RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 1 (Summary and Tutorial)

Topic(s): System of Linear Equations, Inequalities

Summary for System of Linear Equations

This topic involves

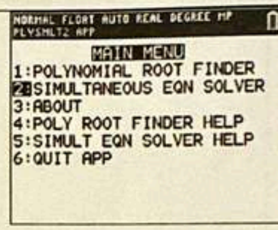
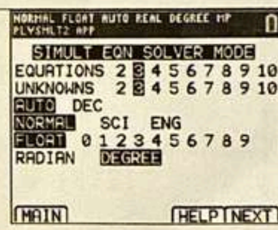
1. Understand the given question
2. Formulate the equations
3. Use of GC to solve the equations

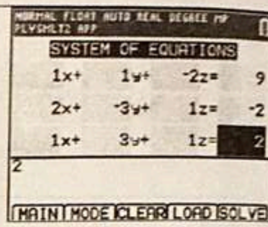
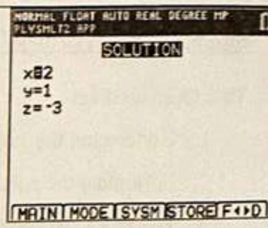
Recap on the use of GC:

Consider the following equations:

$$x+y-2z=9$$
$$2x-3y+z=-2$$
$$x+3y+z=2$$

$x=2$, $y=1$, $z=-3$.

1. Press Œ and select **PlySmlt2**. 2. Select **2: Simultaneous Eqn Solver**.	
3. Adjust the settings as depicted on the screen to solve 3 equations with 3 unknowns. 4. Select **NEXT** by pressing the s button.	

5. Enter the coefficients into the equations. 6. Press s to **SOLVE** the system of equations.	
7. Read off the answers. Hence $x = 2, y = 1, z = -3$.	

Summary for Inequalities

The following basic concepts are useful in solving inequalities.

1. $a > b$ and $c > 0 \Rightarrow ac > bc$.	2. $a > b$ and $c < 0 \Rightarrow ac < bc$.
3. $\frac{a}{b} > 0 \Leftrightarrow ab > 0$.	4. $\frac{a}{b} < 0 \Leftrightarrow ab < 0$.

Note:

- *The inequality sign* ***remains unchanged*** *if a* ***positive*** *constant is multiplied or divided to both sides of an inequality.*
- *The inequality sign is* ***reversed*** *if a* ***negative*** *constant is multiplied or divided to both sides of an inequality.*

General Steps when solving a given inequality in rational form.

Step	Description	Example $\frac{8-5x}{2x+1} \le 2-x$.
0	Are exact answers required? If not, just use GC to solve.	
1	check for values of x which will make the expression(s) undefined	$2x+1 \ne 0 \Rightarrow x \ne -\frac{1}{2}$
*2	bring all terms to LHS or RHS and simplify so that we get $\frac{a}{b} > 0$, $\frac{a}{b} < 0$ or similar form.	$\frac{8-5x}{2x+1} \le 2-x$, $x \ne -\frac{1}{2}$ . . . $\frac{(x-1)(x-3)}{2x+1} \le 0$
*3	Applying properties 3 or 4, multiply the square of the denominator to maintain the inequality sign.	$(x-1)(x-3)(2x+1) \le 0$, $x \ne -\frac{1}{2}$
4	Solve the inequality taking into account the values found in step 1 (If time permits, check answers using GC)	$x < -\frac{1}{2}$ or $1 \le x \le 3$

* Note that sometimes, we may have a case where the numerator (or denominator) is always positive (or negative).

Example, $\frac{4x^2-5x+3}{(x-2)(x+1)} \le 0$ where $4x^2-5x+3 = 4\left(x-\frac{5}{8}\right)^2 + \frac{23}{16}$ is always positive, the inequality reduces to $(x-2)(x+1) \le 0$, $x \ne 2, -1$.

$\left|\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|AC|}\right| \times \frac{\overrightarrow{AC}}{|AC|} = \overrightarrow{AC}$

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 2 (Summary and Tutorial)

Topic(s): Vectors 1 (Vector Algebra, Ratio Theorem, Scalar and Vector Product)

Summary for Vectors 1

Vector Algebra

With reference to an origin $O(0, 0, 0)$, given points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$, we have the corresponding (position vectors) expressed in column form $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

$$\mathbf{a} \pm \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix} \text{ and}$$

$$k\mathbf{a} = k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} \text{ with } k \text{ a real number.}$$

The **magnitude** (or modulus) of a vector, $|\mathbf{a}|$, is the non-negative number

$|\mathbf{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}$. This value is equal to the distance from O to A.

We say that $\mathbf{a}$ is parallel to $\mathbf{b}$, denoted by $\mathbf{a} \mathbin{/\!/} \mathbf{b}$, if and only if $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$, that is, $\mathbf{b}$ is a (non-zero) scalar multiple of $\mathbf{a}$.

- If $\lambda > 0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in the **same** direction.
- If $\lambda < 0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in **opposite** directions.

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O, such that $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$. We then say that the points O, A and B are **collinear**.

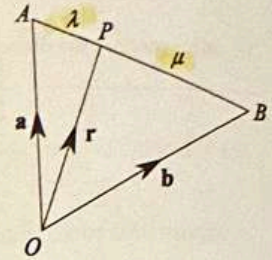
The **unit vector** in the **direction** of **a** denoted by $\hat{\mathbf{a}}$ is obtained by scaling **a** by $\frac{1}{|\mathbf{a}|}$, thus $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.

The vectors $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}-2\\-4\\4\end{pmatrix}$ are parallel since $\begin{pmatrix}-2\\-4\\4\end{pmatrix}$ is a scalar multiple of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ $(k=-2)$ but are in opposite directions since $k<0$. The points $(-2,-4,4)$, $(0,0,0)$ and $(1,2,-2)$ are also said to be collinear. The magnitude of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ is $\sqrt{1^2+2^2+(-2)^2}=3$ so the unit vector in the direction of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ is $\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$

Let **a** and **b** be non-zero and **non-parallel** vectors:
If $\lambda\mathbf{a} = \mu\mathbf{b}$ for some $\lambda, \mu \in \mathbb{R}$, then $\lambda = \mu = 0$.
If $\alpha\mathbf{a} + \beta\mathbf{b} = s\mathbf{a} + t\mathbf{b}$ for some $\alpha, \beta, s, t \in \mathbb{R}$, then $\alpha = s$, $\beta = t$.

Note the importance of **non-parallel** vectors when comparing coefficients.
Suppose $\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\0\\0\end{pmatrix}$ then $6\mathbf{a}+2\mathbf{b} = 4\mathbf{a}+3\mathbf{b}$ however we cannot "compare coefficients" of vectors **a** and **b** (note that **a** is parallel to **b**) as $6 \neq 4$ and $2 \neq 3$.

Ratio Theorem



Consider a triangle OAB with $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$.
So $\mathbf{a}$ and $\mathbf{b}$ are non-zero and **non-parallel** vectors.
Let P be a point which divides AB in the ratio $\lambda:\mu$,
i.e. $\frac{AP}{PB}=\frac{\lambda}{\mu}$. If $\overrightarrow{OP}=\mathbf{p}$, then $\mathbf{p}=\frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$ (MF26)

Note that the Ratio Theorem is an immediate consequence of the addition of vectors.
From diagram, $\mathbf{p}-\mathbf{a}=\frac{\lambda}{\lambda+\mu}(\mathbf{b}-\mathbf{a})$, rearranging we have $\mathbf{p}=\frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$. Sometimes, it is easier to use $\mathbf{p}-\mathbf{a}=\frac{\lambda}{\lambda+\mu}(\mathbf{b}-\mathbf{a})$ like the following example.

Points A, B and P have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ respectively, relative to the origin O. Given that $\mathbf{a}=\begin{pmatrix}2\\2\\5\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\6\\-3\end{pmatrix}$, find $\mathbf{p}$ if P lies on AB produced such that $\frac{AB}{AP}=\frac{2}{5}$.

Solution: Easier to find directly, $\mathbf{p}-\mathbf{a}=\frac{5}{2}(\mathbf{b}-\mathbf{a})$ [DO NOT WRITE $\frac{\mathbf{b}-\mathbf{a}}{\mathbf{p}-\mathbf{a}}=\frac{2}{5}$. We cannot divide vectors]

$$\mathbf{p}=\begin{pmatrix}2\\2\\5\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2-2\\6-2\\-3-5\end{pmatrix}=\begin{pmatrix}2\\12\\-15\end{pmatrix}$$

Scalar (Dot) Product

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O.

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where $\theta = \angle AOB$.

Re-arranging, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

If vectors $\mathbf{a}$ and $\mathbf{b}$ are in the **same direction** then $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$, largest possible value.
If vectors $\mathbf{a}$ and $\mathbf{b}$ are in the **opposite direction** then $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$, smallest possible value.
Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, denoted by $\mathbf{a}\perp\mathbf{b}$, if and only if $\mathbf{a}\cdot\mathbf{b}=0$.
In particular, $\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$ as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular.

If $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, then $\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3$.

Find the cosine of the angle between the vectors $\mathbf{a}=2\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $\mathbf{b}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and determine if the angle is acute or obtuse.

Solution

$$\cos\theta=\frac{(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})}{\sqrt{2^2+3^2+2^2}\sqrt{1^2+2^2+(-1)^2}}$$

$$\cos\theta=\frac{2-6-2}{\sqrt{17}\sqrt{6}}=-\sqrt{\frac{6}{17}}<0$$

θ is an obtuse angle

Properties of Scalar Product

(i) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

(ii) $\mathbf{a}\cdot(\mathbf{b}\pm\mathbf{c})=(\mathbf{a}\cdot\mathbf{b})\pm(\mathbf{a}\cdot\mathbf{c})$

(iii) $\lambda(\mathbf{a}\cdot\mathbf{b})=(\lambda\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(\lambda\mathbf{b})$

(iv) $\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2 \Leftrightarrow |\mathbf{a}|=\sqrt{\mathbf{a}\cdot\mathbf{a}}$ note the relationship between the scalar product and the modulus

Given that $|\mathbf{a}|=2$, $|\mathbf{b}|=3$, $\mathbf{a}\cdot\mathbf{b}=-1$,

(i)
$$\begin{aligned}&(\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})\\&=2\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\&=2|\mathbf{a}|^2-\mathbf{a}\cdot\mathbf{b}-|\mathbf{b}|^2\\&=2(2^2)-(-1)-3^2\\&=0\end{aligned}$$

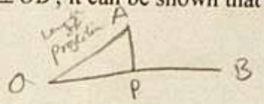
(ii)
$$\begin{aligned}&(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})\\&=4\mathbf{a}\cdot\mathbf{a}+2\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\&=4|\mathbf{a}|^2-|\mathbf{b}|^2\\&=4(2^2)-3^2\\&=7\end{aligned}$$

(iii) $|\mathbf{a}-2\mathbf{b}|=\sqrt{(\mathbf{a}-2\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})}=\sqrt{\mathbf{a}\cdot\mathbf{a}-2\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{a}+4\mathbf{b}\cdot\mathbf{b}}=\sqrt{|\mathbf{a}|^2-4\mathbf{a}\cdot\mathbf{b}+4|\mathbf{b}|^2}=\ldots=2\sqrt{11}$

Applications of Scalar Product

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O.
Given that P is the point on the line OB such that $\overrightarrow{AP} \perp \overrightarrow{OB}$, it can be shown that

$$\overrightarrow{OP} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}$$



P is the foot of perpendicular from A to the line OB and P is also the point on the line OB nearest to A.

Thus, the **length of projection of vector a onto vector b** is given by $\left|\overrightarrow{OP}\right| = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$.

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O. Given that $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, find the length of projection of **a onto vector b**. Find the position vector of the foot of the perpendicular from A to OB. **Solution** $\quad OP = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right| = \frac{\left|(2\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})\right|}{\sqrt{1^2+2^2+(-1)^2}} = \frac{\lvert 2-6-2\rvert}{\sqrt{6}} = \sqrt{6}$ $\overrightarrow{OP} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}} = -\left(\frac{6}{\sqrt{6}}\right)\frac{\mathbf{i}+2\mathbf{j}-\mathbf{k}}{\sqrt{6}} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

If α, β and γ are the angles between **a** and the component vectors **i, j** and **k** respectively, then $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \left(\frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|}, \frac{a_3}{|\mathbf{a}|}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$. Component of $\hat{\mathbf{a}}$ are referred to as the direction cosines.

Vector (Cross) Product

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors that are represented by $\overline{OA}$ and $\overline{OB}$ respectively.

$$\mathbf{a}\times\mathbf{b}=(|\mathbf{a}||\mathbf{b}|\sin\theta)\hat{\mathbf{n}}$$

where $\theta=\angle AOB$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

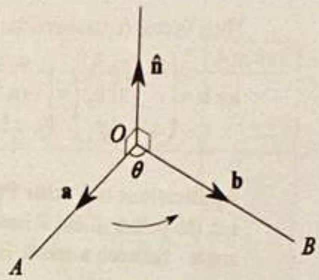


Figure 1

It follows that

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** if and only if $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|$. In particular, $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$.

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are **parallel** if and only if $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.
In particular, $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$.

Given that $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=\mathbf{0}$, what can be deduced about the vectors $\mathbf{u}$ and $\mathbf{v}$?

Solution:

$(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=\mathbf{0}$

$\mathbf{u}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}+\mathbf{v}\times\mathbf{u}-\mathbf{v}\times\mathbf{v}=\mathbf{0}$

$\mathbf{0}+2\mathbf{v}\times\mathbf{u}-\mathbf{0}=\mathbf{0}$

$\mathbf{v}\times\mathbf{u}=\mathbf{0}$

$\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$ or $\mathbf{u}\parallel\mathbf{v}$

Properties of Vector Product

1. $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$.
2. $|\mathbf{a}\times\mathbf{b}|=|\mathbf{b}\times\mathbf{a}|=|\mathbf{a}||\mathbf{b}||\sin\theta|\le|\mathbf{a}||\mathbf{b}|$.
3. $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c})=(\mathbf{a}\times\mathbf{b})\pm(\mathbf{a}\times\mathbf{c})$.
4. $\lambda(\mathbf{a}\times\mathbf{b})=(\lambda\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\lambda\mathbf{b})$.
5. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$ So $\mathbf{a}\times\mathbf{b}\perp\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}\perp\mathbf{b}$.

Examples of how the properties are used

$$\begin{aligned}|(\mathbf{a}+2\mathbf{b})\times(3\mathbf{a}-\mathbf{b})|&=|3\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+6\mathbf{b}\times\mathbf{a}-2\mathbf{b}\times\mathbf{b}|\\&=|0+\mathbf{b}\times\mathbf{a}+6\mathbf{b}\times\mathbf{a}+0|\\&=7|\mathbf{b}\times\mathbf{a}|\end{aligned}$$

$$\begin{aligned}|(\alpha\mathbf{a}+\beta\mathbf{b})\times(\alpha\mathbf{a}-\beta\mathbf{b})|&=|\alpha^2\mathbf{a}\times\mathbf{a}-\alpha\beta\mathbf{a}\times\mathbf{b}+\alpha\beta\mathbf{b}\times\mathbf{a}-\beta^2\mathbf{b}\times\mathbf{b}|\\&=\alpha\beta|0+\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{a}+0|\\&=2\alpha\beta|\mathbf{b}\times\mathbf{a}|\end{aligned}$$

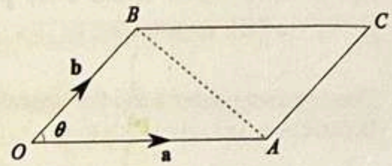
Let $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ be two points in three-dimensional space and let the position vectors of A and B with respect to the origin O be $\mathbf{a}$ and $\mathbf{b}$ respectively.

Then vector (cross) product $\mathbf{a}\times\mathbf{b}$, is the vector given by

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix} \text{ (MF26)}$$

Applications of Vector Product

Let the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to the origin O, and let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Let C be the point such that $OACB$ is a parallelogram.



Then we have

(1) Area of triangle OAB is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

(2) Area of parallelogram $OACB$ is $|\mathbf{a}\times\mathbf{b}|$.

(3) Distance from B to OA is $|\mathbf{b}\times\hat{\mathbf{a}}|$.

(Recall the **length of projection** of $\overrightarrow{OB}$ onto $\overrightarrow{OA}$ is $|\mathbf{b}\cdot\hat{\mathbf{a}}|$)

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2020 Year 6 Term 3 Revision 3 (Summary and Tutorial)

Topic(s): Vectors 2 (Lines and Planes)

Summary for Lines and Planes

Line: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ where
$\mathbf{r}$ is the position vector of a general point on the line,
$\mathbf{a}$ is the position vector of a known point on the line and
$\mathbf{b}$ is the vector that indicate the direction parallel to the line (known as the *direction vector*)

Note : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ is not **UNIQUE.**

Different Forms of Equations of a Line

$$\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ \lambda \in \mathbb{R} \qquad \text{– \textit{Vector Form}}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},\ \lambda \in \mathbb{R}$$

$$\Leftrightarrow x = a_1 + \lambda b_1, y = a_2 + \lambda b_2, z = a_3 + \lambda b_3 \qquad \text{– \textit{Parametric Form}}$$

$$\Leftrightarrow \frac{x-a_1}{b_1} = \lambda,\ \frac{y-a_2}{b_2} = \lambda,\ \frac{z-a_3}{b_3} = \lambda$$

$$\Leftrightarrow \frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} \qquad \text{– \textit{Cartesian Form}}$$

$\frac{2-x}{4} = \frac{x-2}{-4}$

Plane: $\mathbf{r}\,.\,\mathbf{n} = d$ where
$\mathbf{r}$ is the position vector of a general point on the plane,
$\mathbf{n}$ is the normal vector to the plane (a vector perpendicular to the plane)
d is a scalar constant.

Notes:
The position vector of any point A on the plane, $\mathbf{a}$ will always give the result, $\mathbf{a}\,.\,\mathbf{n} = d$

The vector equation of the plane can always be reduced to the form $\mathbf{r}\,.\,\hat{\mathbf{n}} = \frac{d}{|\mathbf{n}|} = D$. In this form, the normal vector is reduced to a unit vector and $|D|$ is the ***shortest distance of the plane from the origin.***

$\frac{d}{|\underset{\sim}{n}|}$ if $d \geq 0$, d is
if $d < 0$, d is $-$

Different Forms of Equations of a Plane

Scalar Product form: $\mathbf{r}.\mathbf{n} = d$

Cartesian form: $n_1x + n_2y + n_3z = d$ where $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

Vector form: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{m}_1 + \mu \mathbf{m}_2, \quad \lambda, \mu \in \mathbb{R}$
a is the position vector of a point on the plane,
$\mathbf{m}_1$ and $\mathbf{m}_2$ are non-parallel vectors that are parallel to the plane. Thus, $\mathbf{m}_1 \times \mathbf{m}_2$ will be a normal to the plane.

We will now summarize the various relationships involving points, lines and planes.

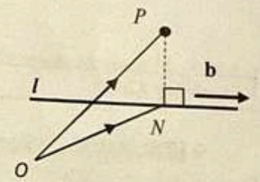
Involving Points and Lines

❖ To check whether the point P with position vector **p** lies on the given line l: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \ \lambda \in \mathbb{R}$

– Let $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$ and find a possible value for λ.
– If there is a unique value for λ from the three possible equations, then the point P lies on the line.

❖ To find the position vector of the *foot of the perpendicular* from a point P with position vector **p** to a given line l : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \ \lambda \in \mathbb{R}$

Let N be the foot of the perpendicular from the point P to the line, l. Make use of the fact that $\overrightarrow{PN}$ is perpendicular to l



Step 1: Since N lies on l, $\overrightarrow{ON} = \mathbf{a} + \lambda\mathbf{b}$ for a unique value of λ.

Step 2: Find $\overrightarrow{PN} = \overrightarrow{ON} - \overrightarrow{OP}$ in terms of λ, where $\overrightarrow{OP} = \mathbf{p}$ is given.

Step 3: Solve for the value of λ using the fact that $\overrightarrow{PN} \cdot \mathbf{b} = 0$
Step 4: Using the value of λ found above, the position vector of N is given by $\mathbf{a} + \lambda\mathbf{b}$.

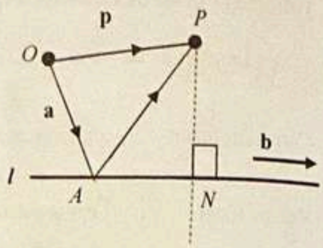
❖ To find the perpendicular distance (shortest distance) from a point, P with position vector, **p**, to a given line l : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \ \lambda \in \mathbb{R}$

Let N be the foot of the perpendicular from the point P to the line, l
Method 1
Step 1 : Find the position vector of the foot of the perpendicular from the point P to the given line l.

Step 2 : Perpendicular distance from the point P to the given line l is $|\overrightarrow{PN}|$.

Method 2 (Cross Product)

Perpendicular distance from the point P to the given line l is $\left|\overrightarrow{PN}\right| = \left|\overrightarrow{AP} \times \hat{\mathbf{b}}\right|$



Method 3 (Dot Product)

Step 1 : Find $\left|\overrightarrow{AN}\right| = \left|\overrightarrow{AP} \cdot \hat{\mathbf{b}}\right|$

Step 2 : Using Pythagoras' Theorem,
Perpendicular distance from the point P to the given line l is $\left|\overrightarrow{PN}\right| = \sqrt{\left|\overrightarrow{AP}\right|^2 - \left|\overrightarrow{AN}\right|^2}$

❖ To find the position vector of the reflection of a point, P, about a line l : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$

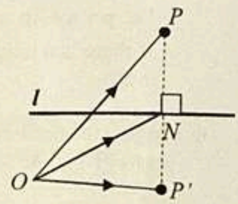
Let P' be the reflection of P about l and that N be foot of the perpendicular from P to l.

Method

– Find the vector $\overrightarrow{PN}$ (**Note that** $\overrightarrow{PN} = \overrightarrow{NP'}$)

– Position vector of P' $= \overrightarrow{OP'} = \overrightarrow{OP} + \overrightarrow{PN} + \overrightarrow{NP'} = \overrightarrow{OP} + 2\overrightarrow{PN}$

(**OR** use Ratio Theorem: $\overrightarrow{ON} = \dfrac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$)



Involving Lines

❖ To check if the lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$ are parallel

If $\mathbf{b} \,//\, \mathbf{d}$ (i.e. $\mathbf{b} = k\mathbf{d}$), then $l_1 \,//\, l_2$

❖ To find the acute angle between 2 lines, $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$

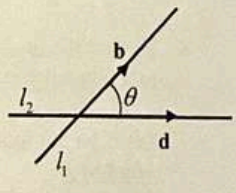
The acute angle between the lines is determined by the two direction vectors **b** and **d**.

Method

Make use of scalar product of the two direction vectors **b** and **d**.

– $|\mathbf{b} \cdot \mathbf{d}| = |\mathbf{b}|\,|\mathbf{d}|\cos\theta$ where θ is the acute angle between the 2 direction vectors, **b** and **d**.

Therefore $\cos\theta = \dfrac{|\mathbf{b}.\mathbf{d}|}{|\mathbf{b}||\mathbf{d}|}$



- To find the position vector of the point of intersection of the two lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$

Let P be the point of intersection of the 2 lines l_1 and l_2, and $\mathbf{p}$ be the position vector of P. Then P lies on both lines and $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{p} = \mathbf{c} + \mu\mathbf{d}$ for some values of λ and μ.

Method

Step 1 : Since the lines intersect, $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$

Step 2 : From Step 1, we will have 3 linear equations in terms of λ and μ

Solve for the values of λ and μ

– If there exist unique values for λ and μ, then the position vector of the intersection point is given by $\mathbf{a} + \lambda\mathbf{b}$ or $\mathbf{c} + \mu\mathbf{d}$.

– If there are no unique values for λ and μ, then the 2 lines are non-intersecting lines. (They can be either parallel or skew lines)

- To check if the lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{d},\ \mu \in \mathbb{R}$ are skew lines (Skew lines are non–parallel and non–intersecting lines.)

Assume that P, the point of intersection of the 2 lines l_1 and l_2, and $\mathbf{p}$ be the position vector of P exists show that it does not exist.
Then P lies on both lines and $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{p} = \mathbf{c} + \mu\mathbf{d}$ for some values of λ and μ.

Method

Step 1 : Show that l_1 and l_2 are not parallel ($\mathbf{b} \neq k\mathbf{d}$)

Step 2 : Then assume $\mathbf{a} + \lambda\mathbf{b} = \mathbf{c} + \mu\mathbf{d}$ and show that there is **NO** unique values for λ and μ that satisfy the equations formed.

- To find the shortest distance between two parallel lines $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{b},\ \mu \in \mathbb{R}$.

The shortest distance between two parallel lines can be found by taking any point, says A on $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and find the shortest distance between point A and $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{b},\ \mu \in \mathbb{R}$.

OR

The shortest distance between two parallel lines can be found by taking any point, says C on $l_2 : \mathbf{r} = \mathbf{c} + \mu\mathbf{b},\ \mu \in \mathbb{R}$ and find the shortest distance between point C and $l_1 : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$.

(Refer above for the shortest distance between a point and a line)

Involving Points, Lines and Planes

- To check whether the point P with position vector $\mathbf{p}$ lies on the given plane $\pi : \mathbf{r}.\mathbf{n} = d$

  Method

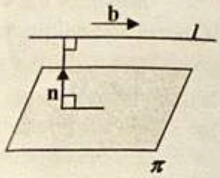
  – If $\mathbf{p}.\mathbf{n} = d$, then the point P lies on the plane.

  If $\mathbf{p}.\mathbf{n} \neq d$, then the point P does not lie on the plane.

- To check if the line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ is parallel to a given plane $\pi : \mathbf{r}.\mathbf{n} = d$

  Method

  – If $\mathbf{b}.\mathbf{n} = 0$, then the line is parallel to the plane.
  (perpendicular to the normal)



- To check if the line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ lies in a given plane $\pi : \mathbf{r}.\mathbf{n} = d$

  Method 1

  – Show that $(\mathbf{a} + \lambda\mathbf{b}).\mathbf{n} = d$ for all values of λ, then the line lies in the plane.

  Method 2

  Step 1 : Show that $\mathbf{b}.\mathbf{n} = 0$, then the line is parallel to the plane.

  Step 2 : Show that $\mathbf{a}.\mathbf{n} = d$, then the point with position vector, $\mathbf{a}$ lies on the plane. Then conclude the line lies in the plane.

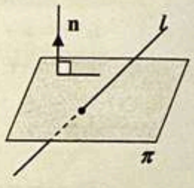
- To find the position vector of point of intersection between the line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and the plane $\pi : \mathbf{r}.\mathbf{n} = d$

  Let P with position vector $\mathbf{p}$ be the point of intersection between the line l and the plane π
  Then $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$, for some $\lambda \in \mathbb{R}$ and $\mathbf{p}.\mathbf{n} = d$

  Method

  – Since $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$, for some $\lambda \in \mathbb{R}$ and $\mathbf{p}.\mathbf{n} = d$, then $(\mathbf{a} + \lambda\mathbf{b}).\mathbf{n} = d$.

  – Solve for λ and use this value to find the position vector of the point of intersection is given by $\mathbf{p} = \mathbf{a} + \lambda\mathbf{b}$



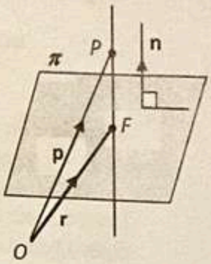
- **To find the position vector of the foot of the perpendicular from the point P to the plane $\pi: \mathbf{r}.\mathbf{n}=d$**

Let F be the foot of the perpendicular from the point P to the plane $\pi : \mathbf{r}.\mathbf{n}=d$.

Method

Step 1: Find vector equation of the line that is perpendicular to the plane (hence direction vector of this line is $\mathbf{n}$) and passing through $P \Rightarrow \mathbf{r}=\mathbf{p}+\lambda\mathbf{n},\ \lambda\in\mathbb{R}$

Step 2: The point of intersection between line, $\mathbf{r}=\mathbf{p}+\lambda\mathbf{n},\ \lambda\in\mathbb{R}$ and the plane $\pi: \mathbf{r}.\mathbf{n}=d$ is the foot the perpendicular from the point P to the plane.
Consider $(\mathbf{p}+\lambda\mathbf{n}).\mathbf{n}=d$ and solve for λ to use this value to find the position vector of the point given by $\mathbf{p}+\lambda\mathbf{n}$.



(Same as above for the point of intersection between a line and a plane)

- To find the distance between point P with position vector $\mathbf{p}$ and the plane $\pi : \mathbf{r}.\mathbf{n}=d$

Let F be the foot of the perpendicular from the point P to the plane $\pi : \mathbf{r}.\mathbf{n}=d$.

Method 1

Step 1: Find the position vector of the foot of the perpendicular from the point P to the plane $\pi : \mathbf{r}.\mathbf{n}=d$.

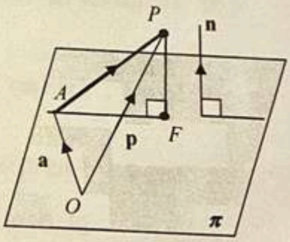
Step 2: The distance between the point and the plane is given by $|\overrightarrow{PF}|$.
(Refer to diagram above)

Method 2

Step 1: Find any point, says A on the plane with position vector $\mathbf{a}$.

Step 2: Find $\overrightarrow{AP}=\mathbf{p}-\mathbf{a}$

The shortest distance between the point P and the plane is given by $\dfrac{|(\mathbf{p}-\mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}$, that is length of projection of AP on the normal vector of the plane.



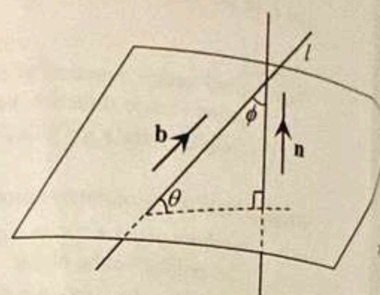
Method 3

The distance between the point and the plane is given by $\dfrac{|\mathbf{p}.\mathbf{n}-d|}{|\mathbf{n}|}$.

Note from method 2: $\dfrac{|(\mathbf{p}-\mathbf{a}).\mathbf{n}|}{|\mathbf{n}|}=\dfrac{|\mathbf{p}.\mathbf{n}-\mathbf{a}.\mathbf{n}|}{|\mathbf{n}|}=\dfrac{|\mathbf{p}.\mathbf{n}-d|}{|\mathbf{n}|}$.

❖ To find the acute angle between the line $l: \mathbf{r} = \mathbf{a} + \lambda \mathbf{b},\ \lambda \in \mathbb{R}$ and the plane $\pi : \mathbf{r} . \mathbf{n} = d$

Let the acute angle between **n** and **b** be ϕ.



__Method 1__

Step 1: Find the acute angle between **n** and **b** using $\cos\phi = \dfrac{|\mathbf{n}.\mathbf{d}|}{|\mathbf{n}||\mathbf{d}|}$

Step 2: The acute angle between the line and the plane is $90° - \phi$.

Let the acute angle between the line and the plane be θ. Then $\phi = 90° - \theta$.

__Method 2__

$- \sin\theta = \dfrac{|\mathbf{n}.\mathbf{d}|}{|\mathbf{n}||\mathbf{d}|}$

(since $\cos\phi = \cos(90° - \theta) = \sin\theta$)

- ***Involving Planes***

❖ To check if the two planes $\pi_1 : \mathbf{r} . \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} . \mathbf{n}_2 = d_2$ are parallel

__Method__

– If $\mathbf{n}_1 \,//\, \mathbf{n}_2$ (i.e. $\mathbf{n}_1 = k\ \mathbf{n}_2$), then $\pi_1 \,//\, \pi_2$

❖ To find the distance between two parallel planes $\pi_1 : \mathbf{r} . \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} . \mathbf{n}_2 = d_2$

__Method__

Step 1 : Find any point, says A with position vector, **a** on the plane π_1.

Step 2 : Find the shortest distance between A and π_2 which is $\dfrac{|\mathbf{a}.\mathbf{n} - d|}{|\mathbf{n}|}$

(Refer to Distance between point and the plane $\pi : \mathbf{r} . \mathbf{n} = d$)

❖ To find the acute angle between two planes $\pi_1 : \mathbf{r} . \mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r} . \mathbf{n}_2 = d_2$

Let the acute angle between the two planes be θ.

__Method__

– To find the acute angle between the two planes , we make use of $\cos\theta = \dfrac{|\mathbf{n}_1 . \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$

- To find vector equation of the line of intersection between two planes π_1: $\mathbf{r}.\mathbf{n}_1 = d_1$ and $\pi_2 : \mathbf{r}.\mathbf{n}_2 = d_2$

Method 1 (using GC)

Step 1: Find the cartesian equations for the planes

Step 2: Using GC to find the line of intersection

Method 2 (if GC is not allowed)

Step 1: Find the cartesian equations for the planes

Step 2: Let one of the variables says $x = 0$. Then solve for y and z using the cartesian equations from **Step 1** to get a common point, A with position vector, $\mathbf{a}$.

Step 3: Compute $\mathbf{b} = \mathbf{n}_1 \times \mathbf{n}_2$ which is the direction vector of the line of intersection. Then the equation of the line of intersection is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$

Method 3 (if GC is not allowed)

Step 1: Find the cartesian equations for the planes

Step 2: Find two common points using the cartesian equations from **Step 1** by first letting $x = 0$ and solve for y and z and then letting $y = 0$ and solve for x and z to get another common point

Step 3: Use the two points to find the equation of the line of intersection .

Example to illustrate Method 2 and 3:

Find the equation of the line of intersection of the two planes whose equations are $\mathbf{r}.\begin{pmatrix}1\\1\\-3\end{pmatrix} = 6$ and $\mathbf{r}.\begin{pmatrix}1\\0\\2\end{pmatrix} = 1$ respectively

Method 2

$$\mathbf{r}.\begin{pmatrix}1\\1\\-3\end{pmatrix} = 6 \Rightarrow x + y - 3z = 6 \text{ ------- (1)}$$

and $$\mathbf{r}.\begin{pmatrix}1\\0\\2\end{pmatrix} = 1 \Rightarrow x + 2z = 1 \text{ -------(2)}$$

Let $z = 0$ and solve, we have $x = 1, y = 5$ and $z = 0$

$$\begin{pmatrix}1\\1\\-3\end{pmatrix} \times \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\-5\\-1\end{pmatrix}$$

$\Rightarrow$ the required vector equation is $\mathbf{r}=\begin{pmatrix}1\\5\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-5\\-1\end{pmatrix},\ \lambda\in\mathbb{R}$

Method 3

$\mathbf{r}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=6\Rightarrow x+y-3z=6$ ------- (1) and $\quad\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=1\Rightarrow x+2z=1$ -------(2)

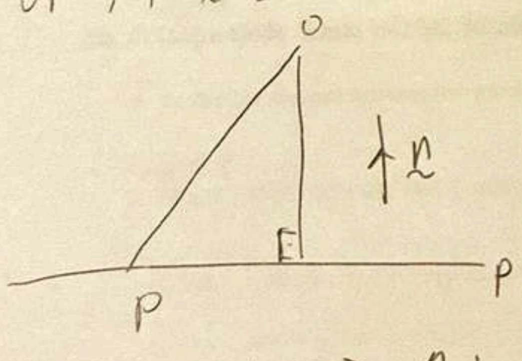
Let $z=0$ and solve, we have $x=1, y=5$ and $z=0$

Let $x=0$ and solve, we have $x=0, y=\dfrac{15}{2}$ and $z=\dfrac{1}{2}$

$$\begin{pmatrix}1\\5\\0\end{pmatrix}-\begin{pmatrix}0\\\frac{15}{2}\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}1\\-\frac{5}{2}\\-\frac{1}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2\\-5\\-1\end{pmatrix}$$

$\Rightarrow$ the required vector equation is $\mathbf{r}=\begin{pmatrix}1\\5\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-5\\-1\end{pmatrix},\ \lambda\in\mathbb{R}$

$\overrightarrow{OP}$, $p: \underset{\sim}{r}\cdot\underset{\sim}{n}=d$



$|\overrightarrow{OP}\cdot\hat{n}|$

Scalar

$$\frac{|\overrightarrow{OP}\cdot\underset{\sim}{n}|}{|\underset{\sim}{n}|}=\frac{|d|}{|\underset{\sim}{n}|}$$

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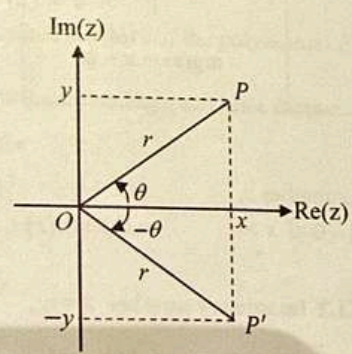
H2 Mathematics 9758
2019 Year 6 Term 3 Revision 5 (Summary and Tutorial)

Topic(s): Complex Numbers

Summary for Complex Numbers

Definition of Imaginary Numbers i $\boxed{\mathrm{i}=\sqrt{-1}}$

Note: $\mathrm{i}^2=-1$ $\mathrm{i}^3=(\mathrm{i}^2)(\mathrm{i})=-\mathrm{i}$ $\mathrm{i}^4=(\mathrm{i}^2)^2=1$

Representation of a complex number		**Conjugate**
Cartesian Form	$z=x+\mathrm{i}y$	$z^*=x-\mathrm{i}y$
Modulus–Argument / Trigonometric / Polar Form / Exponential Form	$z=r\mathrm{e}^{\mathrm{i}\theta}$ $=r(\cos\theta+\mathrm{i}\sin\theta)$	$z^*=r\mathrm{e}^{-\mathrm{i}\theta}$ $=r\left[\cos(-\theta)+\mathrm{i}\sin(-\theta)\right]$ $=r(\cos\theta-\mathrm{i}\sin\theta)$
Argand diagram	$z=x+\mathrm{i}y$ can be represented as • point P with coordinates (x, y) • $\lvert z\rvert=r=\sqrt{x^2+y^2}$ • $\arg(z)=\theta$, where $-\pi<\theta\le\pi$	z^* can be represented as • point P' with coordinates $(x,-y)$, which is the **reflection of P in the real axis** • $\lvert z^*\rvert=r=\sqrt{x^2+y^2}$ • $\arg(z^*)=-\theta$,
	Eg: for the case where $x>0$ and $y>0$:  	

Equality of Complex Numbers

2 complex numbers are equal if and only if

$$\boxed{z_1 = z_2 \iff x_1 = x_2 \text{ and } y_1 = y_2 \quad \text{(equate real and imaginary parts)}}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Useful properties involving Conjugates

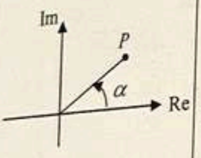
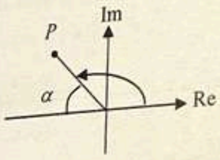
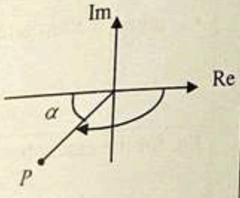
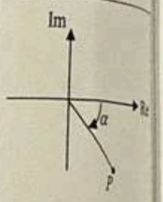
Consider a complex number $z = x + iy$, $x, y \in \mathbb{R}$, then

(i)	$z = z^* \iff z$ is real	(v)	$(z \pm w)^* = z^* \pm w^*$
(ii)	$z + z^* = 2x = 2\mathrm{Re}(z)$	(vi)	$(zw)^* = z^* w^*$
(iii)	$z - z^* = 2iy = 2\mathrm{Im}(z)\,i$	(vii)	$(z^n)^* = (z^*)^n$, where $n \in \mathbb{Z}^+$
(iv)	$zz^* = \lvert z \rvert^2$	(viii)	$\left(\dfrac{z}{w}\right)^* = \dfrac{z^*}{w^*}$

Finding arg(z):

Consider a point P on an Argand diagram representing $z = x + iy$, $x, y \in \mathbb{R}$

Always first identify the quadrant that z lies, then let **basic angle**, $\alpha = \tan^{-1}\left|\dfrac{y}{x}\right|$.

1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
 arg(z) =	 $\arg(z) = \pi - \alpha$	 $\arg(z) = -(\pi - \alpha)$	 $\arg(z) = -\alpha$

Note:

- For **REAL** number x,

  (a) $\arg(x) = 0$ if $x > 0$ (b) $\arg(x) = \pi$ if $x < 0$

- For **PURELY imaginary number** $z = iy$,

  (a) $\arg(z) = \dfrac{\pi}{2}$ if $y > 0$ (b) $\arg(z) = -\dfrac{\pi}{2}$ if $y < 0$

Useful Results involving argument & modulus

Consider complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

	Argument of z	Modulus of z
1	$\arg(z_1 z_2) = \arg z_1 + \arg z_2$	$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$
2	$\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$	$\left\lvert\dfrac{z_1}{z_2}\right\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$
3	$\arg(z^n) = n \times \arg(z)$	$\lvert z^n\rvert = \lvert z\rvert^n$
4	$\arg(z^*) = -\arg(z)$	$\lvert z^*\rvert = \lvert z\rvert$
	Note: You may need to add/subtract 2π when necessary to find the argument within the principal range such that $-\pi < \theta \leq \pi$.	

Geometrical Effect of Multiplying a Complex Numbers z by i

- It is an anti-clockwise rotation of z through an angle of $\dfrac{\pi}{2}$ radians about origin O.

Complex Roots of Polynomials

Reminder: a polynomial of degree n has n roots (complex and/or real).
If a polynomial with **REAL coefficients** has non-real roots, they occur in conjugate pairs.

(Note: This result **does not apply** if at least one of the coefficients is a non-real number.)

Application:

$(-3-2\mathrm{i})$ is a **root** of the equation $P(z) = z^3 + az^2 + bz - 13 = 0$, where $a, b \in \mathbb{R}$

$\Rightarrow$ $(-3+2\mathrm{i})$ is also a **root** of the equation $P(z) = 0$.

Then $(z-(-3-2\mathrm{i}))$ and $(z-(-3+2\mathrm{i}))$ are both **factors** of the polynomial $P(z)$.

Using the above two **linear factors**, we can form the following **quadratic factor:**

$$\begin{aligned}
&\text{Quadratic factor}\\
&= [z-(-3-2\mathrm{i})][z-(-3+2\mathrm{i})]\\
&= (z+3+2\mathrm{i})(z+3-2\mathrm{i})\\
&= (z+3)^2 - (2\mathrm{i})^2\\
&= z^2 + 6z + 13
\end{aligned}$$

SOME COMMON TYPES OF PROBLEMS

1	Solving an equation with complex number as coefficient(s) or simultaneous equations involving complex numbers, by comparing real and imaginary parts respectively.
2	Solving an equation with real coefficient(s): one may start with z and its conjugate z^*, to find roots and/or to find unknown coefficients.
3	Finding modulus and argument of a complex number (could be in the form similar to z^n or $\frac{z^n}{w^*}$ etc.), using the properties or using the polar form.
4	Use of argument to determine conditions when a complex number (could be in the form similar to z^n or $\frac{z^n}{w^*}$ etc.) is (i) real (argument $= k\pi,\ k \in \mathbb{Z}$) (ii) real and positive (argument $= 2k\pi,\ k \in \mathbb{Z}$) (iii) real and negative (argument $= (2k+1)\pi,\ k \in \mathbb{Z}$) (iv) purely imaginary (argument $= \frac{2k+1}{2}\pi,\ k \in \mathbb{Z}$)

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2020 Year 6 Term 3 Revision 6 (Summary and Tutorial)

Topic(s): Functions

Summary for Functions

Relations and Functions

- The set of inputs of a function is known as the **domain** while the corresponding set of outputs is known as the **range**.
- To define a function completely, both the rule and domain must be specified.
- The range of a function can be obtaining by looking at its graph (y-values).
- If a function is such that no two elements in the given domain have the same output, it is a **one-one function**.
- To check if a function is one-one, we use horizontal lines to check.

  If function is one-one, such as $\mathrm{f}: x \mapsto x^2, x > 2$, we write since every horizontal line $y = k$, $k \in [4, \infty)$, cuts the graph of f at one and only one point, f is a one-one function. Note that $[4, \infty)$ is the range of f.

  If function is NOT one-one, such as $\mathrm{f}: x \mapsto x^2, x \in \mathbb{R}$, we choose a specific line $y = k$ where k is a value in range of f. In this case, we can write since the horizontal line $y = 1$ cuts the graph of f at more than one point, f is not a one-one function

Inverse Functions

- Given a function f, the **inverse function** f^{-1} exists if and only if f is one-one.
- To define f^{-1}, you must find the rule and domain.
- To find the rule of f^{-1}, let $y = \mathrm{f}(x)$ and **make x the subject,** i.e $x = \mathrm{f}^{-1}(y)$.

  Example to find the rule of inverse of $\mathrm{g}: x \mapsto 1 + \mathrm{e}^{-x}, \quad x \in \mathbb{R}$

  Let $y = 1 + \mathrm{e}^{-x}$ and make x the subject to get $x = -\ln(y-1)$.
- To find domain of f^{-1}, use the result: $\quad D_{f^{-1}} = R_f$

  Note also that $\quad R_{f^{-1}} = D_f$.
- The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections of each other in the line $y = x$.

Composite Functions

- Given 2 functions f and g, the **composite function** gf exists if and only if $R_f \subseteq D_g$.
- To define gf, you must find the rule and domain.
- To find the rule of gf, find $f(x)$ first, then find $g(f(x))$.
- To find domain of gf, use the result: $D_{gf} = D_f$
- In general, $fg \neq gf$. i.e. composition of functions is not commutative.
- The composite function ff, if it exists, is written as f^2, i.e. $f^2(x) = ff(x) = f(f(x))$.
  - Useful results: (i) $ff^{-1}(x) = x$, where $x \in D_{f^{-1}}$.
    (ii) $f^{-1}f(x) = x$, where $x \in D_f$.
- To find the range of gf, use either

(1) $R_{gf} = R_g$ restricted to R_f

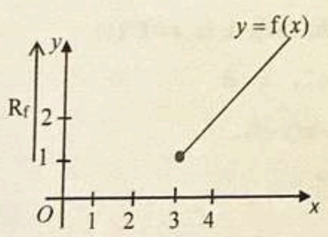
or (2) sketch the graph of $y = gf(x)$ over D_{gf} and find R_{gf}.

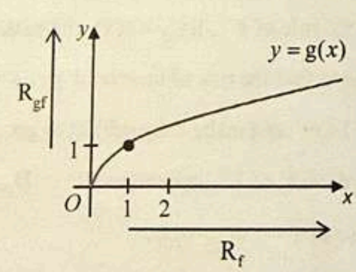
Example
Find the range of the composite gf given that the functions f and g are defined as follows:

$$f : x \mapsto x - 2, \quad x \geq 3,$$
$$g : x \mapsto \sqrt{x}, \quad x \geq 0.$$

<u>Method 1</u>
Sketch the graphs of the functions f and g on separate diagrams.

For the composite function gf, f is first applied to each $x \in D_f$ to find the image $f(x)$, and then the rule for g is applied to $f(x)$ to obtain the image $g(f(x))$. i.e. $gf(x) = g(f(x))$.





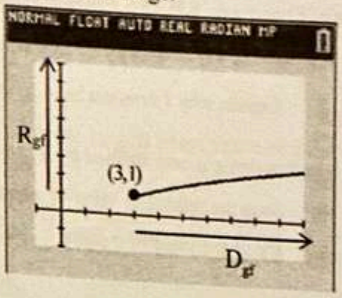
Consequently, we can obtain the range of gf as the range of g whose domain is restricted range of f, i.e $[3, \infty) \xrightarrow{f} [1, \infty) \xrightarrow{g} [1, \infty)$.
Thus $R_{gf} = [1, \infty)$.

Method 2

Find the function gf followed by sketching its graph to find the range.

$gf(x) = g[f(x)] = \sqrt{x-2}$, $D_{gf} = D_f = [3, \infty)$

From sketch, $R_{gf} = [1, \infty)$.



RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 7 (Summary and Tutorial)

Topic(s): Arithmetic and Geometric Progressions, Summations, Method of Difference

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Summary for Arithmetic and Geometric Progressions

Arithmetic and Geometric Progressions

Let a be the first term of the sequence, and d and r be the common difference and common ratio respectively.

	AP (*dna*)	GP (*ran*)
n^{th} term, u_n	$u_n = a+(n-1)d$	$u_n = ar^{n-1}$
	$u_n = S_n - S_{n-1}$, $n \geq 2$ $u_1 = S_1$	
Sum of first n terms, S_n	$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[u_1 + u_n]$	$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$, $r \neq 1$
To show sequence is a AP/GP	$u_{n+1} - u_n =$ constant $(=d)$	$\frac{u_{n+1}}{u_n}$ =constant $(=r)$
Sum to infinity, S_∞	-	$S_\infty = \frac{a}{1-r}$, $\lvert r \rvert < 1$

Summary for Summation

The Σ (Sigma) Notation

The series $u_1 + u_2 + \cdots + u_n$ can be denoted by $\sum_{r=1}^{n} u_r$, i.e. $\sum_{r=1}^{n} u_r = u_1 + u_2 + \cdots + u_n$.

In general, $\sum_{r=k}^{m} u_r$ ← m: **Ending** value of r; u_r: **General Term**; $r=k$: **Starting** value of r

$= u_k + u_{k+1} + \cdots + u_{m-1} + u_m$, where $k, m \in \mathbb{Z}_0^+$ and $k \le m$.

e.g. $\sum_{r=3}^{n+1}(3r+1) = [3(3)+1] + [3(4)+1] + \ldots + [3(n+1)+1] = 10 + 13 + \ldots + (3n+4)$

e.g. $3^3 + 7^3 + 11^3 + \ldots + 187^3 = \sum_{r=0}^{46}(3+4r)^3$ or $\sum_{r=1}^{47}(3+4(r-1))^3$

Properties of the Σ (Sigma) Notation

If u_r and v_r are expressions in terms of r, a and b are constants and k and m are non-negative integers such that $k \le m$, then we have the following results:

(1) $\sum_{r=k}^{m} u_r$ has $(m-k+1)$ terms.	(2) For $2 \le k \le m$, $\sum_{r=k}^{m} u_r = \left(\sum_{r=1}^{m} u_r\right) - \left(\sum_{r=1}^{k-1} u_r\right)$.
(3) $\sum_{r=k}^{m}(au_r) = a\sum_{r=k}^{m} u_r$.	(4) $\sum_{r=k}^{m}(au_r \pm bv_r) = a\left(\sum_{r=k}^{m} u_r\right) \pm b\left(\sum_{r=k}^{m} v_r\right)$.

Note that $\sum_{r=k}^{m}(u_r v_r) \neq \left(\sum_{r=k}^{m} u_r\right)\left(\sum_{r=k}^{m} v_r\right)$.

e.g.

(1): $\sum_{r=3}^{n+1}(3r+1)$ has $(n+1)-3+1 = n-1$ terms, if each $(3r+1)$ is considered as a term.

(2): $\sum_{r=3}^{n+1}(3r+1) = \sum_{r=1}^{n+1}(3r+1) - \sum_{r=1}^{2}(3r+1)$.

(3) and (4): $\sum_{r=3}^{n+1}(3r+1) = 3\sum_{r=3}^{n+1} r + \sum_{r=3}^{n+1} 1$.

Special Sums

Let a, d and r be constants and k and n be positive integers such that $k \le n$, then we have the following results:

(1) $\sum_{k=1}^{n}[a+(k-1)d]=\frac{n}{2}[\text{1st term + last term}]=\frac{n}{2}[2a+(n-1)d]$.

LHS is the sum of 1[st] n terms of an **arithmetic** series with a as its first term and d as its common difference.

(2) $\sum_{k=1}^{n}ar^{k-1}=\frac{a(r^n-1)}{r-1}=\frac{a(1-r^n)}{1-r}, r\neq 1$.

LHS is the sum of 1[st] n terms of a **geometric** series with a as its first term and r as its common ratio.

e.g.

(1): $\sum_{r=1}^{n+1}(3r+1)$ is an arithmetic series with 1st term $3(1)+1=4$ and common difference 3. So the sum is

$$\frac{(n+1)}{2}\{4+[3(n+1)+1]\} \text{ or } \frac{(n+1)}{2}\{2(4)+[(n+1)-1]3\}=\frac{(n+1)}{2}(3n+8).$$

(2): $\sum_{r=1}^{n+1}2(3^r)$ is a geometric series with 1st term $2(3)=6$ and common ratio 3.

So the sum is $\frac{6(3^{n+1}-1)}{3-1}=3(3^{n+1}-1)$.

Summation using Standard Results

Let a be a constant and n be a positive integer.

(1) $\sum_{r=1}^{n} a = na$.	**(2)** $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.
(3) $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.	**(4)** $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left(\frac{1}{2}n(n+1)\right)^2 = \left(\sum_{r=1}^{n} r\right)^2$.

Remarks:

a. The result in **(1)** can be generalized:

$\sum_{r=k}^{m} a = (\textit{number of terms}) \times a = (m-k+1)a$, where k and m are integers such that $k \le m$.

b. The result in **(2)** can be derived using the sum of the first n terms of an AP.

c. The results in **(3)** and **(4)** are not found in the List of Formulae MF26 but these two results will be given when required.

d. The results in **(2)**, **(3)** and **(4)** hold if and only if r starts from 0 or 1.

Summary for Method of Difference (MOD)

If the rth term, u_r, can be expressed as the "difference" $\mathrm{f}(r) - \mathrm{f}(r-1)$, where $\mathrm{f}(r)$ is an expression in terms of r, then

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} (\mathrm{f}(r) - \mathrm{f}(r-1)) = \begin{cases} (\cancel{\mathrm{f}(1)} - \mathrm{f}(0)) \\ +(\cancel{\mathrm{f}(2)} - \cancel{\mathrm{f}(1)}) \\ +(\cancel{\mathrm{f}(3)} - \cancel{\mathrm{f}(2)}) \\ \quad\vdots \\ +(\cancel{\mathrm{f}(n-1)} - \cancel{\mathrm{f}(n-2)}) \\ +(\mathrm{f}(n) - \cancel{\mathrm{f}(n-1)}) \end{cases}$$

$$= \mathrm{f}(n) - \mathrm{f}(0).$$

1st 2 full cancellations

Last full cancellation

Remark:

Sometimes, the "difference" comes in a "different" form, but can be solved using MOD once you list down the terms as above. Examples: $\mathrm{f}(r) - \mathrm{f}(r-2)$, $\mathrm{f}(r+1) - \mathrm{f}(r)$, etc.

Differentiation & Integration

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 8 (Summary and Tutorial)

Topic(s): Differentiation Techniques

Summary for Differentiation Techniques

1. DERIVATIVE AS A LIMIT

If $y=\mathrm{f}(x)$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\lim\limits_{\delta x\to 0}\dfrac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}$.

Eg: To prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)=2x$ from 1st principles,

Let $y=\mathrm{f}(x)=x^2$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\lim_{\delta x\to 0}\frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}\\&=\lim_{\delta x\to 0}\frac{(x+\delta x)^2-x^2}{\delta x}\\&=\lim_{\delta x\to 0}\frac{x^2+2x\delta x+(\delta x)^2-x^2}{\delta x}\\&=\lim_{\delta x\to 0}(2x+\delta x)\\&=2x\end{aligned}$$

2. BASIC RULES OF DIFFERENTIATION

Type of Rule	How it works	Examples
Distributive Rule	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[a\mathrm{f}(x)\pm b\mathrm{g}(x)\right]=a\mathrm{f}'(x)\pm b\mathrm{g}'(x)$, where a and b are constants.	
Product Rule	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\mathrm{g}(x)\right]=\mathrm{g}(x)\mathrm{f}'(x)+\mathrm{f}(x)\mathrm{g}'(x)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left((2x-3)\sqrt{x-5}\right)$ $=2\sqrt{x-5}+(2x-3)\dfrac{1}{2\sqrt{x-5}}$ $=\cdots=\dfrac{6x-23}{2\sqrt{x-5}}$
Quotient Rule	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}\right]=\dfrac{\mathrm{g}(x)\mathrm{f}'(x)-\mathrm{f}(x)\mathrm{g}'(x)}{\left(\mathrm{g}(x)\right)^2}$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x+1}{x^2-3}\right)$

		$=\frac{(x^2-3)(1)-(x+1)(2x)}{(x^2-3)^2}$ $=\cdots=-\frac{x^2+2x+3}{(x^2-3)^2}$
Chain Rule	1. $\frac{d}{dx}fg(x)=\frac{d}{dx}[f(g(x))]=[f'(g(x))]g'(x)$ or 2. $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ where $y=f(u)$ and $u=g(x)$	$\frac{d}{dx}(\sin\sqrt{x})$ $=\cos\sqrt{x}\times\frac{1}{2\sqrt{x}}$ $=\frac{1}{2\sqrt{x}}\cos\sqrt{x}.$
Relationship between $\frac{dx}{dy}$ **and** $\frac{dy}{dx}$	$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$	

Properties of logarithm (useful for simplifying expressions)

	Examples
a) $\ln(x^y)=y\ln x,\ x>0$ b) $x^y=e^{\ln(x^y)},\ x^y>0$	$\frac{d}{dx}(2^{x+3})=\frac{d}{dx}(e^{\ln 2^{x+3}})$ $=\frac{d}{dx}(e^{(x+3)\ln 2})$ $=e^{(x+3)\ln 2}\ln 2=2^{x+3}\ln 2$
c) $\log_x y=\frac{\log_z y}{\log_z x},\quad x,y,z>0,\ x,z\neq 1$	$\frac{d}{dx}(\log_5(x^2+1))$ $=\frac{d}{dx}\left(\frac{\ln(x^2+1)}{\ln 5}\right)$ $=\frac{1}{\ln 5}\times\frac{1}{x^2+1}\times 2x=\frac{2x}{(x^2+1)\ln 5}$

3. DERIVATIVES OF STANDARD FUNCTIONS

y	$\frac{dy}{dx}$
x^n for fixed n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n[f(x)]^{n-1}f'(x)$
$e^{f(x)}$	$e^{f(x)}f'(x)$
$\ln[f(x)],\ f(x)>0$	$\frac{1}{f(x)}f'(x)$

Trigonometric Functions (Highlighted formulae are in MF26)

The following results hold for x measured in radians only.

If x is measured in degrees, then convert $x°$ to $\frac{\pi x}{180}$ radians.

y	$\frac{dy}{dx}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

y	$\frac{dy}{dx}$
$\sin[f(x)]$	$[\cos(f(x))]\, f'(x)$
$\cos[f(x)]$	$[-\sin(f(x))]\, f'(x)$
$\tan[f(x)]$	$[\sec^2(f(x))]\, f'(x)$
$\cot[f(x)]$	$[-\operatorname{cosec}^2(f(x))]\, f'(x)$
$\sec[f(x)]$	$[\sec(f(x))\tan(f(x))]\, f'(x)$
$\operatorname{cosec}[f(x)]$	$[-\operatorname{cosec}(f(x))\cot(f(x))]\, f'(x)$

Inverse Trigonometric Functions

The following results hold for x measured in radians only.

If x is measured in degrees, then convert $x°$ to $\frac{\pi x}{180}$ radians.

y	$\frac{dy}{dx}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert<1$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}, \quad \lvert x\rvert<1$
$\tan^{-1} x$	$\frac{1}{1+x^2}, \quad x\in\mathbb{R}$

y	$\frac{dy}{dx}$
$\sin^{-1}[f(x)]$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\cos^{-1}[f(x)]$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \quad \lvert f(x)\rvert<1$
$\tan^{-1}[f(x)]$	$\frac{f'(x)}{1+[f(x)]^2}, \quad f(x)\in\mathbb{R}$

4. HIGHER ORDER DERIVATIVES

In general, $\frac{d^n y}{dx^n} = f^{(n)}(x) = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right)$ for $n = 2, 3, 4, \ldots$.

Eg: If $y = x^n$, where n is a negative integer, then $\frac{dy}{dx} = nx^{n-1}$

$$\frac{d^2 y}{dx^2} = \frac{d}{d}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(nx^{n-1}\right) = n(n-1)x^{n-2}$$

$$\frac{d^3 y}{dx^3} = \frac{d}{d}\left(\frac{d^2 y}{dx^2}\right) = \ldots = n(n-1)(n-2)x^{n-3}$$

5. IMPLICIT DIFFERENTIATION

For a relation involving x and y, for example $y^3 - x + y = 0$, which cannot be expressed in the form $y = f(x)$, we can obtain $\frac{dy}{dx}$ by differentiating every term in the equation with respect to x. Very often, Chain Rule will be applied.

Eg, to differentiate $y^3 - x + y = 0$ implicitly w.r.t. x,

$$\frac{d}{dx}\left(y^3 - x + y\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(x) + \frac{d}{dx}(y) = 0$$

Applying Chain Rule, $\frac{d}{dx}\left(y^3\right) = \frac{d}{dy}\left(y^3\right) \times \frac{dy}{dx} = 3y^2\frac{dy}{dx}$

So we have $3y^2\frac{dy}{dx} - 1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{1+3y^2}$

6. PARAMETRIC DIFFERENTIATION

Suppose $x = f(t)$ and $y = g(t)$, where $t \in \mathbb{R}$.

By Chain rule : $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ Recall $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$.

$$\boxed{\therefore \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}}$$

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 9 (Summary and Tutorial)

Topic(s): Applications of Differentiation

Summary for Applications of Differentiation

1 Strictly Increasing/Strictly Decreasing & Concavity

IF	Then f is	
$f'(x)>0$	Strictly increasing	over the interval (a,b)
$f'(x)<0$	Strictly decreasing	
$f''(x)>0$	Concave up	
$f''(x)<0$	Concave down	

To show "<0" or ">0", we can complete the square of the expression if it is quadratic otherwise we can try to "build up the expression" from the given interval.

For example to show $f'(x)=2-\frac{1}{x}>0$ for $x>\frac{1}{2}$.

We can rewrite the expression: $f'(x)=\frac{2x-1}{x}$

If $x>\frac{1}{2}$, then $2x>1$ which implies that $2x-1>0$

If $x>\frac{1}{2}$, then the denominator must be ">0"

Therefore if the numerator is ">0" and the denominator is also ">0", we can conclude that for $x>\frac{1}{2}$, $f'(x)>0$

So the **strategy** here is to try to examine the signs the numerator and denominator from the given condition of $x>\frac{1}{2}$.

2 Methods for determining the nature of a stationary point

Method 1: Second Derivative Test

IF		Then $(k,\ f(k))$ is
$f''(x)>0$	at $x=k$	a minimum turning point
$f''(x)<0$		a maximum turning point

Note that if $f''(x)=0$ then there is no conclusion about the nature of the stationary point. The first derivative test must then be used.

Method 2: First Derivative Test

To determine the nature of the stationary point at $(k,\ \mathrm{f}(k))$, we need to check the signs of $\mathrm{f}'(x)$ for $x = k^-$ and $x = k^+$.

Important note: you need to fully factorise $\mathrm{f}'(x)$ where possible BEFORE discussing the signs of $\mathrm{f}'(x)$.

Example: $\mathrm{f}(x) = 2x^5 - 5x^3$. It is not sufficient to just differentiate and get $\mathrm{f}'(x) = 10x^4 - 15x^2$. You need to factorise further to get $\mathrm{f}'(x) = 10x^2\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right)$, BEFORE discussing stationary points at $(0,0)$, $\left(\sqrt{\frac{3}{2}}, -3\sqrt{\frac{3}{2}}\right)$ and $\left(-\sqrt{\frac{3}{2}}, 3\sqrt{\frac{3}{2}}\right)$ making reference to the table below.

x	k^-	k	k^+	k^-	k	k^+	k^-	k	k^+
$\mathrm{f}'(x)$	+ve	0	−ve	−ve	0	+ve	+ve or −ve	0 or 0	+ve or −ve
Nature of Stationary Point	Maximum			Minimum			Stationary point of inflexion		

3 TANGENTS AND NORMALS

To find the equations of the tangent or normal at a point $(k,\ \mathrm{f}(k))$ on a curve, we need:

Equation of a (straight) line $y - \mathrm{f}(k) = m\ (x - k)$ with gradient m passing through $(k,\ \mathrm{f}(k))$.

- Obtain m by finding $\mathrm{f}'(k)$ which will the **gradient of the tangent** at $(k,\ \mathrm{f}(k))$.
- If the question needs equation of the normal at $(k,\ \mathrm{f}(k))$, **gradient of the normal** at $(k,\ \mathrm{f}(k))$ is $-\dfrac{1}{\mathrm{f}'(k)}$.

When gradient at a point on a curve $\mathrm{f}(x)$ is **parallel** to the		
x-axis	$\mathrm{f}'(x) = 0$	(Usually mean that the **numerator** of $\mathrm{f}'(x) = 0$)
y-axis	$\mathrm{f}'(x)$ is undefined.	(Usually mean that the **denominator** of $\mathrm{f}'(x) = 0$)

4 MAXIMIZATION AND MINIMIZATION

Some guidelines to solve problems involving maximization and minimization :

- Denote each changing quantity by a variable.
- Write down a formula for the quantity to be maximized or minimized.
- Express the quantity to be maximized or minimized in terms of 1 variable only.
- Differentiate and equate derivative to zero for stationary values.
- Justify if quantity is a maximum or minimum (using 2^{nd} or 1^{st} derivative test).
- Answer the question.

Some strategies

1. Identify any right angle triangle by drawing a diagram. Use Pythagoras theorem to relate the two variables.
2. Identify similar triangles to relate the two variables.
3. Read questions carefully to identify the **constant(s)** used in the questions. That will reduce confusion on what variable to differentiate with respect to.

5 CONNECTED RATES OF CHANGE

Some guidelines to solve questions involving rate of change:

- Denote each changing quantity by a variable.
- Find the equations relating the variables.
- Use the **chain rule** to link up the derivatives.
- Write down the values of the variables and the given rates of change.
- Solve for the unknown rate.

Some strategies

1. Write down the rate of change that is given in the question. Use the units to guide you. For example $\mathrm{m^3s^{-1}}$ is the rate of change of volume per unit second.
2. Write down the rate of change required by the question and use the chain rule to link up the derivative.

   For example to find $\frac{\mathrm{d}V}{\mathrm{d}t}$ and you are given $\frac{\mathrm{d}V}{\mathrm{d}r}$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$. Then using chain rule, we can get $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$.

6 **MACLAURIN SERIES**

Reminder: the formulas are found in MF26.

If $\mathrm{f}(x)$ can be expanded as a power series for a given range of x including zero, then

$$\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \frac{\mathrm{f}''(0)}{2!}x^2 + \frac{\mathrm{f}^{(3)}(0)}{3!}x^3 + \ldots + \frac{\mathrm{f}^{(n)}(0)}{n!}x^n + \ldots \text{ [in MF26]}$$

Binomial Series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

The binomial expansion is valid for $|x|<1$, i.e. $-1<x<1$. [in MF26]

Note that we need to have "1" before applying the formula, so if we need to expand $(3+x)^{\frac{1}{2}}$, we can do the following depending on what is required.

- For **ascending** powers of x, we rewrite

$$(3+x)^{\frac{1}{2}} = 3^{\frac{1}{2}}\left(1+\frac{x}{3}\right)^{\frac{1}{2}}$$

before applying the binomial expansion.

In this case, the expansion is valid for $\left|\frac{x}{3}\right|<1$ (i.e. small x)

- For **descending** powers of x, we rewrite

$$(3+x)^{\frac{1}{2}} = (x+3)^{\frac{1}{2}} = x^{\frac{1}{2}}\left(1+\frac{3}{x}\right)^{\frac{1}{2}}$$

before applying the binomial expansion.

In this case, the expansion is valid for $\left|\frac{3}{x}\right|<1$ (i.e. large x)

Small angle approximations

For all small angles, positive or negative, we have

(1) $\sin x \approx x$

(2) $\cos x \approx 1-\frac{x^2}{2}$

(3) $\tan x \approx x$, where x is measured in **radians**.

Note that

- If angle x is small, addition or subtraction to angle x may not remain small, i.e $\sin(x \pm a) \not\approx x \pm a$. We need to use addition formula to simplify the expression before applying the small angle approximations.
  - For example: $\sin\left(x+\frac{\pi}{4}\right) = \sin x\cos\frac{\pi}{4} + \cos x\sin\frac{\pi}{4} \approx x\left(\frac{1}{\sqrt{2}}\right)+\left(1-\frac{x^2}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$
- If x is small, multiple of x is still small.
  - For example: $\cos(2x) \approx 1-\frac{(2x)^2}{2}$

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 10 (Summary and Tutorial)

Topic(s): Graphing Techniques and Transformations

Summary for Graphing Techniques

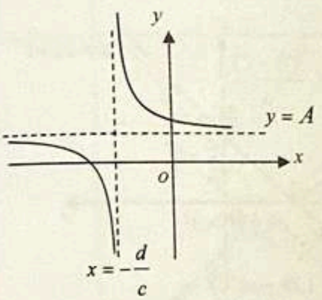
Recall using SIA as reference when sketching and labelling the curves you draw
S: Shape (including coordinates of stationary points if any)
I: Intercepts with the Axes (let $x = 0$ to find y, let $y = 0$ to find x)
A: Asymptotes (vertical, horizontal and oblique) – remember to label the equations of asymptotes.
Obviously, GC is there for us to use where applicable.

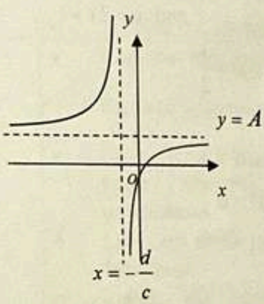
A. Rational functions

Type 1: $y = \dfrac{ax+b}{cx+d}$, $c \neq 0$, numerator and denominator have no common factor

$$y = \frac{ax+b}{cx+d} = A + \frac{B}{cx+d}$$

This is also known as rectangular hyperbola; the shape is either one of the following





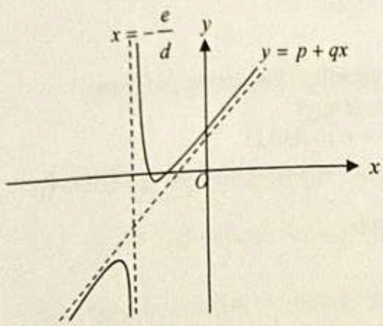
- Vertical asymptote : $x = -\dfrac{d}{c}$
- Horizontal asymptote : $y = A = \dfrac{a}{c}$

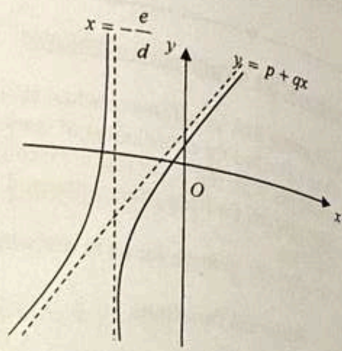
If $a = 0$, the horizontal asymptote is $y = 0$, i.e. x-axis

Useful to note: This graph has no turning points.

Type 2: $y=\dfrac{ax^2+bx+c}{dx+e}$, $a\neq 0,\ d\neq 0$, numerator and denominator have no common factor

Note that $y=\dfrac{ax^2+bx+c}{dx+e}$ can be re-written as $(p+qx)+\dfrac{r}{dx+e}$





- Vertical asymptote : $x=-\dfrac{e}{d}$
- Oblique asymptote : $y=px+q$

Example: $y=\dfrac{2x^2+5x+3}{x+2}\Rightarrow y=2x+1+\dfrac{1}{x+2}$

Asymptotes: $x=-2$ and $y=2x+1$

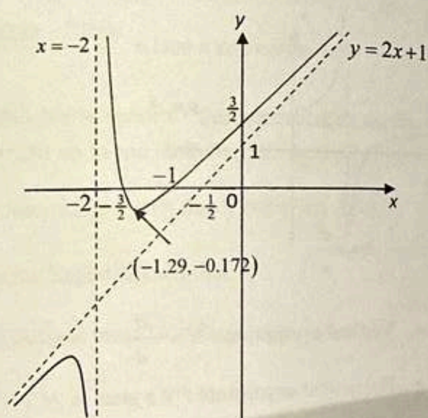
When $x=0$, $y=\dfrac{3}{2}$

When $y=0$,

$2x^2+5x+3=0$

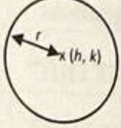
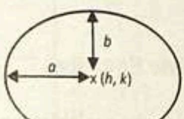
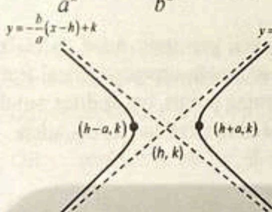
$(2x+3)(x+1)=0$

$x=-\dfrac{3}{2}$ or -1



B. Conics

Note that you should use the **same scale for both axes**. If unsure about the shape of the curve, refer to the CONICS app in your GC. You can also use the app to sketch and/or check your answers.

Shape	Equation(s) in Standard Form	Point(s) to note when sketching
Circle	$(x-h)^2+(y-k)^2=r^2$, where $r>0$. 	• Note that this is just a translation of the basic circle $x^2+y^2=r^2$. • Use a compass • Label the coordinates of the centre and radius clearly • Check if the origin is in the circle, on the circle, or outside the circle
Ellipse	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a>0$, $b>0$. 	• Note that this is just a translation of the basic ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. • Graph is symmetrical about the lines $x=h$ and $y=k$ • Label the centre and lengths of semi-major and semi-minor axes clearly • Check if the origin is in the ellipse, on the ellipse, or outside the ellipse • If $a=b$, the equation becomes the equation of a circle with centre (h,k) and radius a units. • A circle is a special type of ellipse.
Hyperbola	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, 	• Note that this is just a translation of the basic hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. • Curve should be drawn such that it tends **towards the oblique asymptotes** • Label the centre and vertices of the hyperbola • Label the equations of the asymptotes • The oblique asymptotes intersect at the centre of the hyperbola, (h,k). • <u>Finding equations of asymptotes</u> **Step 1:** Rewrite the equation in the form $\frac{(y-k)^2}{b^2}=\frac{(x-h)^2}{a^2}-1$. **Step 2:** Ignore the term '-1' since it is insignificant when $x\to\pm\infty$. So we have $\frac{(y-k)^2}{b^2}=\frac{(x-h)^2}{a^2}$.

Shape	Equation(s) in Standard Form	Point(s) to note when sketching
		Step 3: Take square root on both sides of equation and add "$\pm$" on RHS of equation. So we have $\frac{y-k}{b}=\pm\frac{x-h}{a}$. **Step 4:** Proceed to make y the subject of the formula.

C. Using GC to sketch graphs of Parametric Equations

Step 1: Ensure that the mode is set to '**PARAMETRIC**'.

Step 2: Press $\boxed{\text{y=}}$ and enter the equations.

Step 3: Press $\boxed{\text{graph}}$ to visualize the graph. The default setting is $T_{min}=0, T_{max}=2\pi$.

Step 4: If the range of T is not between 0 and 2π inclusive, press $\boxed{\text{window}}$ and **adjust the values for T_{min} and T_{max} according to the domain of the parameter** as stated in the question.

D. Finding the Cartesian equation of Parametric Equations

The idea is to obtain a single equation involving the variables x and y only. This can often be done by eliminating the parameter via substitution method. However for parametric equations involving trigonometrical functions, you should consider using the trigonometric identities such as $\sin^2 x+\cos^2 x=1$ and $\tan^2 x+1=\sec^2 x$ to formulate the Cartesian equation.

Important note on the use of GC

The GC gives the shape of the graph. In many cases, you **may need to resize the window to capture the entire graph**, or to change window settings to capture critical features of the graph. We can use it to find intersections with axes and turning points, but **it does not draw asymptotes**! To determine if a graph has any asymptotes, you need to examine its equation.

Summary for Transformations

Standard Transformations

The effects of standard transformations (translation, scaling and reflection) on the graph of $y = f(x)$ are summarised in the table below.

Note that a is taken to be a positive constant.

Replacement of x / y	Transformed Equation	Description of transformation (to be used when question requires description of transformations)
Replace x by $x-a$	$y = f(x-a)$	Translation of a units in the positive x-direction
Replace x by $x+a$	$y = f(x+a)$	Translation of a units in the negative x-direction
Replace y by $y-a$	$y-a = f(x)$ i.e. $y = f(x)+a$	Translation of a units in the positive y-direction
Replace y by $y+a$	$y+a = f(x)$ i.e. $y = f(x)-a$	Translation of a units in the negative y-direction
Replace x by $\frac{x}{a}$	$y = f\left(\frac{x}{a}\right)$	Scaling parallel to the x-axis by factor of a (in terms of coordinates, no change in corresponding y-coordinate)
Replace y by $\frac{y}{a}$	$\frac{y}{a} = f(x)$ i.e. $y = af(x)$	Scaling parallel to the y-axis by factor of a (in terms of coordinates, no change in corresponding x-coordinate)
Replace x by $-x$	$y = f(-x)$	Reflection in the y-axis
Replace y by $-y$	$-y = f(x)$ i.e. $y = -f(x)$	Reflection in the x-axis

Remarks:

For questions involving only standard transformations, there may be more than one sequence of transformations. For example, to transform $f(x)$ into $f(ax+b)$, you may

Either replace x by $x+b$, then replace x by ax

OR replace x by ax, then replace x by $x+\frac{b}{a}$.

However, for questions involving non-standard transformation(s) such as modulus, the sequence would be restricted. For example, to transform $f(x)$ into $f(-|x|)$, you can only

replace x by $-x$, then replace x by $|x|$.

Sketching the graph of $y=|\mathbf{f}(x)|$ and $y=\mathbf{f}(|x|)$ from the graph of $y=\mathbf{f}(x)$

$y=\lvert f(x)\rvert=\begin{cases} f(x) & \text{, if } f(x)\geq 0 \\ -f(x) & \text{, if } f(x)<0 \end{cases}$	$y=f(\lvert x\rvert)=\begin{cases} f(x), & x\geq 0, \\ f(-x), & x<0. \end{cases}$
The graph of $y=\lvert f(x)\rvert$ lies above (or on) the x-axis.	The graph of $y=f(\lvert x\rvert)$ is symmetrical about the y-axis.
To obtain this graph from the graph of $y=f(x)$:	
1. Retain the part of the graph of $y=f(x)$ where $f(x)\geq 0$.	**1. Retain** the part of the graph of $y=f(x)$ where $x\geq 0$
2. Reflect in the x-axis the part of the graph of $y=f(x)$ where $f(x)<0$.	**2. Reflect** the part drawn in **1.** in the y-axis so that the graph of $y=f(\lvert x\rvert)$ is symmetrical about the y-axis.

Sketching the graph of $y=\mathbf{f}'(x)$ from the graph of $y=\mathbf{f}(x)$

Note that we are sketching the **gradient** of y = f(x), that is, we need to study the **gradient** of y = f(x).
Regardless of whether f(x) < 0 (or > 0) in a particular interval, as long as the gradient is positive, $y=\mathbf{f}'(x)$ will still be above the x-axis for that interval.
Similar argument follows if gradient is negative in an interval.

The following shows the details for your reference if necessary.

Graph of $y=\mathbf{f}(x)$ (Note the description of the gradient)	**Graph of $y=\mathbf{f}'(x)$, that is graph of gradient of $y=\mathbf{f}(x)$**
Stationary point at $x=k$ (Gradient of stationary point is 0)	$x-$intercept at $x=k$
Strictly increasing over (a,b) (Gradient of graph is positive for (a,b))	Above $x-$axis over (a,b)
Strictly decreasing over (a,b) (Gradient of graph is negative for (a,b))	Below $x-$axis over (a,b)
Vertical asymptote $x=k$ (Gradient of vertical line is undefined)	Vertical asymptote $x=k$
Horizontal asymptote $y=k$ (Gradient of any horizontal line is 0)	Horizontal asymptote $y=0$
Oblique asymptote $y=kx+c$ (Gradient of $y=kx+c$ is k)	Horizontal asymptote $y=k$

Sketching the graph of $y=\frac{1}{\mathrm{f}(x)}$ from the graph of $y=\mathrm{f}(x)$

We will still use SIA to guide us in sketching $y=\frac{1}{\mathrm{f}(x)}$,

- identify any x-intercepts (vertical asymptotes) as it gives vertical asymptotes (x-intercepts).
- any turning point(s) for $\mathrm{f}(x)$, as max point would becomes min point and vice versa.

Note that if $\mathrm{f}(x)>0$ (or <0) over a certain interval, then $\frac{1}{\mathrm{f}(x)}>0$ (correspondingly <0) for that interval.

The following shows the details for your reference if necessary.

SIA	Features of $y=\mathrm{f}(x)$	$y=\mathrm{f}(x)$	$y=\frac{1}{\mathrm{f}(x)}$
Intercepts and Asymptotes	x-intercepts	$(a,0)$	Vertical asymptote $x=a$
	Asymptotes	Vertical: $x=a$	x-intercept
		Horizontal: $y=b,\ b\neq 0$	Horizontal: $y=\frac{1}{b}$
		Horizontal: $y=0$	$\frac{1}{\mathrm{f}(x)}\to\pm\infty$
		Oblique: $y=ax+b$	Horizontal: $y=0$
Shape	Turning points	Max pt (h,k)	Min pt $\left(h,\frac{1}{k}\right)$
		Min pt (h,k)	Max pt $\left(h,\frac{1}{k}\right)$
	y-values	(x,y)	$\left(x,\frac{1}{y}\right)$
		Above x-axis	Remains
		Below x-axis	Remains
		$\mathrm{f}(x)=\pm 1$	$\frac{1}{\mathrm{f}(x)}=\pm 1$
		$\mathrm{f}(x)$ increases	$\frac{1}{\mathrm{f}(x)}$ decreases
		$\mathrm{f}(x)$ decreases	$\frac{1}{\mathrm{f}(x)}$ increases
	Behaviour of graph	$\mathrm{f}(x)\to+\infty$	$\frac{1}{\mathrm{f}(x)}\to 0^+$
		$\mathrm{f}(x)\to-\infty$	$\frac{1}{\mathrm{f}(x)}\to 0^-$

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H2 Mathematics 9758
2020 Year 6 Term 3 Revision 11 (Summary and Tutorial)

Topic(s): Integration Techniques

Summary for Integration Techniques

General Tips

- For indefinite integrals, do not forget the $+C$
- Can always check your answer by differentiating / use GC for definite integrals
- Before using integration by parts when you see a product of two functions, ask yourself if the expression is of the form $\mathrm{f}'(x)(\mathrm{f}(x))^n$ or $\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$.

Overview of Integration Techniques

In general, you should know how to find the anti-derivatives of the following classes of functions:

- Logarithmic ($\ln x$)
- Inverse trigo ($\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$)
- Algebraic (polynomials, rational functions $\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$, where P and Q are polynomials)
- Trigonometric (6 basic trigonometric functions, and rational functions of them)
- Exponential (e^x)

This is also the 'order' for integration by parts (except for the rational functions); with the functions at the end of the list the functions you would usually choose to integrate first when doing integration by parts.

Before we talk about integration by parts, let us review the integration techniques involving the Exponential, Trigonometric and Algebraic functions first.

Exponential functions

Nothing much to say here, you should know how to find the anti-derivatives of such functions.

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$ is your staple here.

Trigonometric functions

You need to know the anti-derivatives of the 6 basic trigo functions (sin and cos obviously, the other 4 are in MF26). For higher powers of the trigo functions and product of several of them, we have to use trigonometric identities to help us.

In general, the idea is to **linearize products** (using the double angle formulae as well as the reverse factor formulae)

- $\sin^2 x, \cos^2 x$ can be reduced to cos 2x (can be done repeatedly)
- $\tan^2 x, \cot^2 x$ can be expressed in terms of $\sec^2 x, \operatorname{cosec}^2 x$ using the trigonometric identities $(1+\tan^2 x = \sec^2 x, 1+\cot^2 x = \operatorname{cosec}^2 x)$. Their respective antiderivatives are tan x and –cot x.

Basic trigonometric functions (highlighted in MF 26)

f(x)	$\int f(x)\,dx$	f(x)	$\int f(x)\,dx$
$\sin x$	$-\cos x$	$\sin^2 x = \dfrac{1-\cos 2x}{2}$	$\dfrac{1}{2}\left(x - \dfrac{\sin 2x}{2}\right)$
$\cos x$	$\sin x$	$\cos^2 x = \dfrac{\cos 2x + 1}{2}$	$\dfrac{1}{2}\left(\dfrac{\sin 2x}{2} + x\right)$
$\tan x$	$-\ln\lvert\cos x\rvert = \ln\lvert\sec x\rvert$	$\tan^2 x = \sec^2 x - 1$	$\tan x - x$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	$\sec^2 x$	$\tan x$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert$	$\operatorname{cosec}^2 x$	$-\cot x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\cot^2 x = \operatorname{cosec}^2 x - 1$	$-\cot x - x$

- Product of trigonometric functions involving sine and cosine

  Usually the idea is to split the product into a sum, and would thus involve the factor and/or double angle formulae.

  - Integrals like $\int \sin mx \sin nx\,dx$, $\int \cos mx \cos nx\,dx$, $\int \sin mx \cos nx\,dx$ and $\int \cos mx \sin nx\,dx$ involve the factor formula if $m \neq n$. If $m = n$, then it reduces to the double angle formula. To apply the factor formula, first look at MF26

$$\boxed{\begin{array}{ll} \sin P + \sin Q \equiv 2\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q), & \cos P + \cos Q \equiv 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q) \\ \sin P - \sin Q \equiv 2\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q), & \cos P - \cos Q \equiv -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q) \end{array}}$$

  Identify which of the 4 products you have (Note that $\int \sin mx \cos nx\,dx$ and $\int \cos mx \sin nx\,dx$ are the same, but we usually place the 'larger' angle before the 'smaller' angle before identifying the product).

Example

$$\int \sin 2x \cos 5x \, dx = \int \cos 5x \sin 2x \, dx \quad \text{('bigger' angle in front)}$$

$$= \frac{1}{2}\int \sin \underbrace{(5x+2x)}_{\text{sum}} - \sin \underbrace{(5x-2x)}_{\text{difference}} \, dx \quad \text{(identifying which of the 4 to use)}$$

You can always check by using the factor formula given in the MF26 that you have done the 'reverse' factor formula correctly.

- Be careful of certain products like $\int \sec x \tan x \, dx = \sec x + C$ and $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$.
- $f'(x)(f(x))^n$ or $\frac{f'(x)}{f(x)}$.

Some integrals like $\int \frac{\sin x}{2+\cos x} \, dx$ are of the form $\int \frac{f'(x)}{f(x)} \, dx$. Some like $\int \sin x \sec^5 x \, dx$ are of the form $\int f'(x)(f(x))^n \, dx$. Try to 'spot' these forms first before resorting to brute force (converting everything to sine and cosine and using double angle formula/factor formula)! In these questions, it may be helpful to write down the $f'(x)$ first (it is usually sine or cosine, and in general would be the derivative of a trigonometric function).

Algebraic functions

You should know, $\int x^n \, \mathrm{d}x = \begin{cases} \dfrac{1}{n+1}x^{n+1}+C, & \text{if } n \neq -1 \\ \ln|x|+C, & \text{if } n=-1 \end{cases}$.

This section is dedicated to find the anti-derivatives for functions of the form $\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$.

- Long division (if the fraction is improper, i.e. deg P $\geq$ deg Q)
- Factorize $\mathrm{Q}(x)$ into a product of linear and quadratic factors
- Partial fractions based on the linear/quadratic factors
- There will be 3 types of terms $\dfrac{1}{x-a}$, $\dfrac{1}{(x-a)^2}$ and $\dfrac{Ax+B}{x^2+a^2}$. The anti-derivatives are respectively of the form $\ln|x-a|$, $-\dfrac{1}{x-a}$ and $\dfrac{A}{2}\ln(x^2+a^2)+\dfrac{B}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$.

Example

$$\int \frac{1}{1+x^3}\,\mathrm{d}x = \int \frac{1}{(1+x)(x^2-x+1)}\,\mathrm{d}x \quad \text{(factorize denominator into linear and quadratic factors)}$$

$$= \int \frac{\frac{1}{3}}{(1+x)} + \frac{-\frac{x}{3}+\frac{2}{3}}{(x^2-x+1)}\,\mathrm{d}x \quad \text{(partial fractions)}$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{3}\int \frac{1}{2}\frac{(2x-1)-3}{x^2-x+1}\,\mathrm{d}x \quad \text{(split into the ln and } \tan^{-1} \text{ terms)}$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}x \quad \text{(complete the square for the } \tan^{-1} \text{ term)}$$

$$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\,\mathrm{d}x + C$$

There is another algebraic form $\int \dfrac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x$ which you will need to know how to evaluate, and it is also given in the MF 26, $\int \dfrac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \sin^{-1}\left(\dfrac{x}{a}\right)+C$.

Before we discuss how to find the antiderivatives of **L**ogarithmic and **I**nverse Trigonometric functions, we need to talk about Integration by Parts.

Integration by Parts

This is a useful tool when dealing with anti-derivatives of products of functions (as it is essentially the reverse of the product rule).

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \underbrace{v}_{\text{Integrate}}\ \underbrace{u}_{\text{Keep}} - \int \underbrace{v}_{\text{Integrate}}\ \underbrace{\frac{\mathrm{d}u}{\mathrm{d}x}}_{\text{Differentiate}}\,\mathrm{d}x$$

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \left[v(x)u(x)\right]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

When we have a combination of 2 or more (usually 2 for H2) different types of functions, the LIATE rule can help us determine which function we should integrate first. The following is non-exhaustive, but shows typical combinations asked at the A-levels:

Logarithmic + Algebraic, Inverse Trig + Algebraic

$\int P(x)\ln x\,\mathrm{d}x, \int P(x)\tan^{-1}x\,\mathrm{d}x$. Clearly the only choice here is to differentiate the logarithmic or inverse trigonometric function and integrate the polynomial.

Example:

$$\int \underbrace{x^2}_{\frac{dv}{dx}}\underbrace{\ln x}_{u}\,\mathrm{d}x = \underbrace{\frac{x^3}{3}}_{\text{Integrate}}\ \underbrace{\ln x}_{\text{Keep}} - \int \underbrace{\frac{x^3}{3}}_{\text{Integrate}}\ \underbrace{\left(\frac{1}{x}\right)}_{\text{Differentiate}}\,\mathrm{d}x$$

$$= \frac{x^3\ln x}{3} - \frac{x^3}{9} + C$$

$$\int \underbrace{x}_{\frac{dv}{dx}}\underbrace{\tan^{-1}x}_{u}\,\mathrm{d}x = \underbrace{\frac{x^2}{2}}_{\text{Integrate}}\ \underbrace{\tan^{-1}x}_{\text{Keep}} - \int \underbrace{\frac{x^2}{2}}_{\text{Integrate}}\ \underbrace{\left(\frac{1}{1+x^2}\right)}_{\text{Differentiate}}\,\mathrm{d}x$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,\mathrm{d}x \quad \text{(long division since improper rational function)}$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\left(x - \tan^{-1}x\right) + C$$

Algebraic + Trigonometric, Algebraic + Exponential

$\int P(x)\cos x\,\mathrm{d}x, \int P(x)e^x\,\mathrm{d}x$, where P is a polynomial. We repeatedly integrate the trigonometric/exponential function, and differentiate the polynomial until the degree of the polynomial reaches 0.

Example

$$\int \underbrace{x^2}_{u}\underbrace{\cos x}_{\frac{dv}{dx}}\,\mathrm{d}x = \underbrace{(\sin x)}_{\text{Integrate}}\underbrace{x^2}_{\text{Keep}} - \int \underbrace{(\sin x)}_{\text{Integrate}}\ \underbrace{(2x)}_{\text{Differentiate}}\ \mathrm{d}x$$
$$= x^2\sin x - 2\int x\sin x\,\mathrm{d}x$$
$$= x^2\sin x - 2\left(-x\cos x - \int(-\cos x)\,\mathrm{d}x\right)$$
$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

It is highly advisable that you use ( ) and also factorize/multiply coefficients (especially those negative signs) with great prudence.

Trigonometric + Exponential $\int e^{ax}\cos bx\,\mathrm{d}x, \int e^{ax}\sin bx\,\mathrm{d}x$

Integration by parts 2 times (you can choose to **either integrate the exponential or trigonometric function,** but it must be the same function that you integrate 2 times)

Example

$$\int e^{2x}\cos 3x\,\mathrm{d}x = \frac{1}{2}e^{2x}\cos 3x - \int \frac{1}{2}e^{2x}(-3\sin 3x)\,\mathrm{d}x$$
$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,\mathrm{d}x$$
$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \int\frac{1}{2}e^{2x}(3\cos 3x)\,\mathrm{d}x\right)$$
$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\,\mathrm{d}x$$

Now rearranging all the terms involving the integral to the LHS, we will have

$$\frac{13}{4}\int e^{2x}\cos 3x\,\mathrm{d}x = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C'$$
$$\Rightarrow \int e^{2x}\cos 3x\,\mathrm{d}x = \frac{2}{13}e^{2x}\cos 3x + \frac{3}{13}e^{2x}\sin 3x + C$$

Now let us discuss the techniques that allow us to find the antiderivatives of Logarithmic and Inverse Trigonometric functions. In essence, it is Integration by Parts, using a 'trick' that you need to be familiar with. When faced with finding the anti-derivative of a logarithmic or inverse trigonometric function by itself, we often write $\int f(x)\,\mathrm{d}x = \int 1\cdot f(x)\,\mathrm{d}x$ so as to allow us to integrate 1 and differentiate the logarithmic or inverse trigonometric function when using integration by parts.

Logarithmic functions

These are typically done by writing $\int f(x)\,dx = \int \underbrace{1}_{\frac{dv}{dx}} \cdot \underbrace{f(x)}_{u}\,dx = xf(x) - \int xf'(x)\,dx$.

Example: $\int \ln x\,dx = \int \underbrace{1}_{\frac{dv}{dx}} \cdot \underbrace{\ln x}_{u}\,dx = x\ln|x| - \int x\left(\frac{1}{x}\right)dx = x\ln|x| - x + C$.

Inverse trigonometric functions

Same as above (for the logarithmic functions).

Example: $\int \tan^{-1} x\,dx = \int \underbrace{1}_{\frac{dv}{dx}} \cdot \underbrace{\tan^{-1} x}_{u}\,dx = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$.

Integration by Substitution

When a question requires you to do a substitution, the substitution will be provided for you. You need to remember that there are generally 3 steps (for either indefinite/definite integrals).

Sub given $x = 2\tan\theta$	Indefinite integral $\int \frac{1}{\sqrt{x^2+4}}\,dx$		Definite integral $\int_0^2 \frac{1}{\sqrt{x^2+4}}\,dx$
	There are no upper/lower limits in an indefinite integral	Substitute limits	Upper limit: $x = 2$ So $2 = 2\tan\theta \Rightarrow \theta = \frac{\pi}{4}$ Lower limit: $x = 0$ So $0 = 2\tan\theta \Rightarrow \theta = 0$
Substitute integrand	$\frac{1}{\sqrt{x^2+4}} = \frac{1}{\sqrt{4\tan^2\theta+4}} = \frac{1}{2\sec\theta}$	Substitute integrand	$\frac{1}{\sqrt{x^2+4}} = \frac{1}{\sqrt{4\tan^2\theta+4}} = \frac{1}{2\sec\theta}$
Substitute dx (diff. the relation given)	$x = 2\tan\theta \Rightarrow \frac{dx}{d\theta} = 2\sec^2\theta$ Therefore, $dx = 2\sec^2\theta\,d\theta$	Substitute dx (diff. the relation given)	$x = 2\tan\theta \Rightarrow \frac{dx}{d\theta} = 2\sec^2\theta$ Therefore, $dx = 2\sec^2\theta\,d\theta$
	$\int \frac{1}{\sqrt{x^2+4}}\,dx$ $= \int \frac{1}{2\sec\theta} 2\sec^2\theta\,d\theta$ $= \int \sec\theta\,d\theta$ $= \ln\lvert\sec\theta + \tan\theta\rvert + C$		$\int_0^2 \frac{1}{\sqrt{x^2+4}}\,dx$ $= \int_0^{\frac{\pi}{4}} \frac{1}{2\sec\theta} 2\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta$ $= \left[\ln\lvert\sec\theta + \tan\theta\rvert\right]_0^{\frac{\pi}{4}} = \ln(1+\sqrt{2})$
Substitute back to original variable	$\ln\lvert\sec\theta + \tan\theta\rvert + C$ $= \ln\left\lvert\frac{\sqrt{x^2+4}+x}{2}\right\rvert + C$		No need since definite integral gives a real value (if it exists)

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2020 Year 6 Term 3 Revision 12 (Summary and Tutorial)

Topic(s): Applications of Integration

Summary for Applications of Integration

1 Evaluating Areas Bounded by Curves and Axes

1.1 Area bounded by the curve and the x-axis

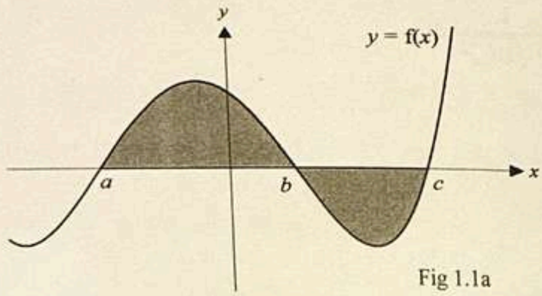


Fig 1.1a

In Fig. 1.1a,

(i) $\int_a^b \mathrm{f}(x)\,\mathrm{d}x > 0$. Thus, area under $y = \mathrm{f}(x)$ from $x = a$ to $x = b$ is given by $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$.

(ii) $\int_b^c \mathrm{f}(x)\,\mathrm{d}x < 0$. Thus, area under $y = \mathrm{f}(x)$ from $x = b$ to $x = c$ is given by $-\int_b^c \mathrm{f}(x)\,\mathrm{d}x$

or $\int_b^c \left|\mathrm{f}(x)\right|\,\mathrm{d}x$ (see Fig. 1.1b).

(iii) The area under $y = \mathrm{f}(x)$ from $x = a$ to $x = c$ is given by

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \left(-\int_b^c \mathrm{f}(x)\,\mathrm{d}x\right) \quad \text{or} \quad \int_a^c \left|\mathrm{f}(x)\right|\,\mathrm{d}x \text{ (see Fig. 1.1b).}$$

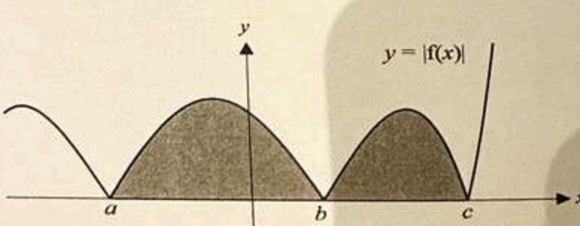


Fig 1.1b

1.2 Area bounded by the curve and the y-axis

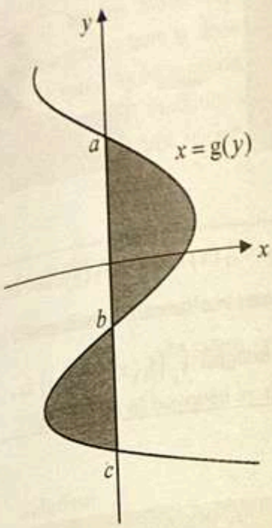


Fig 1.2

In Fig.1.2,

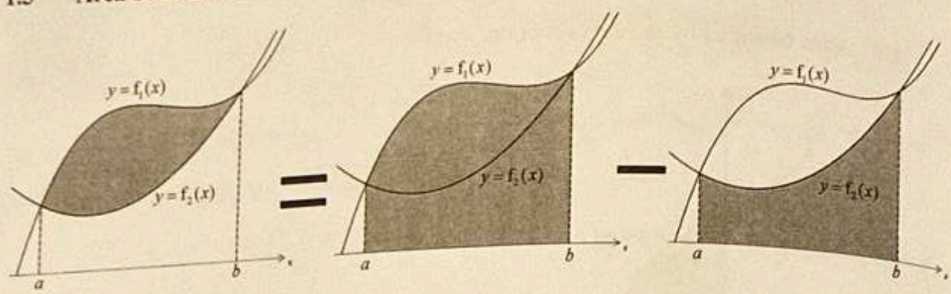
(i) $\int_b^a g(y)\,dy > 0$. Thus, area bounded by $x = g(y)$ and the y-axis from $y = b$ to $y = a$ is given by $\int_b^a g(y)\,dy$.

(ii) $\int_c^b g(y)\,dy < 0$. Thus, area bounded by $x = g(y)$ and the y-axis from $y = c$ to $y = b$ is given by $-\int_c^b g(y)\,dy$.

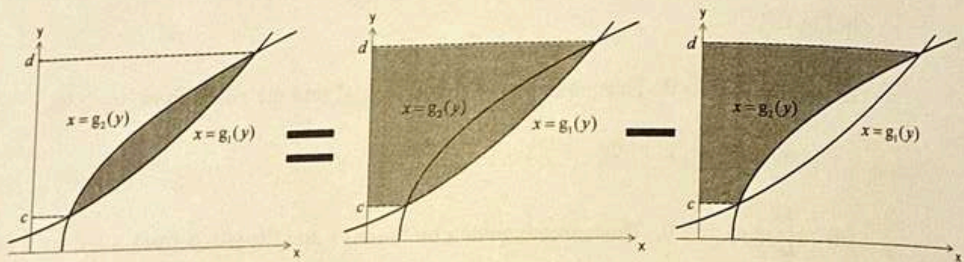
(iii) The shaded area bounded by the curve $x = g(y)$ and the y-axis from $y = c$ to $y = a$ is given by

$$\int_b^a g(y)\,dy + \left(-\int_c^b g(y)\,dy\right).$$

1.3 Area bounded between two curves



In general, given 2 curves defined by the equations $y = \mathrm{f}_1(x)$ and $y = \mathrm{f}_2(x)$, and an interval $[a, b]$ where $\mathrm{f}_1(x) \geq \mathrm{f}_2(x)$ within this interval, we can evaluate the area bounded between them from $x = a$ to $x = b$ by evaluating the definite integral $\int_a^b (\mathrm{f}_1(x) - \mathrm{f}_2(x))\, \mathrm{d}x$.



Similarly, given 2 curves defined by the equations $x = \mathrm{g}_1(y)$ and $x = \mathrm{g}_2(y)$, and an interval $[c, d]$ where $\mathrm{g}_1(y) \geq \mathrm{g}_2(y)$ within this interval, we can evaluate the area bounded between them from $y = c$ to $y = d$ by evaluating the definite integral $\int_c^d (\mathrm{g}_1(y) - \mathrm{g}_2(y))\, \mathrm{d}y$.

1.4 Evaluating areas involving curves which are defined parametrically

Essentially, we write the required area as $\int_a^b f(x)\,dx$ or $\int_c^d g(y)\,dy$ like how we did in the earlier sections. However, since both y and x are now expressed in terms of a parameter t, the integration here is done with respect to the parameter t instead of the variable x. To do so, we apply the integration by substitution technique to change the integral to one involving the parameter t. Students are encouraged to **sketch the curve with their GC** in order to identify the required region.

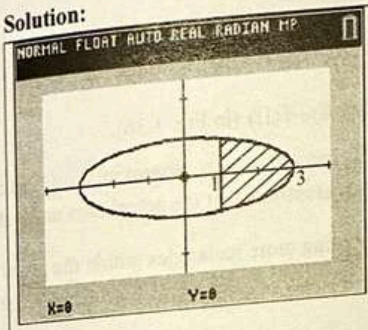
Example

The parametric equations of a curve C is given by

$$x = 3\sin t,\quad y = \cos t,\text{ where } 0 \le t \le 2\pi.$$

Find the area of bounded by the curve C, the lines $x = 1$ and $x = 3$.

Solution:



$$\begin{aligned}\text{Required area} &= 2\int_1^3 y\,dx \\ &= 2\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}} \left(\cos t\ \frac{dx}{dt}\right) dt \\ &= 2\int_{\sin^{-1}\frac{1}{3}}^{\frac{\pi}{2}} \left(\cos t\ (3\cos t)\right) dt\end{aligned}$$

and we proceed to solve this integral.

*note the substitution of $y = \cos t$, $dx = \dfrac{dx}{dt}dt$, and also the change of limits.

1.5 Area under a curve as the limiting sum of the areas of the rectangles

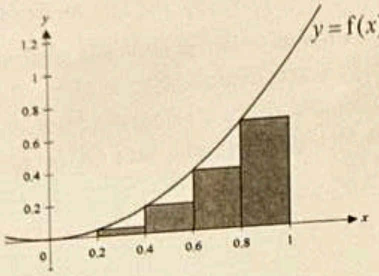


Fig 1.3a

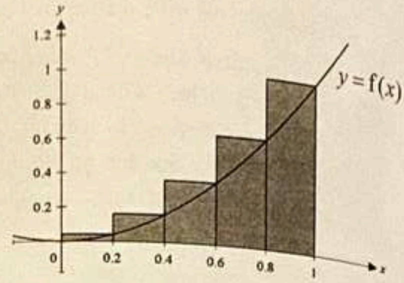


Fig 1.3b

Other than using integration, students are required to be able to estimate the area under a curve using the sum of the areas of the rectangles of equal width.

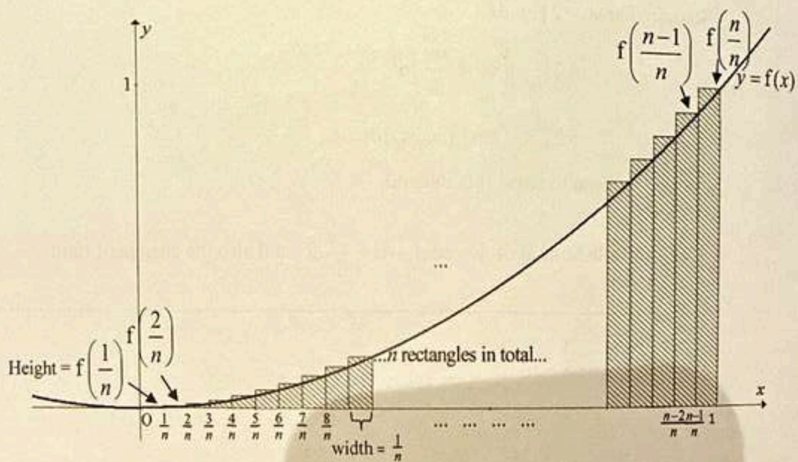
For example, to obtain an estimate for the area under the curve $y = f(x)$ between $x = 0$ and $x = 1$, we can consider finding the sum of the areas of the rectangles as shown in Fig. 1.3a or Fig. 1.3b.

$\int_0^1 f(x)\,dx \approx (0.2)(f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8))$ (in Fig. 1.3a) or

$\int_0^1 f(x)\,dx \approx (0.2)(f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1))$ (in Fig. 1.3b).

From the diagrams, we see that the former gives an approximation which is an **underestimate** while the latter provides an **overestimate** of the actual area under the curve.

To obtain a better estimate, we can consider using more rectangles within the given interval. Let us consider the case in Fig. 1.3b, but instead of using just 5 rectangles, let's use n rectangles.



Thus $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x \approx \frac{1}{n}\left(\mathrm{f}\left(\frac{1}{n}\right)+\mathrm{f}\left(\frac{2}{n}\right)+\mathrm{f}\left(\frac{3}{n}\right)+\ldots+\mathrm{f}\left(\frac{n}{n}\right)\right)$.

As we increase the number of rectangles n, we find that the sum of the areas of the n rectangles will approach the value of the integral $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$.

Thus, the **limiting sum of the areas of the *n* rectangles** can be used here to evaluate the area under the graph of $y=\mathrm{f}(x)$ from $x=0$ to $x=1$.

i.e. $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\mathrm{f}\left(\frac{r}{n}\right)$.

Note: This idea can also be similarly applied if we choose to consider the case in Fig. 1.3a.

2 Volume of Solid of Revolution

2.1 Rotation about the x-axis ($y = 0$)

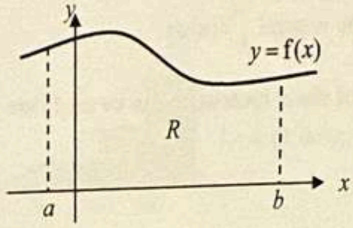


Fig. 2.1a

In general, if the region R bounded by the curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$, as shown in Fig.2.1a is rotated completely about the x-axis, the volume of solid of revolution formed is given by $\pi \int_a^b y^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx$.

If the region R is bounded by the curves $y = f_1(x)$, $y = f_2(x)$, the lines $x = a$ and $x = b$ as shown in Fig. 2.1b,

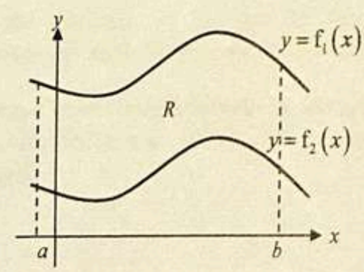


Fig. 2.1b

then the volume of solid of revolution formed when R is rotated completely about the x-axis is given by $\pi \int_a^b [f_1(x)]^2 \, dx - \pi \int_a^b [f_2(x)]^2 \, dx = \pi \int_a^b [f_1(x)]^2 - [f_2(x)]^2 \, dx$

2.2 Rotation about the y-axis ($x = 0$)

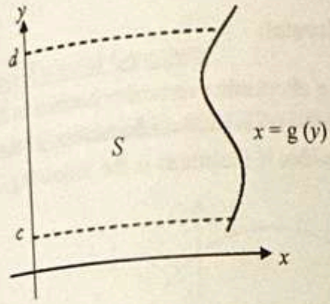


Fig. 2.2a

In general, if the region S bounded by the curve $x = g(y)$, the y-axis, the lines $y = c$ and $y = d$, as shown in Fig. 2.2a is rotated completely about the y-axis, the volume of solid of revolution formed is given by $\pi\int_c^d x^2 \, dy = \pi\int_c^d [g(y)]^2 \, dy$

If the region S is bounded by the curves $x = g_1(y)$, $x = g_2(y)$, the lines $y = c$ and $y = d$ as shown in Fig. 2.2b,

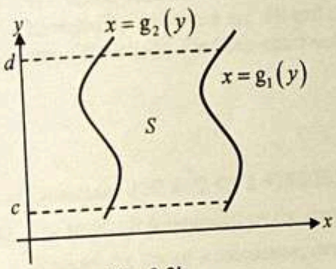


Fig. 2.2b

then the volume of solid of revolution formed when S is rotated completely about the y-axis is given by $\pi\int_c^d [g_1(y)]^2 \, dy - \pi\int_c^d [g_2(y)]^2 \, dy$

$$= \pi\int_c^d [g_1(y)]^2 - [g_2(y)]^2 \, dy$$

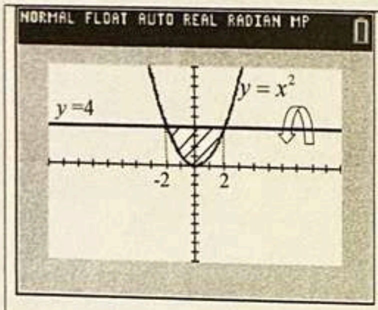
2.3 Rotation about other lines (vertical or horizontal)

When the required region is rotated through 2π about other vertical or horizontal line instead of one of the axes, we will need rewrite the problem (usually via translation of the graph) so that the line of rotation is one of the axis. This idea is illustrated in the following example.

Example

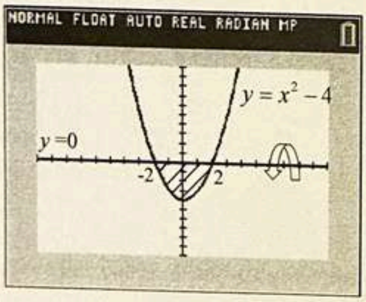
The region R is bounded by the curve $y = x^2$ and the line $y = 4$. Find the volume of the solid formed when R is rotated completely about the line $y = 4$.

Solution:



In order to find the required volume, the region is translated 4 units in the negative y-direction so that the line of rotation is now the x-axis.

Thus, the equation of the curve in consideration is now $y = x^2 - 4$ instead of $y = x^2$.



Required volume $= \pi \int_{-2}^{2} (y-4)^2 \, dx = \pi \int_{-2}^{2} (x^2 - 4)^2 \, dx$

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2020 Year 6 Term 3 Revision 13 (Summary and Tutorial)

Topic(s): Differential Equations

Summary for Differential Equations

Differential equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$

- solved by "direct" integration, i.e. integrate $\mathrm{f}(x)$ with respect to x.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x) \Rightarrow y = \int \mathrm{f}(x)\,\mathrm{d}x$$

Example [9740/2008/01/Q4(i)(ii))]

(i) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{x^2+1}.$$

(ii) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$.

Solution:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x}{x^2+1}$

$y = \displaystyle\int \frac{3x}{x^2+1}\,\mathrm{d}x$ [integrate the $\mathrm{f}(x) = \dfrac{3x}{x^2+1}$ with respect to x]

$= \dfrac{3}{2}\displaystyle\int \frac{2x}{x^2+1}\,\mathrm{d}x$

$= \dfrac{3}{2}\ln(x^2+1) + c$

(ii) When $x = 0, y = 2$

$2 = \dfrac{3}{2}\ln(0^2+1) + c$

$c = 2$

Thus the particular solution is $y = \dfrac{3}{2}\ln(x^2+1) + 2$.

Differential equations of the form $\frac{dy}{dx} = f(y)$

- solved by separating the variables, i.e.

$$\frac{dy}{dx} = f(y) \Rightarrow \int \frac{1}{f(y)}\, dy = \int 1\, dx$$

Example [9740/2012/02/Q1(b)]
Given that u and t are related by

$$\frac{du}{dt} = 16 - 9u^2$$

and that $u = 1$ when $t = 0$, find t in terms of u, simplifying your answer.

Solution:

$\frac{du}{dt} = 16 - 9u^2$

$\int \frac{1}{16-9u^2}\, du = \int 1\, dt$ [separate the variables to obtain $\int \frac{1}{f(u)}\, du = \int 1\, dt$]

$\frac{1}{3}\int \frac{3}{4^2-(3u)^2}\, du = \int 1\, dt$

$\frac{1}{3} \times \frac{1}{2(4)} \ln\left|\frac{4+3u}{4-3u}\right| = t + c$

$\frac{1}{24} \ln\left|\frac{4+3u}{4-3u}\right| = t + c$

When $t = 0, u = 1$

$c = \frac{1}{24}\ln 7$

Thus $t = \frac{1}{24}\ln\left|\frac{4+3u}{4-3u}\right| - \frac{1}{24}\ln 7 = \frac{1}{24}\ln\left|\frac{4+3u}{7(4-3u)}\right|$

Differential equations of the form $\frac{d^2y}{dx^2} = f(x)$

- solved by integrating the right hand side twice with respect to x. Note that

$$\frac{d^2y}{dx^2} = f(x) \Rightarrow \frac{dy}{dx} = \int f(x)\ dx$$

Now suppose that $\int f(x)\ dx = F(x) + c$, then we have

$$\frac{dy}{dx} = F(x) + c$$

which gives

$$y = \int F(x) + c\ dx$$

Example [9740/2012/02/Q1(a)]

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} = 16 - 9x^2$$

Giving your answer in the form $y = f(x)$.

Solution:

$\frac{d^2y}{dx^2} = 16 - 9x^2$

$\frac{dy}{dx} = \int 16 - 9x^2\ dx$ [integrate with respect to x]

$= 16x - 3x^3 + c$

$y = \int 16x - 3x^3 + c\ dx$ [integrate with respect to x]

$= 8x^2 - \frac{3}{4}x^4 + cx + d$ [note that there are two constants c and d]

Solving Differential Equations using a *given* substitution

- the substitution will be given in the question
- this substitution allows us to reduce the differential equation to familiar forms which we discussed earlier, namely $\frac{dy}{dx} = f(x)$ or $\frac{dy}{dx} = f(y)$.

Example [9233/N87/02/Q14(a)]
The variables x and y are connected by the differential equation

$$\frac{dy}{dx} = \frac{1+x+y}{1-x-y}.$$

Show that the substitution $u = x + y$ reduces the equation to

$$\frac{du}{dx} = \frac{2}{1-u},$$

and solve this differential equation.

Solution:

$u = x + y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$ [differentiating both sides with respect to x]

$\therefore \frac{dy}{dx} = \frac{du}{dx} - 1$

$$\frac{dy}{dx} = \frac{1+x+y}{1-x-y}$$

$$\frac{du}{dx} - 1 = \frac{1+u}{1-u}$$

$$\frac{du}{dx} = \frac{1+u}{1-u} + 1$$

$$= \frac{1+u+1-u}{1-u}$$

$$= \frac{2}{1-u} \quad \text{(shown)}$$

$\int (1-u)\, du = \int 2\, dx$ [separate the variables]

$$u - \frac{1}{2}u^2 = 2x + c$$

Differential Equations in Real-World Context

- this includes formulating the differential equation in the given context, solving the equation and interpreting the solution of the differential equation.

In general, one can use Rate of change = Rate increase – Rate decrease to formulate the differential equation for a given real-world context problem.

Example [9740/2010/01/Q7]

A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant $20\,°\mathrm{C}$. As the liquid warms up, the rate of increase of its temperature θ °C after time t minutes is proportional to the temperature difference $(20-\theta)$ °C. Initially the temperature of the liquid is $10\,°\mathrm{C}$ and the rate of increase of the temperature is $1\,°\mathrm{C}$ per minute.

By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-\frac{1}{10}t}$.
Find the time it takes the liquid to reach a temperature of $15\,°\mathrm{C}$, and state what happens to θ for large values of t. Sketch a graph of θ against t.

interpreting

Solution:

formulating

$\frac{\mathrm{d}\theta}{\mathrm{d}t} = k(20-\theta)$ [rate of change $= \frac{\mathrm{d}\theta}{\mathrm{d}t}$, rate increase $= k(20-\theta)$ where k is the constant of proportionality]

When $\theta = 10, \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$

$1 = k(20-10) \Rightarrow k = \frac{1}{10}$

$\int \frac{1}{20-\theta}\,\mathrm{d}\theta = \int \frac{1}{10}\,\mathrm{d}t$ [solving DE of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$]

$-\ln|20-\theta| = \frac{1}{10}t + c$

$20-\theta = \pm e^{-\frac{1}{10}t-c}$

$20-\theta = Ae^{-\frac{1}{10}t}$ where $A = \pm e^{-c}$

$\theta = 20 - Ae^{-\frac{1}{10}t}$

When $t = 0, \theta = 10$

$10 = 20 - A \Rightarrow A = 10$

$\therefore \theta = 20 - 10e^{-\frac{1}{10}t}$

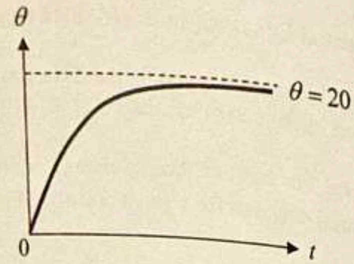
When $\theta = 15$,

$15 = 20 - 10e^{-\frac{1}{10}t}$

$e^{-\frac{1}{10}t} = \frac{1}{2}$

$t = -10\ln\frac{1}{2} = 6.93$ mins

As $t \to \infty$, $e^{-\frac{1}{10}t} \to 0$ and so $\theta \to 20$.



Example [9233/N74/02/Q20]
A race called the Matrices live on an isolated island called Vector. The number of births per unit time is proportional to the population at any time and the number of deaths per unit time is proportional to the square of the population. If the population at time t is p, show that $\frac{dp}{dt} = ap - bp^2$, where a and b are positive constants. Solve the equation for p in terms of t, given that $p = \frac{2a}{3b}$ when $t = 0$. Show that there is a limit to the size of the population.

Solution:

Rate increase $\propto p$
$\therefore$ Rate increase $= ap$ where a is a positive constant

Rate decrease $\propto p^2$
$\therefore$ Rate decrease $= bp^2$ where b is a positive constant

Thus $\frac{dp}{dt} = ap - bp^2$ [formulating]

$\int \frac{1}{ap - bp^2}\, dp = \int 1\, dt$ [solving DE of the form $\frac{dy}{dx} = f(y)$]

...

Refer to Q12 of your C9 Differential Equations tutorial for the remaining parts of the solution.

Probability & Statistics

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 14 (Summary and Tutorial)

Topic(s): Permutations and Combinations, Probability

Summary for Permutations and Combinations

Definitions - A **permutation** is an ***ordered arrangement*** of objects
- A **combination** is a ***selection of objects*** in which the order of selection ***does not matter***.

Add or multiply?

Mutually exclusive operations → Add (**Case** 1, 2, 3 etc…)

Doing one operation **followed by** another to get to the result. → Multiply (**Step** 1, 2, 3 etc…)

Permutations

Given	Objects taken	No of Permutations
n distinct objects	n	$n!$
n distinct objects	r ***no repetitions***	${}^nP_r$ or ${}^nC_r r!$
n distinct objects	r ***with repetitions***	n^r
n objects, not all distinct (n_1 *of type 1,* n_2 *of type 2, ...,* n_k *of type k, where* $n = n_1 + n_2 + ... + n_k$)	n	$\dfrac{n!}{n_1!n_2!...n_k!}$
n objects, not all distinct	r	Involves combinations & permutations See section on ***Combinations***.

Useful Techniques when dealing with restrictions:

1. Grouping or Slotting.
   (a) Grouping ("must be together", "cannot be separated" etc.).
   Eg. No of ways to arrange the letters a, b, c, d, e, f, g such that a, b & c are together $= (5!)\times(3!)$
   [abc, d, e, f, g: 5 items → 5! ways, within abc → 3! ways]
   (b) Slotting ("cannot be together", "must be separated" etc.)
   Eg. No of ways to arrange the letters a, b, c, d, e, f, g such that a, b & c are not adjacent to each other $= (4!)\times({}^5P_3)$
   [arrange d, e, f, g first → 4! ways, slot and permute a, b, c → ${}^5P_3$ ways]

2. Taking Complement ("two items cannot be together" or "at least 1").
Use this technique with care. Note that for Eg in 1(b) above, a, b, c not adjacent to each other is NOT the complement of abc together.

Combinations

Notation: ${}^nC_r$ can also be written as $\binom{n}{r}$

Note:

- $\binom{n}{r} = \binom{n}{n-r} = \dfrac{n!}{r!(n-r)!}$. E.g. $\binom{9}{2} = \binom{9}{7}$
- ${}^nP_r = {}^nC_r r!$

No of ways to select *r objects*	Given : ***n distinct*** objects $\binom{n}{r}$	Given: ***n*** objects, not all distinct ***No direct way to calculate.*** Need to consider different cases: (i) Start with case where the *r* selected objects are ***all distinct.*** (ii) Next, consider case(s) with pair(s) of identical objects, and so on. **(iii) No of ways = Sum of the different cases**
Eg: To choose 3 letters (***arrangements*** not required)	Given: Letters a,b,c,d,e,f No of ways = $\binom{6}{3} = 20$	Given: a, a, a, b, b, c Case 1: All distinct (abc) – 1 way. Case 2: Contains an identical pair (aab, aac, bba, bbc) – 2 x 2 = 4 ways. Case 3: All identical (aaa) – 1 way. Total no of ways = 1 + 4 +1 = 6
Eg. Find the no of 3-letter codes (***arrangements*** to be considered)	Given: Letters a,b,c,d,e,f No of ways = $\binom{6}{3}(3!) = 120$ OR: ${}^6P_3 = 6\times5\times4 = 120$	Given: a, a, a, b, b, c Case 1: All distinct (abc) – 1x3! = 6 ways. Case 2: Contains an identical pair (aab, aac, bba, bbc) – $2\times2\times\dfrac{3!}{2!} = 12$ ways. Case 3: All identical (aaa) – 1 way. Total no of ways = 6 + 12 +1 = 19

Remark: Before doing any calculations, it is important to establish if

- the problem involves ***permutations only***, or ***combinations only***, or ***both***.
- the objects or items given are ***all distinct***. (see Eg in above table).
- ***repetitions*** are allowed.

Circular Permutations – Arranging *n* distinct objects in a Circle

- In row arrangement, such as in a queue, there is a *first* & a *last* position. No of ways = $n!$
- In a circle, such as a round table, there is no starting or ending point.

  No of ways $= \dfrac{n!}{n} = (n-1)!$
  - No. of ways to seat 2 couples and a boy at a round table = $(5-1)! = 24$
  - No. of ways to seat 2 couples and a boy at a round table if each couple is seated together $= (3-1)!2!2! = 8$

    [couple 1, couple 2, boy round the round → $(3-1)!$ ways, each couple → $2!$ ways]
- If the seats of the round table are numbered, we will perform the necessary calculations assuming the seats are not numbered, and multiply the value with the number of seats
  - No. of ways to seat 2 couples and a boy at a round table with numbered seats $= (5-1)! \times 5 = 120$
  - No. of ways to seat 2 couples and a boy at a round table if each couple is seated together, and the seats are numbered = $(3-1)!2!2! \times 5 = 40$

Reminder: Questions on P&C may involve probability. Read the questions carefully.

Summary for Probability

If the sample space S consists of a finite number of **equally likely** outcomes, then the probability of event A written as $P(A)$ is defined as

$$\mathrm{P}(A)=\frac{\text{no. of elements in } A}{\text{no. of elements in } S}=\frac{\mathrm{n}(A)}{\mathrm{n}(S)}.$$

In layman definition, Required probability $=\dfrac{\text{outcomes we are interested in}}{\text{total possible outcomes}}$.

Important Results

1. $0 \le \mathrm{P}(A) \le 1$, answers should always be between 0 and 1 inclusive AND exact where possible.
2. $\mathrm{P}(A')=1-\mathrm{P}(A)$
3. $\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \cap B)$
   With the help of venn diagram, others (besides Results 2 and 3) can also be easily deduced. Some examples as follow:
   $\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(A' \cap B)$
   $\mathrm{P}(A)=\mathrm{P}(A \cap B)+\mathrm{P}(A \cap B')$
   or any others involving $A' \cup B, A \cup B', A' \cup B', A \cap B, A' \cap B, A \cap B', A' \cap B'$ and even 3 sets A, B, C.
4. If A and B are **mutually exclusive**, then $\mathrm{P}(A \cap B)=\mathrm{P}(\phi)=0$, thus $\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(B)$

Conditional Probability

If A and B are two events and $\mathrm{P}(B) \neq 0$, then the probability of A, given that B has already occurred, is written as $\mathrm{P}(A \mid B)$ and is calculated using the formula:

$$\mathrm{P}(A \mid B)=\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Note:

1. $\mathrm{P}(B|A)=\dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$, where $\mathrm{P}(A) \neq 0$.
2. In general, $\mathrm{P}(A|B) \neq \mathrm{P}(B|A)$.
3. $\mathrm{P}(A|B)$ can be considered as the probability of occurrence of event $A \cap B$ with respect to the sample space B i.e. reduced sample space.
   Example (part of **AJC Prelim 9740/2008/02/Q7**):

   3 machines A, B and C produce 25%, 35% and 40% respectively of the golf balls manufactured by a factory. These balls are either yellow or white. Of the balls produced by A and B, 20% and 30% respectively are yellow. It is known that the probability of picking a yellow ball is 0.355. If 3 balls

are picked randomly, find the probability that at least 1 is yellow given that all the balls picked are from machine A.

Using reduced sample space, P(at least 1 is yellow|all from A) $= 1 - 0.8^3 = 0.488$

Independent Events

Two events A and B are said to be independent if the occurrence of A does not affect that of B and vice versa.

To prove independence of events A and B, we can show either one of the following

1. $P(A|B) = P(A)$
2. $P(B|A) = P(B)$
3. $P(A \cap B) = P(A)P(B)$

It can be easily shown that if A and B are independent, then the following pairs are also independent A and B', A' and B', A' and B

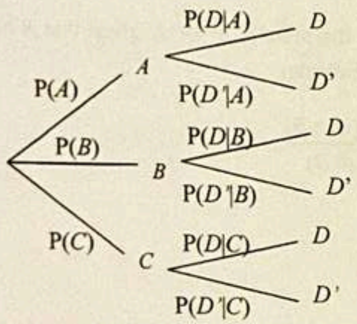
We DO NOT assume independence unless we have proven it or condition is given in the question.

Common mistakes:

Mixed up the tests of "mutually exclusive" and "independence".
Taking $P(A \cap B) = P(A)P(B)$ even when A and B are NOT independent.

Techniques in calculating probabilities

1. Systematic listing – example 9740/2014/P2/Q10.
2. Venn diagram – example 9740/2015/P2/Q9. This question involves 3 sets.
3. Tree diagram – though a useful technique, many students are lazy to draw the diagram and hence made careless mistakes.



$P(A) + P(B) + P(C) = 1$

$P(A \cap D) = P(A) \times P(D|A)$

4. P&C – do not be confused when applying this technique. The counting for denominator has to be consistent with that of numerator. If order matters for denominator, then it has to matter for numerator.

Reminder: Probability could easily be embedded in another Statistics question due to its nature. Read question carefully.

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 15 (Summary and Tutorial)

Topics: Discrete Random Variable, Binomial Distribution, Normal Distribution

Summary for Discrete Random Variable

Probability Distribution

A table or formula giving the values of $\mathrm{P}(X=x)$ for every x in sample space is called the **probability distribution** of X. For the experiment of tossing a fair coin 3 times where X is the number of heads obtained, the **probability distribution** of X is as follows

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The probability distribution of X satisfy the following:

1. $0 \le \mathrm{P}(X=x) \le 1$ for all x in S.
2. $\sum_{x \in S} \mathrm{P}(X=x)=1$ where the summation is over all values of x in S.

Expectation of Discrete Random Variable

The expectation (or mean, or expected value) of a discrete random variable X taking values from a set S is given by $\mathrm{E}(X)=\sum_{x \in S} x\mathrm{P}(X=x)$.

Independent Discrete Random Variables

Let X and Y be two discrete random variables taking on possible values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ respectively. The random variables X and Y are said to be **independent** if for all i and j,

$$\mathrm{P}(X=x_i \text{ and } Y=y_j)=\mathrm{P}(X=x_i)\mathrm{P}(Y=y_j).$$

Functions of a Discrete Random Variable

The expectation of $\mathrm{g}(X)$, where g is a function of X, is denoted by $\mathrm{E}(\mathrm{g}(X))=\sum_{x \in S} \mathrm{g}(x)\mathrm{P}(X=x)$.

In particular, $\mathrm{E}(X^2)=\sum_{x \in S} x^2\,\mathrm{P}(X=x)$.

Variance and Standard Deviation of a Discrete Random Variable

The variance of a discrete random variable X is given by

$$\mathrm{Var}(X) = \mathrm{E}(X-\mu)^2 = \sum (x-\mu)^2 \, \mathrm{P}(X=x)$$

$$= \mathrm{E}(X^2) - \mu^2 = \left(\sum x^2 \mathrm{P}(X=x)\right) - \mu^2$$

The standard deviation of X, denoted by σ, is defined as $\sqrt{\mathrm{Var}(X)}$.

Properties of Expectation and Variance

(*Note that these properties hold for discrete and continuous random variables*)

Let X and Y be random variables and a and b be constants. We have

Expectation	Variance
(1) $\mathrm{E}(a) = a$ (2) $\mathrm{E}(aX) = a\mathrm{E}(X)$ (3) $\mathrm{E}(aX \pm b) = a\mathrm{E}(X) \pm b$ (4) $\mathrm{E}(aX \pm bY) = a\mathrm{E}(X) \pm b\mathrm{E}(Y)$	(1) $\mathrm{Var}(a) = 0$ (2) $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$ (3) $\mathrm{Var}(aX \pm b) = a^2\mathrm{Var}(X)$ If X and Y are *independent* random variables, then (4) $\mathrm{Var}(aX \pm bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

Important Results

In general, $\mathrm{E}(nX) = n\mathrm{E}(X)$

$$\mathrm{E}(X_1 + X_2 + \dots + X_n) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \dots + \mathrm{E}(X_n) = n\mathrm{E}(X)$$

but $\mathrm{Var}(nX) = n^2\,\mathrm{Var}(X)$

$$\mathrm{Var}(X_1 + X_2 + \dots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \dots + \mathrm{Var}(X_n) = n\,\mathrm{Var}(X).$$

Summary for Binomial Distribution

Note that Binomial random variable is a special **discrete random variable.**

Conditions for an experiment to follow binomial distribution

1. It consists of n independent trials.
2. The outcome of each trial is either a success or a failure.
3. The probability of success for each trial, denoted by p, remains constant.

Note that the above must be stated **in context** of a given question.

For example, a biased coin has probability 0.56 of obtaining a head in any toss. Find the probability of getting 6 heads if the coin is tossed 8 times.

Conditions in context as follow

1. The coin is tossed independently 8 times.
2. The outcome of each toss is either a head (success) or tail (failure).
3. The probability of obtaining a head (success) remains constant at 0.56.

Binomial Distribution

If a discrete random variable X follows a **binomial distribution**, we write $X \sim \mathrm{B}(n, p)$

where the parameters of the distribution n and p refer to the number of trials and the probability of success for each trial respectively.

The probability distribution of X is given by (in MF26)

$$\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, 2, \ldots, n, \text{ where } \binom{n}{x} = {}^nC_x = \frac{n!}{(n-x)!\,x!}$$

The mean and variance of X are given by

(i) $\mathrm{E}(X) = np$ (in MF26)

(ii) $\mathrm{Var}(X) = np(1-p)$ (in MF26)

Summary for Normal Distribution

Note that Normal random variable is a special continuous random variable. For continuous random variable, probability is calculated using the area under its probability density function.

In H2 Math syllabus, we just need to know how to use GC to evaluate these probabilities.

We also need to be aware that

$\mathrm{P}(X < x) \neq \mathrm{P}(X \le x)$ for discrete random variable (Binomial included)

BUT

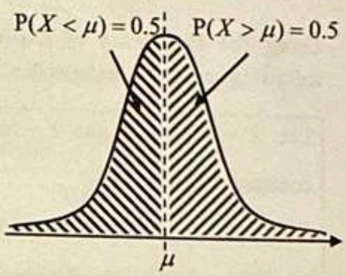
$\mathrm{P}(X < x) = \mathrm{P}(X \le x)$ for continuous random variable

Normal Distribution

If a continuous random variable X follows a normal distribution, we write $X \sim \mathrm{N}(\mu, \sigma^2)$, where $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

Properties of a Normal Curve

Let $X \sim \mathrm{N}(\mu, \sigma^2)$.



(1) It is symmetrical about the line $x = \mu$.

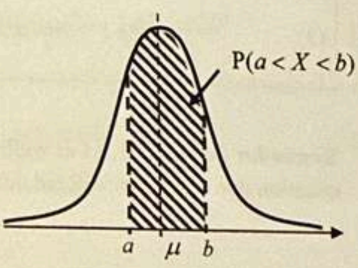
(2) The mean, median and mode are all equal to μ.

(3) It approaches the $x-$axis as $x \to \pm\infty$.

(4) Area under the graph gives the probabilities,

i.e., $\mathrm{P}(a < X < b) = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$

where $y = \mathrm{f}(x)$ represents the probability density function of the normal curve.



Hence $\mathrm{P}(a < X < b)$ is given by the area under the graph from $x = a$ to $x = b$.

(5) Total area under the curve is 1.

(6) $P(\mu-\sigma < X < \mu+\sigma) \approx 0.68$

$P(\mu-2\sigma < X < \mu+2\sigma) \approx 0.95$

$P(\mu-3\sigma < X < \mu+3\sigma) \approx 0.997$

i.e., approximately 68%, 95% and 99.7% of the values drawn from a normal distribution lies within 1, 2 and 3 standard deviations of the mean respectively.

Standard Normal Distribution

Let $X \sim N(\mu,\sigma^2)$. The random variable Z, which is the standard normal variable, is defined by $Z = \dfrac{X-\mu}{\sigma}$. The standard normal distribution is $Z \sim N(0,1)$.

The process of converting $X \sim N(\mu,\sigma^2)$ into $Z \sim N(0,1)$ is known as **standardization.**

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

Remarks: Standardization is usually applied when there are unknown parameter(s) when we can't use the commands normalcdf(or invNorm(of our GC directly.

Using the Properties of Expectation and Variance of Random Variables, we have the following results for independent Normal Random Variables

Let $X \sim N(\mu_1,\sigma_1^2)$ and $Y \sim N(\mu_2,\sigma_2^2)$ be **independent** random variables and a and b be constants. We have

(1) $X \pm Y \sim N(\mu_1 \pm \mu_2, \sigma_1^2 + \sigma_2^2)$

(2) $aX \pm b \sim N(a\mu_1 \pm b, a^2\sigma_1^2)$

(3) $aX \pm bY \sim N(a\mu_1 \pm b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

Reminder (in revision 14 as well): Probability could easily be embedded in another Statistics question due to its nature. Read question carefully.

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 16 (Summary and Tutorial)

Topic(s): Sampling and Estimations, Hypothesis Testing

Summary for Sampling and Estimations

Definitions - A **population** is the entire collection of data (persons or items or individuals) that we want to study e.g. apples produced by a farm.

- A sample is **random** if every element in the population has *an equal chance* of being selected, and the selection of an element is independent of another. e.g. 'Every biscuit bar has an equal chance of being selected, and the selection of one biscuit bar is not affected or influenced by the selection of another biscuit bar'.

[Note that it is **not** sufficient to say 'each biscuit bar has an equal chance of being selected']

- A sample is **non-random** if each element in the population *does not have an equal chance of being selected*, resulting in certain segments of the population being over-represented, as some members are "systematically or deliberately excluded" from the study and the sample being biased.

The Sample Mean, $\overline{X}$ as a Random Variable

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n taken from an infinite population (or finite population if sampling is done with replacement) with mean μ and variance σ^2. Then the **sample mean** $\overline{X}$, defined by

$$\overline{X} = \frac{1}{n}\sum X = \frac{X_1 + X_2 + X_3 + \ldots + X_n}{n},$$

is a random variable with $\mathrm{E}(\overline{X}) = \mu$ and $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

The Distribution of the Sample Mean

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n taken from a **normal** population with mean μ and variance σ^2. Then (1) $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ (2) $X_1 + X_2 + X_3 + \ldots + X_n \sim \mathrm{N}(n\mu, n\sigma^2)$	Let $X_1, X_2, X_3, \ldots, X_n$ be a **large random** sample of size n taken from a **non-normal** population with mean μ and variance σ^2. Then since **sample size n is large,** (1) $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ ***approximately*** by **Central Limit Theorem** (2) $X_1 + X_2 + X_3 + \ldots + X_n \sim \mathrm{N}(n\mu, n\sigma^2)$ ***approximately*** by **Central Limit Theorem**

General Tips

1. To identify questions on Central Limit Theorem, look out when there is no mention of 'normally distributed' or 'normal population' e.g. "The random variable X is thought to have a mean of 50 but it is known that the standard deviation is 14.5." and the question asks for 'find the probability that the **sample mean / average value of X / sum** …', 'by using a **suitable approximation**' or '**estimate** the probability'.
2. Do not make the mistake of writing $X \sim \mathrm{N}(\mu, \sigma^2)$ at the first sighting of mean and variance/standard deviation in the question. Mean and variance applies to any population, not just normal, so look out for the phrases in point 1 above.
3. Also, writing $X \sim \mathrm{N}(\mu, \sigma^2)$ approximately by Central Limit Theorem is **wrong** as the theorem applies to $\bar{X}$.
4. When the population variance σ^2 is unknown, the unbiased estimate of population variance s^2 will be used and the notation will change accordingly. This is especially important when we write the distribution of $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\cancel{\sigma^2}\, s^2}{n}\right)$ approximately under hypothesis testing.

Unbiased Estimates of Population Mean and Population Variance.

In statistics, an estimate is considered to be "good" if it's **unbiased**, i.e. the average value of the sample statistic (used to estimate the population parameter) for all possible samples gives the true value of the population parameter.

In particular, the average value of $\overline{X}$ gives the true value of μ, that is, $\mathrm{E}\left(\overline{X}\right)=\mu$.

Population	Estimate from sample	Unbiased estimate?	How?
μ	Sample mean $\overline{x}$	Yes, sample mean is an unbiased estimate of population mean.	$\overline{x}=\frac{\sum x}{n}$ or $\overline{x}=\frac{\sum(x-a)}{n}+a$
σ^2	Sample variance	No, sample variance is NOT an unbiased estimate of population variance.	We will have to calculate the unbiased estimate of population variance s^2 by using one of the following: (1) $s^2=\frac{n}{n-1}\left[\frac{\sum(x-\overline{x})^2}{n}\right]$ (2) $s^2=\frac{1}{n-1}\left[\sum x^2-\frac{\left(\sum x\right)^2}{n}\right]$ (3) $s^2=\frac{n}{n-1}(\text{sample variance})$ (4) $s^2=\frac{1}{n-1}\left[\sum(x-a)^2-\frac{\left(\sum(x-a)\right)^2}{n}\right]$

Summary for Hypothesis Testing

Performing a Hypothesis Test

Step 1	Understand the given question and write down the **null hypothesis** H_0 and the **alternative hypothesis** H_1
Step 2	Write down the **level of significance** α (usually given in the question)
Step 3	Decide on the **test statistic** to be used and determine its **distribution**
Step 4	Use the GC to calculate the ***p*-value**
Step 5	**Reject** H_0 if p-value $\le \alpha$, OR **Do not reject** H_0 if p-value $> \alpha$ Write down the **conclusion** in the context of the question

Example:

Let X be the IQ of a student in ABC University.

Remember to always define X if there are not defined in the question.

Step 1: To test $H_0: \mu = 118$ vs $H_1: \mu > 118$

This is an example. Read question carefully to decide >, < or ≠.

Step 2: Perform a 1-tail / 2-tail test at 5% level of significance.

Step 3: (Sample from a Normal population of known variance)

Under H_0, $\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$, where $\mu_0 = 118$ and $\sigma = 12$.

From the sample, $\bar{x} = 121$, $n = 50$.

OR

Step 3 (Large sample from a Normal population of unknown variance):

Under H_0, $\bar{X} \sim N\left(\mu_0, \frac{s^2}{n}\right)$ **approximately**, with $\mu_0 = 118$.

From the sample, $\bar{x} = 121$, $s = \sqrt{97.454}$

OR

Step 3 (Large sample from a non-Normal population of unknown variance):

Under H_0, **since** $n = 60$ **is large,** $\bar{X} \sim N\left(\mu_0, \frac{s^2}{n}\right)$ **approximately, by Central Limit Theorem**, with $\mu_0 = 118$.

From the sample, $\bar{x}=121$, $s=\sqrt{97.454}$

$$\boxed{\begin{aligned} &"\mathrm{H}_1:\mu>\mu_0" \Rightarrow \mathrm{P}(\bar{X}\geq\bar{x}) \\ &"\mathrm{H}_1:\mu<\mu_0" \Rightarrow \mathrm{P}(\bar{X}\leq\bar{x}) \\ &"\mathrm{H}_1:\mu\neq\mu_0" \Rightarrow 2\mathrm{P}(\bar{X}\leq\bar{x}) \text{ if } \bar{x}<\mu_0 \\ &\qquad\qquad \text{or } 2\mathrm{P}(\bar{X}\geq\bar{x}) \text{ if } \bar{x}>\mu_0 \end{aligned}}$$

Step 4: Using a z-test, p-value $= \mathrm{P}(\bar{X}\geq 121) = 0.0385$ (3 s.f.)

Step 5: Since $p\text{-value} = 0.0385 < 0.05$, we **reject** H_0 and conclude that there is **sufficient** evidence, at 5% level of significance, to support the claim that the mean IQ of students in ABC University is greater than 118.

OR

(if test is performed at 1% level of significance)

Step 5: Since $p\text{-value} = 0.0385 > 0.01$, we **do not reject** H_0 and conclude that there is **insufficient** evidence, at 5% level of significance, to support the claim that the mean IQ of students in ABC University is greater than 118.

[Notice the only difference are the two phrases in bold and the end of the sentence is to describe the alternative hypothesis H_1.]

General Tips

- Take note of the nature of the "claim" in the question and the respective conclusion e.g. if the claim is "the mean IQ is equal to 118 or at least 118 or at most 118", then when $\mathrm{H}_0:\mu=118$ is rejected, there is "sufficient evidence at … to conclude that the claim is **invalid**"; whereas if the claim is "the mean IQ is higher or lower than 118", then when $\mathrm{H}_0:\mu=118$ is rejected, there is "sufficient evidence at … to conclude that the claim is **valid**".
- If H_0 is rejected at 5% level of significance for a certain p-value, H_0 will also be rejected at 10% level of significance or any level higher than 5%. On the contrary, H_0 may or may not be rejected at 1% or any level lower than 5%.

Definitions

1. The **level of significance** (or significance level) of a hypothesis test, denoted by α, is defined as the **probability of rejecting H_0 when H_0 is true** e.g. "5% level of significance" means "there is a 0.05 probability of wrongly concluding that the mean IQ of the students is more than 118 when in fact it is 118".
2. The p-value $= P(\bar{X} \geq 121)$ in this context refers to the probability of getting a sample with average IQ at least 121. This is also the value we will put down on our script if question asked for the smallest level of significance at which H_0 can be rejected in favour of H_1.

Important Cases where conclusion is given and we are asked to find:

Case 1: Level of significance $\alpha\%$

Carry on as if we are performing a test. After finding the p-value:

If given H_0 is rejected, p-value $\leq \frac{\alpha}{100}$; if given H_0 is not rejected, p-value $> \frac{\alpha}{100}$.

Case 2: Sample mean $\bar{x}$ (No standardization required)

Example: For a test at 5% level of significance and under H_0, $\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$, where $\mu_0 = 118$ and $\sigma = 12$. From the sample, $n = 50$.

If given H_0 **is rejected**, use: [GC invNorm: key in standard deviation $= \frac{12}{\sqrt{50}}$, not 12]

1-tail (e.g. $H_1: \mu > 118$)	2-tail (e.g. $H_1: \mu \neq 118$)
p-value ≤ 0.05 $P(\bar{X} \geq \bar{x}) \leq 0.05$ Using GC, $P(\bar{X} \geq 137.738) = 0.05$ $\bar{x} \geq 138$	p-value ≤ 0.05 $2P(\bar{X} \geq \bar{x}) \leq 0.05$ or $2P(\bar{X} \leq \bar{x}) \leq 0.05$ $P(\bar{X} \geq \bar{x}) \leq 0.025$ or $P(\bar{X} \leq \bar{x}) \leq 0.025$ ⋮

Case 3: μ_0 or n or σ^2 or s^2

All steps are similar to that of Case 2 above except we have to standardize in order to solve for the unknown parameter.

Example: (to find μ_0)

1-tail (e.g. $H_1 : \mu > \mu_0$; from sample, $\bar{x} = 121$)
p-value ≤ 0.05 $P(\bar{X} \geq 121) \leq 0.05$ $P\left(Z \geq \dfrac{121-\mu_0}{\sqrt{\dfrac{12^2}{50}}}\right) \leq 0.05$ Using GC, $P(Z \geq 1.6449) = 0.05$ $\dfrac{121-\mu_0}{\sqrt{\dfrac{12^2}{50}}} \geq 1.6449$ $\vdots$

RAFFLES INSTITUTION
H2 Mathematics 9758
2020 Year 6 Term 3 Revision 17

Topic(s): Review the Use of Graphing Calculators

In this revision, we will review the *use of graphing calculators* for the following topics. There may be tricks that you may not be aware of.

1: Graphing – Functions
2: Graphing – Parametric Equations
3: Equations & Inequalities
4: Complex Numbers
5: Discrete Random Variables

1: Graphing – Functions

Skills Required: By using the **FUNCTION** mode,

1. Sketch the graphs of functions and composite functions [Y=, WINDOW, ZOOM].
2. Determine the y-value for a given x-value [[CALC] value].
3. Determine the x-intercepts [[CALC] zero].
4. Determine the coordinates of the minimum & maximum points [[CALC] minimum/maximum]
5. Determine the points of intersection of 2 graphs [[CALC] intersect].
6. Determine the gradient of the function [[CALC] $\frac{dy}{dx}$ OR nDeriv].
7. Determine the area bounded by the function with the x-axis [[CALC] $\int f(x)\,dx$ OR fnInt].
8. Sketch piecewise functions (OS 5.3.0 and 5.3.1) [[MATH] B: piecewise]

Exercise 1

(a) Functions f and g are defined as follows:

$$f(x)=\frac{x^2-2x-1}{x+2}, \qquad \text{for } x\in\mathbb{R},\ x\neq -2,$$

$$g(x)=x-3-e^{-x}-\ln(x+1), \qquad \text{for } x>-1.$$

Sketch, on separate diagrams, showing clearly any asymptotes, intersections with the axes and coordinates of stationary points, the graphs of

(i) $y=f(x)$, (ii) $y=g(x)$.

The function h is defined as

$$h : x \mapsto f(x), \quad \text{for } x \in \mathbb{R},\ x > -2.$$

(iii) Find the equation of the tangent to $y = gh(x)$ at the point where $x = 0$.

(iv) Find the area bounded by the graph of $y = gh(x)$ with the x-axis.

(v) Solve the inequality $gh(x) < 2\ln(x+1)$.

(b) Sketch the curve $y = 1 + \dfrac{1-10x}{x(x+1)}$, $x \in \mathbb{R}$, $x \neq -1$, $x \neq 0$, stating clearly equations of any asymptotes, intersections with the axes and coordinates of turning points.

(c) Sketch the graph of $y = f(x)$ where $f(x) = \begin{cases} \sin x + \cos x - 1 & \text{for } -\frac{7}{12}\pi \le x < 0, \\ -\sqrt{1 - \dfrac{(x-2)^2}{4}} & \text{for } 0 \le x \le 2. \end{cases}$

State clearly the coordinates of the endpoints.

2: Graphing – Parametric Equations

Skills Required: By using the **PARAMETRIC** mode,

1. Sketch the curve for the given domain [[Y=], [WINDOW]].
2. Determine the coordinates of a point on the curve [[TRACE], [CALC] value].
3. Determine the gradient of the curve at a point [[CALC] $\dfrac{dy}{dx}$ OR nDeriv].
4. Obtain the value of an integral with parametric equations [fnInt].

Exercise 2

(a) A curve has parametric equations $x = \sin 2\theta + \sin\theta$, $y = \cos\theta$, where $0 \le \theta \le \dfrac{2\pi}{3}$.

(i) Find the equation of the tangent to the curve at the point where $\theta = \dfrac{\pi}{2}$.

(ii) Sketch the curve, giving the coordinates of any intercepts.

(b) A curve has parametric equations $x = \cos^2 t$, $y = \sin^3 t$, for $0 \le t \le \dfrac{\pi}{2}$.

(i) Sketch the curve, giving the coordinates of any intercepts.
(ii) Find the area bounded by the curve and the axes.

3: Equations & Inequalities

Skills Required:

1. Solve equations & inequalities.

- By using [APPS] → **PlySmlt2** → SIMULTANEOUS EQN SOLVER,

2. Solve system of linear equations.

Exercise 3

(a) Find the smallest positive integer n such that $10000\left(1-0.9^n\right)<25n(n+3)$.

(b) Find, for $-2 \le x \le 2$, the set of values of x such that $\left|e^x \sin x-\left(x+x^2+\frac{x^3}{3}\right)\right|<0.05$.

(c) Use your calculator to find a vector equation of the line of intersection between
π_1: $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}+2\mathbf{k})=4$ and π_2: $\mathbf{r}\cdot(\mathbf{i}-\mathbf{j}-\mathbf{k})=4$.

(d) A finite arithmetic progression $u_1, u_2, u_3, \ldots, u_{2n}$ has first term a and common difference d.
It is given that $\frac{u_{n+1}}{u_n}=\frac{4027}{4025}$, $u_n+u_{2n}=36228$ and the average of all the odd-numbered terms is 12 075. By formulating three equations involving a, d and nd, find the values of a, d and n.

4: Complex Numbers

Skills Required:

1. Perform operations on complex numbers.

- By using [APPS] → **PlySmlt2** → POLYNOMIAL ROOT FINDER,

2. Find the roots of a polynomial equation with real coefficients.

- By using [MATH] → **CMPLX**,

3. Find the conjugate, modulus and argument of a complex number.
4. Convert a complex number from the Cartesian form to the Polar form and vice versa.

Exercise 4

(a) Evaluate the following: (i) $\frac{(3+4\mathrm{i})^3}{(1-\mathrm{i})^5}$, (ii) $\sqrt{5-12\mathrm{i}}$. [(i) $\frac{161}{8}+\frac{73}{8}\mathrm{i}$; (ii) $3-2\mathrm{i}$]

(b) Find the roots of the following equations:
(i) $z^2+(-1+4\mathrm{i})z+(-5+\mathrm{i})=0$, (ii) $z^4-2z^3+8z-16=0$.

(c) Find the conjugate, modulus and argument of
(i) $(3-\mathrm{i})(\mathrm{i}-3)$, (ii) $\frac{(5-\mathrm{i})}{(3+2\mathrm{i})}$. [(i) $-8-6\mathrm{i}$, 10, 2.50; (ii) $1+\mathrm{i}$, $\sqrt{2}$, $-\frac{\pi}{4}$]

(d) Express the following complex numbers in cartesian form.

(i) $\sqrt{2}e^{\left(\frac{\pi}{4}\right)i}$, (ii) $2e^{\left(\frac{\pi}{3}\right)i}$.

[(i) $1+\mathrm{i}$; (ii) $1+\sqrt{3}\mathrm{i}$]

(e) Express the following complex numbers in polar form.

(i) $\dfrac{(2+2\mathrm{i})}{(\mathrm{i}-1)}$, (ii) $\left(-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}\mathrm{i}\right)(\mathrm{i}-\sqrt{3})$.

[(i) $2e^{-\left(\frac{\pi}{2}\right)i}$; (ii) $2e^{\left(\frac{\pi}{6}\right)i}$]

5: Discrete Random Variables

Skills Required: Use of $\boxed{\text{STAT}}$, summation, [TABLE].

Exercise 5

(a) Find the mean and standard deviation of the random variable X with the following probability distribution:

x	-4	-3	-2	-1	0	1	2	3
$\mathrm{P}(X=x)$	0.04	0.16	0.24	0.16	0.15	0.1	0.1	0.05

(b) A committee of 12 people is chosen at random from a group consisting of 24 women and 16 men. The number of women on the committee is denoted by R.

Use your calculator to find

(i) $\mathrm{P}(R\geq 6)$,

(ii) the probability that R is divisible by 3,

(iii) $\mathrm{E}(R)$,

(iv) $\mathrm{Var}(R)$,

(v) the most probable number of women on the committee.