to be isolated in space and to have its mass concentrated at its centre. The planet spins on its axis with angular speed ω, as illustrated in the figure.



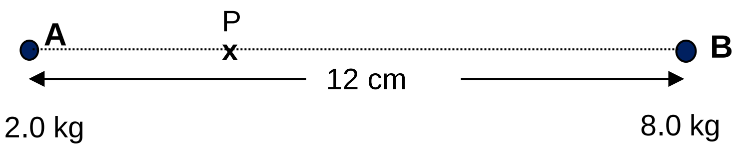
A small object of mass m rests on the equator of the planet. The surface of the planet exerts a normal reaction force on the mass.

(a) State the equations, in terms of M, m, R and ω, for
   (i) the gravitational force between the planet and the object,
   (ii) the centripetal force required for circular motion of the small mass,
   (iii) the normal contact force exerted by the planet on the mass.

(b) Explain why the apparent weight of the mass will have different values at the equator and at the poles.

Gravitational Potential and Potential Energy

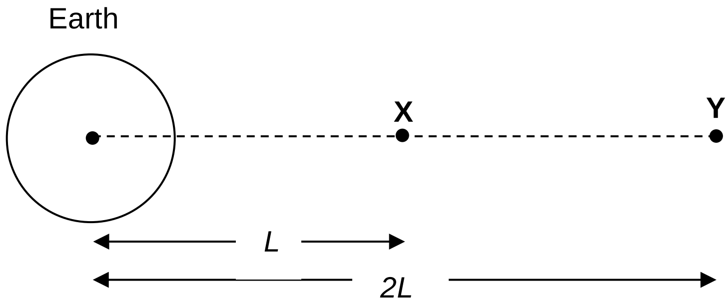
7 The figure below shows body A of mass 2.0 kg and body B of mass 8.0 kg placed 12 cm apart.



(a) Determine the gravitational potential at a point P which is 4.0 cm from A and 8.0 cm from B along the line joining the centres of A and B.

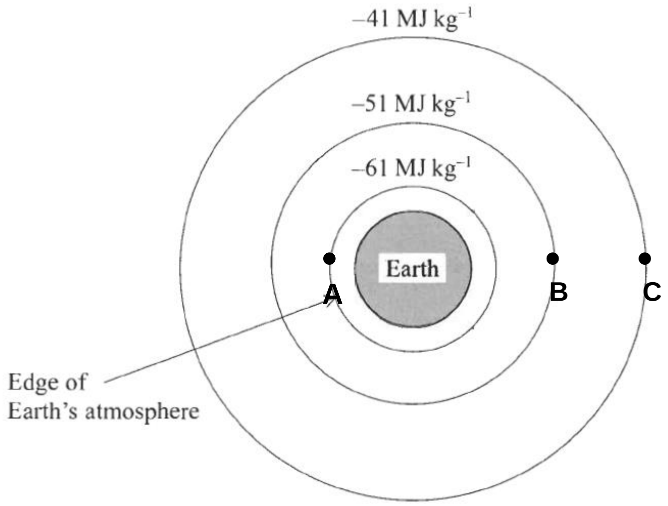
(b) If another mass C of 1.0 kg is placed at rest at P, determine the amount of work needed to bring it to infinity. [-1.0×10^{-8} J kg^{-1}, 1.0×10^{-8} J kg^{-1}]

8 Figure shows two points **X** and **Y** at distances *L* and *2L* from the centre of the earth. The gravitational potential at **X** is – 8.0 kJ kg^{-1}.



Calculate
(a) the gravitational potential at Y,
(b) the work done in moving a 2.0 kg mass from X to Y. [-4.0 KJ kg^{-1}, 8.0 kJ]

9 The diagram shows three equipotential surfaces centred about the Earth with their values marked. Points A, B and C are marked on the surfaces.



(a) State two deductions about the gravitational field strength of the Earth that can be made from the diagram.
(b) Calculate the energy needed to project a 10 kg mass from point A to point C.
(c) A 50 kg rock falls radially from rest towards the Earth from point C to point B. Calculate the final speed of the rock at point B.

[200 MJ, 4.5×10^3 m s^{-1}]

Solution:

(a)1. The direction of g is radial towards the earth since field lines are $\perp$ to equipotential surface and directed towards decreasing potential.

2. For equal $\Delta\phi$ between the equipotential surfaces, the separation Δr of the surfaces increases as *r* increases. Since g = $\left|\frac{\Delta\phi}{\Delta r}\right|$, g decreases with increasing r.

(b) Energy needed W = $m\Delta\phi = 10(-41-(-61)) = 200$ MJ

(c) Gain in KE = Loss in PE

$= m(\phi_i - \phi_f)$

$\frac{1}{2}mv^2 = m(-41-(-51)) \times 10^6 \Rightarrow v = 4.5 \times 10^3$ m s^{-1}

10 Values for the gravitational potential due to the Earth are given in the table.

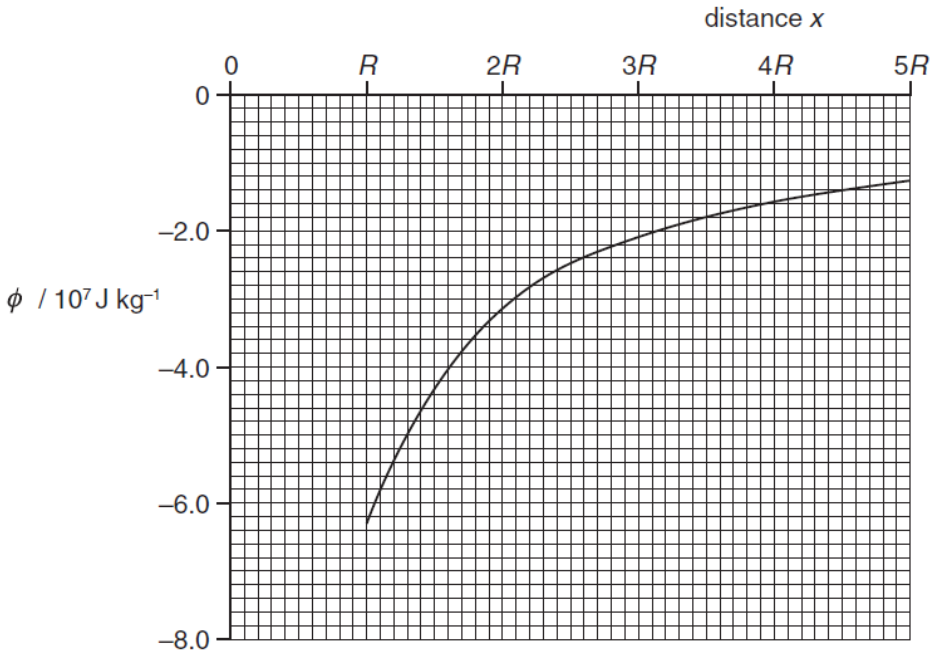
Distance from Earth's surface/ m	Gravitational potential/ MJ kg^{-1}
0	-62.72
390 000	-59.12
400 000	-59.03
410 000	-58.94
Infinity	0

(a) If a satellite of mass 700 kg falls from a height of 400 000 m to the Earth's surface, how much potential energy does it lose?

(b) (i) State the relationship between gravitational field strength and gravitational potential.
(ii) Hence deduce a value for the Earth's gravitational field at a height of 400 000 m.

[2.58×10^9 J, 9.0 m s^{-2}]

11 The Earth may be considered to be an isolated sphere of radius R with its mass concentrated at its centre. The variation of the gravitational potential ϕ with distance x from the centre of the Earth is shown in figure.



The radius R of the Earth is 6.4×10^6 m.

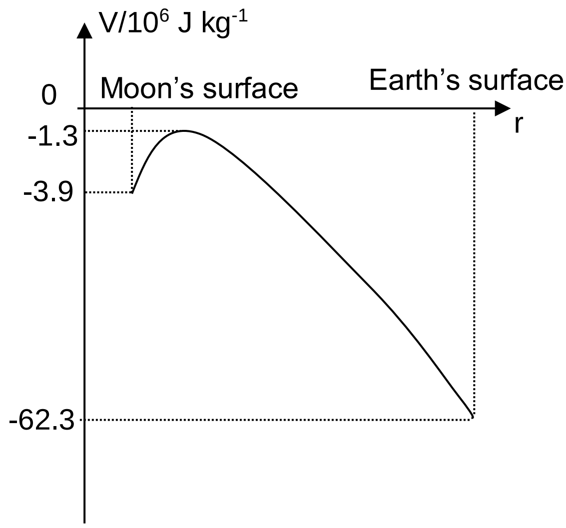
(a) By considering the gravitational potential at the Earth's surface, determine a value for the mass of the Earth.

(b) A meteorite is at rest at infinity. The meteorite travels from infinity towards the Earth. Calculate the speed of the meteorite when it is at a distance of 2R above the Earth's surface.

[6.0×10^{24} kg, 6.5×10^3 ms^{-1}]

12 The graph shows how the total potential V between the Moon and the Earth varies along the line of centres.

(a) By considering the separate contributions of the Earth and the Moon to the gravitational potential, explain qualitatively why the graph has a maximum and why the curve is asymmetrical.



A spacecraft on the Moon is to return to the Earth. After leaving the surface of the Moon, its engine will be shut off. Assume no friction loss, determine

(b) the minimum energy per unit mass that has to be supplied.

(c) the speeds of the spacecraft

(i) during take off, and

(ii) when it reaches the Earth.

[2.6×10^6 J kg^{-1}, 2.3×10^3 m s^{-1}, 1.1×10^4 m s^{-1}]

Planetary and Satellite Motion

13 (a) What is meant by a geostationary orbit and give one advantage of placing a satellite there.

(b) Explain why the orbit of the satellite must lie in a plane that includes the centre of the Earth.

(c) Many systems, such as the Global Positioning System (GPS), use several satellites in low orbits that pass over the Earth's poles. Suggest two advantages of these low polar orbits.

Solution:

(a) Period of satellite = period of Earth's rotation about its axis. Satellite always directly over the same geographical location. Useful for telecommunication purposes.

(b) Centripetal force is provided by the gravitational force between the satellite and the Earth.- its direction is towards centre of Earth. If centre of orbit is not at centre of Earth, a component of the gravitational force will tend to alter its orbit until it lies in the plane that includes the centre of Earth.

(c) 1. Satellite can offer global coverage of the Earth, useful for GPS, Google Earth etc

2. Low altitude ensures good resolution of the images.

14 The Earth may be considered to be a sphere of radius 6400 km with its mass of 6.0×10^{24} kg concentrated at its centre. A satellite of mass 650 kg is to be launched from the Equator and put into geostationary orbit.

Calculate

(a) the angular velocity of the satellite at the orbit,

(b) the radius of the satellite's orbit.

(c) the increase in gravitational potential energy of the satellite during its launch from the Earth's surface to the geostationary orbit.

[7.3×10^{-5} rad s^{-1},4.2×10^7 m, 3.45×10^{10} J]

15 A satellite of mass *m* orbits the Earth of mass *M* in a circular path with radius *R*.

(a) Show that the linear speed v_s of the satellite is given by the expression

$$v_s = \sqrt{\frac{GM}{R}}$$

(b) For this satellite, write down expressions, in terms of the symbols given, for

(i) its kinetic energy *T*,

(ii) its gravitational potential energy *U*,

(iii) its total energy *E*.

(c) As the satellite orbits the Earth, it loses energy because of air resistance.

(i) State whether the total energy *E* becomes more or less negative.

(ii) Hence state and explain the effect of this change on

1. the radius of the orbit,
2. the speed of the satellite.

Solution:

(a) Gravitational force provides centripetal force $\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$

(b) (i) *Kinetic Energy of satellite* $T = \frac{1}{2} mv^2 = \frac{GMm}{2r}$

(ii) *Potential Energy of satellite* $U = -\frac{GMm}{r}$

(iii) *Total Energy of satellite* $E = -\frac{GMm}{2r}$

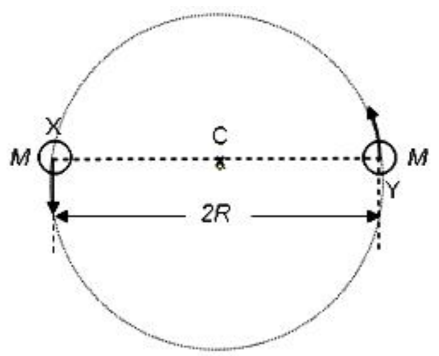
(c) (i) total energy of satellite decreases due to work done against friction, so it becomes more negative.

(i) 1. the radius r decreases according to the equation $E = -\frac{GMm}{2r}$.

Therefore the height of satellite decreases

2. the speed v increases according to the equation $v = \sqrt{\frac{GM}{r}}$. The satellite spirals with **increasing speed** towards the Earth.

16 A particular binary-star system consists of two stars X and Y, of equal mass *M* that move in a circular orbit about a common centre of mass C, the mid-point between the stars, as shown in the diagram. The distance between X and Y is 2*R*.



In terms of *G, M* and *R*, deduce the expression for

(a) (i) the gravitational force between the two stars,

(ii) the angular speed for each star.

(b) For a particular star system R = 6.5×10^8 m and M = 3.0×10^{30} kg, calculate the period of this system. [$\frac{GM^2}{4R^2}$, $\sqrt{\frac{GM}{4R^3}}$, 1.47×10^4 s (4.1 hrs)]

17 (a) Calculate the escape speed of a helium atom from the surface of the Earth, given that Mass of Earth = 6.0 x 10^{24} kg and Radius of Earth = 6400 km, mass of Helium atom = 6.64 x 10^{-27} kg.

(b) The mean kinetic energy of a helium atom at thermodynamic temperature T is given by the expression E_K = 3/2 kT where k = Boltzmann constant = 1.38 x 10^{-23} JK^{-1}.
Determine the surface temperature of the Earth such that helium-4 atoms on the surface of Earth have the escape speed calculated in (a).
Use your results to deduce why there is atmosphere on the surface of Earth.

[1.12 x 10^4 m s^{-1}, 20,000 K]

Solution:

(a) Escape speed from the Earth $v = \sqrt{\frac{2GM_E}{R_E}}$

$$= \sqrt{\frac{2x6.67x10^{-11}x5.98x10^{24}}{6.37x10^6}} = 1.12 \times 10^4 \text{ m s}^{-1}$$

(b) using $E_k = ½ mv^2 = 3/2 kT$

$½ \times 6.64 \times 10^{-27} \times (1.12 \times 10^4)^2 = 3/2 \times 1.38 \times 10^{-23} T$

T = 20 000 K

Since the surface temperature of Earth is much less than 20000 K, helium and other air molecules have lesser k.e. and are unlikely to escape from surface of Earth to infinity $\Rightarrow$ there is atmosphere on Earth.

18 In July 2015, the New Horizons space probe made its closest approach of Pluto.
Charon is one of the moons of Pluto. Data including the radius r and period of rotation about the axis T, are given for Pluto and Charon below:

	r / km	mass / kg	T / days
Pluto	1.20 x 10^3	1.31 x 10^{22}	6.36
Charon	0.600 x 10^3	1.52 x 10^{21}	6.36

When viewed from above, Pluto and Charon rotate in the same direction about their axes.

(a)(i) The orbital speed of Charon is 0.200 km s^{-1} and its orbital radius is 1.75 x 10^4 km. show that the period of the orbit of Charon around Pluto is 6.36 days.

(ii) A space probe on the surface of Pluto is able to observe Charon over a time of several days. Suggest what the space probe would observe as a result of
1. the period of rotation of Pluto about its axis equalling the orbital period of Charon,
2. equal periods of rotation about their axes for both Pluto and Charon.

(b) Pluto may be assumed to be an isolated sphere with its mass concentrated at its centre. Use data from the table to determine the gravitational field strength g_P on the surface of Pluto.

(c) (i) A rock of mass M, initially at rest at infinity, falls towards Pluto. There is negligible resistance to the motion of the rock from Pluto's atmosphere.
Show that the speed v of the rock as its hits the surface of Pluto is given by $v = \sqrt{2g_P r}$
, where r is the radius of Pluto.

(ii) As a result of bombardment of Pluto by a meteor, a rock is ejected from the surface. Use the information from the table and (b) to calculate the minimum speed of the rock so that it eventually reaches infinity.

[0.607 N kg^{-1}, 1210 m s^{-1}]