9 Oscillations

H2 Physics 9478

Content **Page**

Learning Outcomes

Candidates should be able to:

(a) describe simple examples of free oscillations, where particles periodically return to an equilibrium position without gaining energy from or losing energy to the environment.

(b) investigate the motion of an oscillator using experimental and graphical methods.

(c) show an understanding of and use the terms amplitude, period, frequency, angular frequency, phase and phase difference, and express the period in terms of both frequency and angular frequency.

(d) show an understanding that $a = -\omega^2 x$ is the defining equation of simple harmonic motion, where acceleration is (directly) proportional to displacement from an equilibrium position and acceleration is always directed towards the equilibrium position.

(e) recognise and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$.

(f) recognise and use the equations $v = v_0 \cos \omega t$ and $v = \pm\omega\sqrt{\left(x_0^2 - x^2\right)}$.

(g) describe, with graphical illustrations, the relationships between displacement, velocity, and acceleration during simple harmonic motion.

(h) describe the interchange between kinetic and potential energy during simple harmonic motion.

(i) describe practical examples of damped oscillations, with particular reference to the effects of the degree of damping (light/under, critical, heavy/over), and to the importance of critical damping in applications such as a car suspension system.

(j) describe graphically how the amplitude of a forced oscillation changes with driving frequency, resulting in maximum amplitude at resonance when the driving frequency is close to or at the natural frequency of the system (natural frequency refers to the frequency when the system is in free oscillation).

(k) show a qualitative understanding of the effects of damping on the frequency response and sharpness of the resonance.

(l) describe practical examples of forced oscillations and resonance, and show an appreciation that there are some circumstances in which resonance is useful, and other circumstances in which resonance should be avoided.

9.1 Introduction to Oscillations

A **periodic** motion is one in which a body continually retraces its path at **equal** time intervals. Many systems exhibit periodic motion. The molecules in a solid oscillate about their equilibrium positions; electromagnetic waves are characterised by oscillating electric and magnetic field vectors; and in alternating-current electrical circuits, voltage and current vary periodically with time.

An oscillation is a special periodic motion in which the oscillator moves to and fro about an equilibrium position. This is also called harmonic motion. Simple harmonic motion is a type of such a motion.

Consider a body attached to the end of a fixed spring in Fig. 9.1(a), or a pendulum in Fig. 9.1(b). In both cases, the body and the pendulum are at their equilibrium positions. The **equilibrium position** is the position where the resultant force on the body is zero.

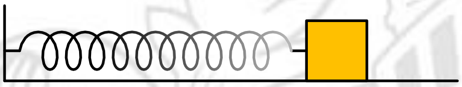


Fig. 9.1(a)

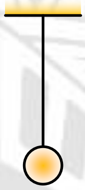


Fig. 9.1(b)

If the body or pendulum is displaced from its equilibrium position (i.e. pulled either to the left or right), it will experience a resultant force that tries to restore it to its equilibrium position. This is called the **restoring force**.

The mass-spring system and pendulum systems described above are examples of oscillators undergoing harmonic motion. We will examine these two systems in greater detail later in this chapter.

❖ Free Oscillation

If a body is displaced from its equilibrium position and then released, it oscillates at its *natural frequency* about the equilibrium position.

A free oscillation occurs when a body oscillates with no driving and/or resistive forces acting on it. Since the oscillating system does not gain energy from or lose energy to the surroundings, the total energy and amplitude remain constant with time.

❖ Definitions of Physical Quantities

Amplitude

Amplitude is the magnitude of the *maximum displacement* of the particle from its equilibrium position.

Period and Frequency

Period T is the time taken for the body to *complete one oscillation*.

Frequency f is the *number of oscillations per unit time*.

T and f are related by the equation

$$T = \frac{1}{f}$$

S.I. unit for T: s

S.I. unit for f: Hz (Hertz) or s^{-1}

Phase (Angle)

The **phase** (angle) ϕ of an oscillation that a body is in, is the *fraction of an oscillation cycle* it is at relative to its starting displacement.

A complete cycle has a phase of 2π.

In one period T, the body undergoing a complete oscillation through a phase angle of 2π. Hence the angular frequency of the oscillation

$$\omega = \frac{2\pi}{T} = 2\pi f$$

S.I. unit of angular frequency ω: rad s^{-1}

Angular frequency is not the same as angular velocity, even though both have the same units of rad s^{-1} and are represented by the same symbol ω.

In oscillations, ω represents angular frequency while in circular motion, it represents angular velocity (rate of change of angular displacement).

Angular Frequency

Angular frequency ω is the *rate of change of phase angle* of the oscillation and is equal to the product of 2π and its frequency f.

Phase Difference

Phase difference $\Delta\phi$ is the difference in the positions of two oscillating bodies in their cycles.

The phase difference is a measure of how "out of step" two oscillating bodies are with each other.

Phase and phase difference are usually measured in degrees or radians.

9.2 Simple Harmonic Motion

When a particle experiences a force F that is directly proportional to its displacement x from a fixed point and that it is always directed towards that fixed point,

$$F = ma = -kx$$

The negative sign indicates that the force is directed opposite to its displacement.

The acceleration of the particle

$$a = -\frac{k}{m}x = -\omega^2 x$$

where ω is the angular frequency of oscillation.

This is the defining equation for simple harmonic motion (S.H.M.).

A particle whose motion satisfies this equation is said to be in S.H.M. The "fixed point" in S.H.M. refers to the equilibrium position of the particle.

By convention, vector quantities directed to the right are positive.

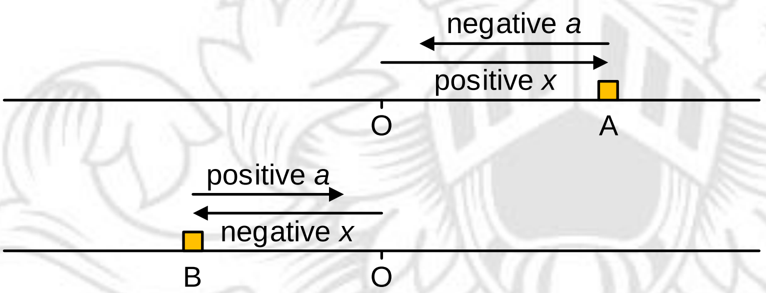


Fig. 9.2

The displacement x of the particle is measured from the fixed point and is directed away from the fixed point while the acceleration a of the particle is directed towards the fixed point. This means that x and a are always opposite in direction as implied by the negative sign in the equation.

Since the acceleration of the particle is directly proportional to its displacement from equilibrium position, the acceleration is **not** constant. Therefore, kinematics equations **cannot** be used to analyse the motion of particles in S.H.M.

Simple Harmonic Motion (S.H.M.)

Simple harmonic motion is the motion of a particle about a fixed point such that its *acceleration* is *proportional* to its *displacement from the fixed point* and is *always directed towards the point*.

❖ Characteristics of S.H.M.

Consider a particle N undergoing S.H.M. between two points A and B, about an equilibrium position O. Throughout the motion, N experiences a restoring force and an acceleration towards O.

Fig. 9.3 illustrates the motion of N at various instances of the motion.

The displacement *x* from point O, the velocity *v* and acceleration *a* of particle N are also indicated. Vector quantities directed to the right is taken to be positive.

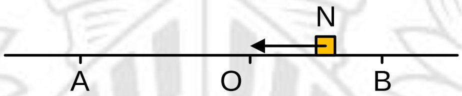
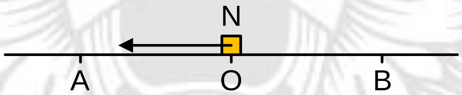
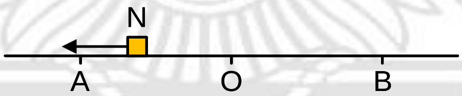
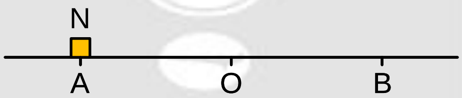
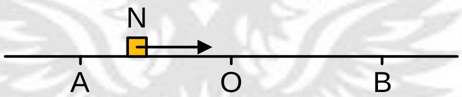
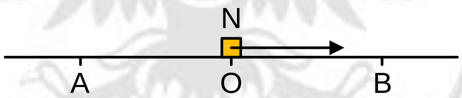
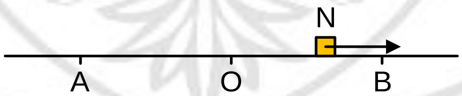
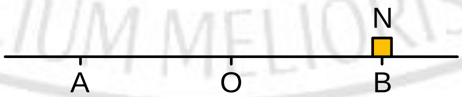
Description	**Diagram**	***x***	***v***	***a***
N at rest at maximum positive displacement		+ → max	0	− ← max
N moves towards A while accelerating towards O (speed up)		+ →	− ←	− ←
N moves towards A with maximum speed and zero acceleration		0	− ← max	0
N moves towards A while accelerating towards O (slow down)		− ←	− ←	+ →
N at rest at maximum negative displacement		− ← max	0	+ → max
N moves towards B while accelerating towards O (speed up)		− ←	+ →	+ →
N moves towards B with maximum speed and zero acceleration		0	+ → max	0
N moves towards B while accelerating towards O (slow down)		+ →	+ →	− ←
N at rest at maximum positive displacement		+ → max	0	− ← max

Fig. 9.3

The motion of particle N has the following characteristics:

- It is symmetrical about the equilibrium position O.
- $|\text{OA}| = |\text{OB}| = x_0$ is the **amplitude** of the oscillation.
- The **period** T of oscillation is the time taken for N to move from B to A and back to B.
- Its acceleration is always directed towards O.
- Its acceleration has maximum magnitude at A and B and zero is zero at O.
- Its speed is a maximum at O and zero at A and B.

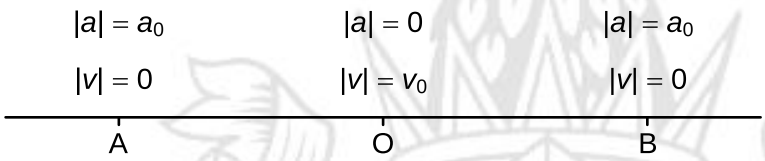


Fig. 9.4

❖ Displacement-Time Relationship

The variation with time t of the displacement x from the equilibrium position of particle N is sinusoidal.

If at $t = 0$, $x = +x_0$ (at B), the motion of N is given by $x = x_0 \cos \omega t$. The displacement-time graph is shown in Fig. 9.5.

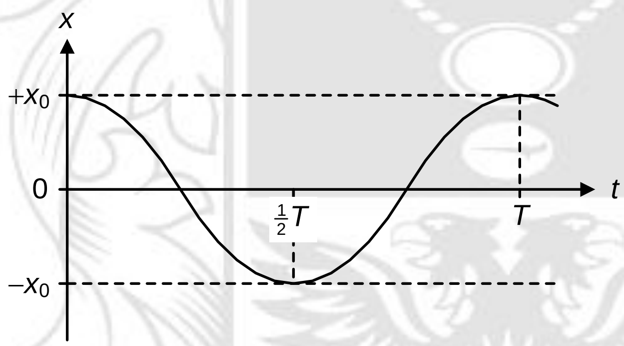


Fig. 9.5

If at $t = 0$, $x = 0$ (at O and moving towards B), the motion of N is given by $x = x_0 \sin \omega t$. The displacement-time graph is shown in Fig. 9.6.

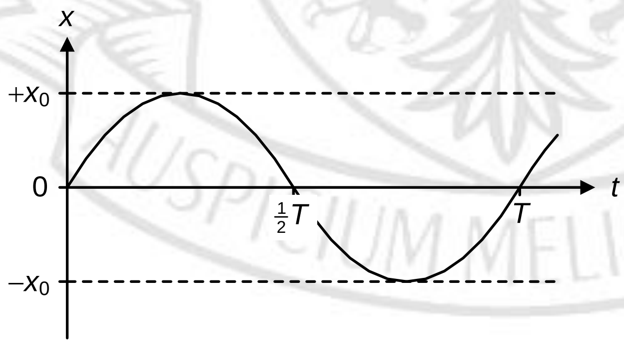


Fig. 9.6

❖ Changes in *x*, *v* and *a* during S.H.M.

Consider the case of $x = +x_0$ at $t = 0$, i.e., particle N is at point B (Fig. 9.3), the displacement of N is given by

$$x = x_0 \cos \omega t$$ Refer to Fig. 9.7

Velocity of N

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(x_0 \cos \omega t\right) v = -\omega x_0 \sin \omega t$$ Refer to Fig. 9.8

and acceleration of N

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(-x_0 \omega \sin \omega t\right) a = -\omega^2 x_0 \cos \omega t$$ Refer to Fig. 9.9

Also,

$$v = -\omega x_0 \sin \omega t$$

$$= -\omega x_0 \left(\pm\sqrt{1-\cos^2 \omega t}\right)$$

$$= \pm\omega\sqrt{x_0^2 - x_0^2 \cos^2 \omega t}$$

$$v = \pm\omega\sqrt{x_0^2 - x^2}$$ Refer to Fig. 9.10

and

$$a = -\omega^2 x_0 \cos \omega t = -\omega^2 x$$ Refer to Fig. 9.11

❖ Graphical Analysis

The following graphs show the variation with time of the displacement, velocity and acceleration of N in Fig. 9.3 for two complete cycles.

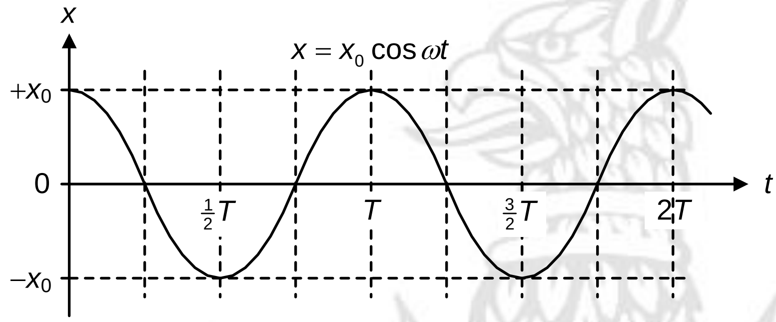


Fig. 9.7

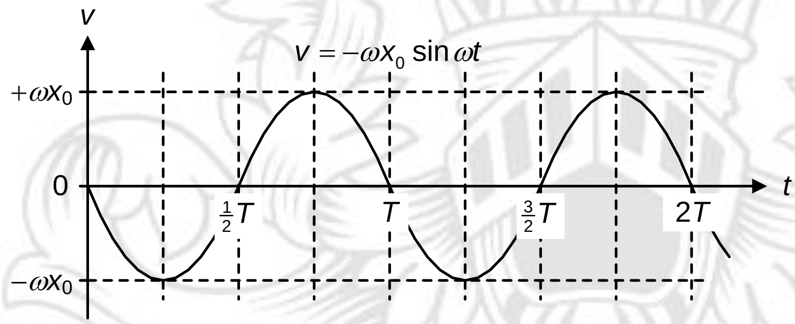


Fig. 9.8

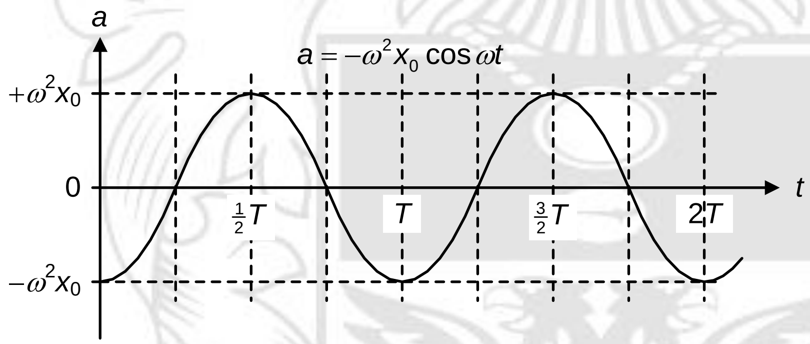


Fig. 9.9

Phase	**Description**
1st quarter of cycle	When N is speeding up from B towards O, both *v* and *a* are directed towards O (both negative).
2nd quarter of cycle	When N is slowing down from O to A, *v* and *a* are in opposite directions: *v* is negative but *a* is positive.
3rd quarter of cycle	When N is speeding up from A towards O, both *v* and *a* are directed towards O (both positive).
4th quarter of cycle	When N is slowing down from O to B, *v* and *a* are in opposite directions: *v* is positive but *a* is negative. The cycle then repeats.

The following graphs show how the velocity *v* and acceleration *a* of N in Fig. 9.3 vary with displacement *x*.

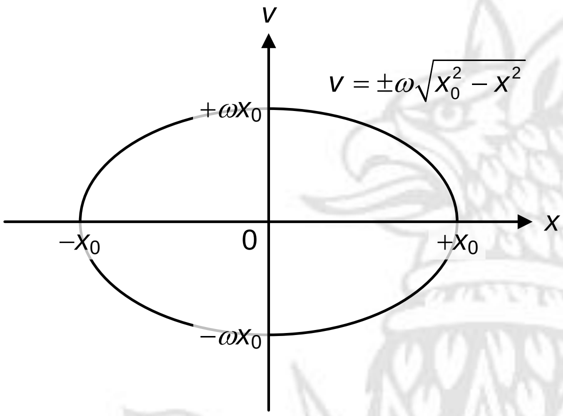


Fig. 9.10

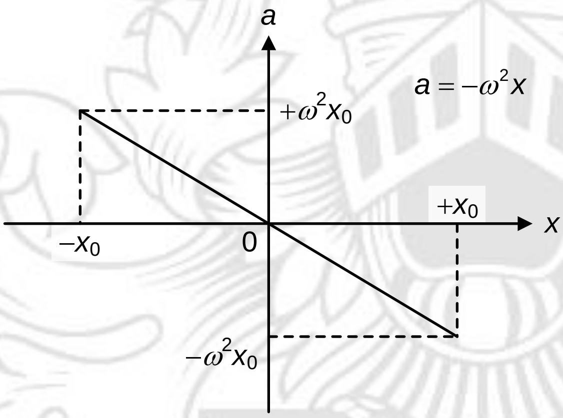


Fig. 9.11

Fig. 9.11 is the graph for the defining equation for S.H.M., $a = -\omega^2 x$.

It is the acceleration-time graph for any particle moving in S.H.M.

Example 1

A particle describes S.H.M. in which the displacement is given by

$$x = 0.050\cos(4\pi t)$$

where x is in metres and t in seconds.

Determine

(a) the amplitude of the motion,

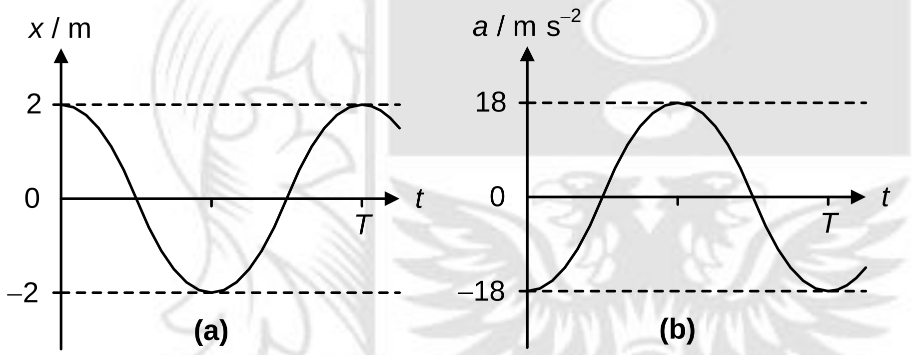
(b) the period of the motion,

(c) the maximum velocity of the particle,

(d) the maximum acceleration of the particle.

Example 2 (J82/II/10)

Figures (a) and (b) below show how the displacement x and the acceleration a of a body vary with time t when it is oscillating with simple harmonic motion.



What is the value of T?

❖ Models of S.H.M.

Two common mechanical systems are used to illustrate simple harmonic motion: mass-spring system and simple pendulum.

Horizontal Mass-Spring System

Consider a block of mass m attached to the end of a spring of negligible mass and force constant k, with the block free to move on a horizontal frictionless surface. When the spring is neither stretched nor compressed it is at its equilibrium position as shown in Fig. 9.12.

Horizontal mass-spring

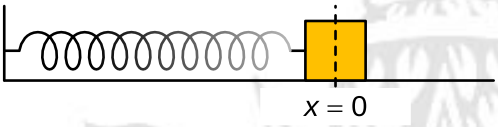


Fig. 9.12

The block is displaced a distance x to the right in Fig. 9.13.

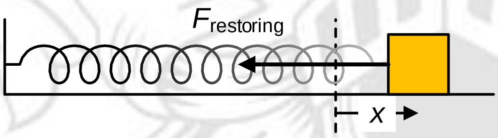


Fig. 9.13

The restoring force exerted towards the left by the spring on the block is

$$F_{restoring} = -kx$$

The negative sign implies that the force is directed towards the left.

This is the resultant force acting on the block.

Applying Newton's 2nd law,

$$F_{restoring} = -kx = ma$$

$$a = -\left(\frac{k}{m}\right)x$$

Defining equation of S.H.M.

The negative sign implies a and x are always opposite in direction.

Since k and m are constants, $a \propto -x$. So, the block oscillates with S.H.M.

Comparing with $a = -\omega^2 x$, angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

and period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Vertical Mass-Spring System

A spring of negligible mass and force constant k is suspended vertically from the ceiling in Fig. 9.14(a).

Vertical mass-spring

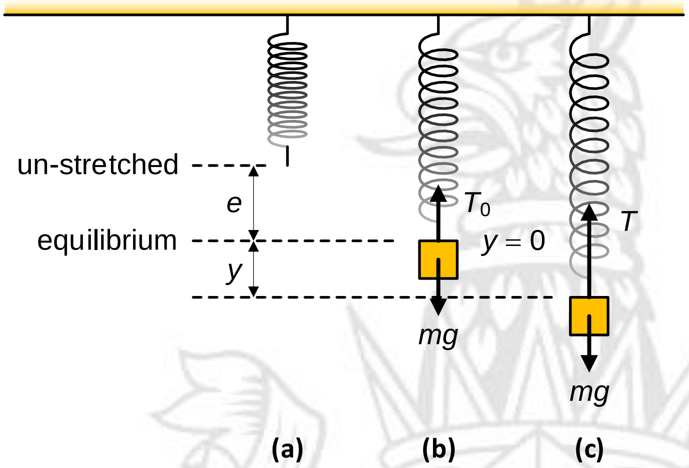


Fig. 9.14

When a block of mass m is hung from the spring and gently lowered, the spring is stretched by an extension e in Fig. 9.14(b). The block is now in equilibrium and this is the equilibrium position of the mass-spring system.

Since the mass is in equilibrium, the tension T_0 in the spring must balance the weight of the block so that the resultant force is zero.

$$mg = T_0 = ke$$

The block is then given an additional downward displacement y in Fig. 9.14(c). At this position, the tension in the spring is

$$T = k(e + y)$$

Taking downwards to be positive, the resultant restoring force is

$$F_{restoring} = mg - T = mg - k(e + y) = mg - ke - ky = -ky$$

Since $mg = ke$.

Applying Newton's 2nd law,

$$F_{restoring} = -ky = ma$$

$$a = -\left(\frac{k}{m}\right)y$$

Defining equation of S.H.M.

The negative sign implies a and x are always opposite in direction.

Since k and m are constants, $a \propto -x$. So, the block oscillates with S.H.M.

Comparing with $a = -\omega^2 x$, angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

and period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Comparing the horizontal and vertical mass-spring systems, it can be concluded that the motion of the mass-spring system is **only dependent** on the **mass** of the block and **force constant** of the spring.

Simple Pendulum

A simple pendulum consists of a bob suspended by a light string. The forces acting on the bob are the tension T in the string and the weight mg of the bob.

When the bob is displaced by a small angle $\theta < 10°$, it is displaced by a distance $s = L\theta$ as shown in Fig. 9.15(a).

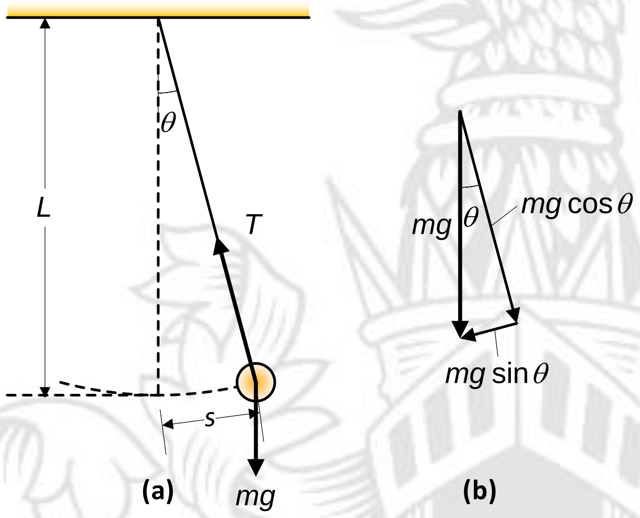


Fig. 9.15

The restoring force is the tangential component of mg:

$$F_{restoring} = -mg\sin\theta$$

Since θ is small, by small angle approximation, $\sin\theta \approx \theta$.

$$F_{restoring} \approx -mg\theta = -mg \times \frac{s}{L}$$

Applying Newton's 2nd law,

$$F_{restoring} = -mg \times \frac{s}{L} = ma$$

$$a = -\left(\frac{g}{L}\right)s$$

Defining equation of S.H.M.

The negative sign implies a and x are always opposite in direction.

Since g and L are constants, $a \propto -x$. So, the bob oscillates with S.H.M.

Comparing with $a = -\omega^2 x$, angular frequency

$$\omega = \sqrt{\frac{g}{L}}$$

and period

$$T = 2\pi\sqrt{\frac{L}{g}}$$

The motion of the system is **only dependent** on the **length** of the pendulum and the **acceleration of free fall**.

Example 3

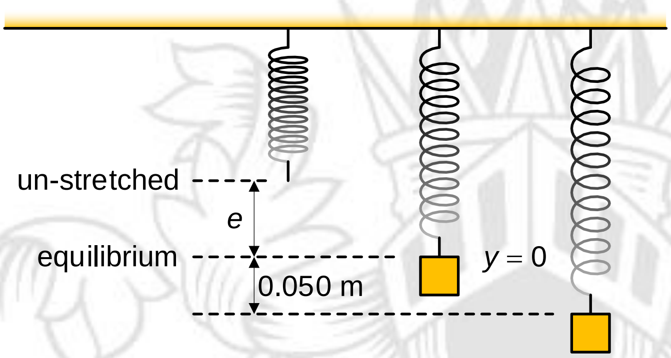
An object of mass 0.20 kg is hung from the lower end of a spring. When the object is pulled down 5.0 cm below its equilibrium position O and released, it vibrates with S.H.M. with a period of 0.40 s.

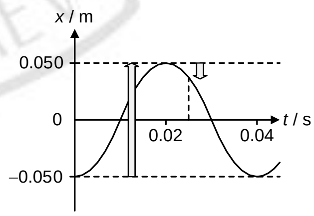
(a) What is the extension of the spring when the mass is hung at rest from the lower end of the spring?

(b) What is the speed of the mass as it passes through O?

(c) What is the magnitude of its acceleration when it is 2.5 cm above O?

(d) Through what distance will the object move in the first 0.25 s?



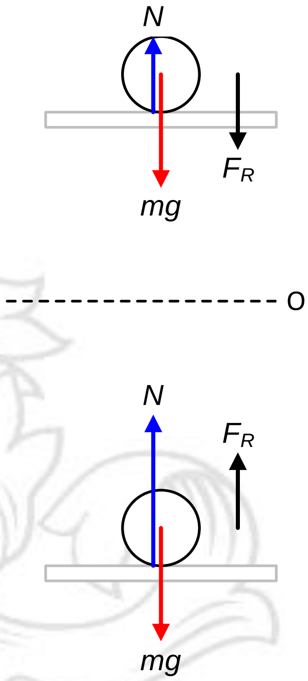


Example 4

A horizontal plate is vibrating vertically in S.H.M. at a frequency of 5.0 Hz.

What is the maximum amplitude of vibration so that fine sand on the plate always remains in contact with it?

[Solution]



We will consider three positions where the sand is above, below or at its equilibrium position O.

For the sand to oscillate in S.H.M., the resultant force F_R must always be directed towards O: below O, F_R is directed upwards while above O, F_R is directed downwards and at O, F_R is zero.

The forces acting on the sand are weight mg and normal contact force N.

Below O,

At O,

Above O,

So, we conclude that N decreases as the sand travels upward during oscillations. Lowest N occurs at its highest position.

With a sufficiently large amplitude, it is possible for N to become zero, i.e., sand loses contact with the plate.

Therefore, for the sand to remain in contact with the plate,

9.3 Energy in Simple Harmonic Motion

❖ **Changes in Energy during S.H.M.**

Consider a body of mass *m*, attached to a light spring, oscillating freely between A and B on a horizontal frictionless surface about the equilibrium position O as shown in Fig. 9.16.

This is free oscillation.

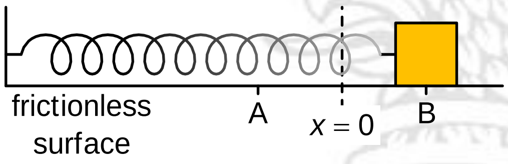


Fig. 9.16

Since there are no dissipative forces, there is no energy loss and the system will oscillate forever once it is set in motion.

During oscillations, there is a continual change of kinetic energy to potential energy and vice-versa. At any instant during the motion, the total energy of the system

$$E_T = E_k + E_p$$

remains constant with time.

❖ **Variation of Energy with Displacement from Equilibrium**

Kinetic energy E_k at displacement *x* from the equilibrium position

$$E_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\pm\omega\sqrt{x_0^2 - x^2}\right)^2$$
$$= \tfrac{1}{2}m\omega^2\left(x_0^2 - x^2\right)$$

where x_0 is the amplitude of oscillation.

Potential energy E_p at displacement *x* from the equilibrium position

$$E_p = \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2x^2$$

Total energy E_T

Since $\omega = \sqrt{k/m}$.

$$E_T = E_k + E_p = \tfrac{1}{2}m\omega^2\left(x_0^2 - x^2\right) + \tfrac{1}{2}m\omega^2x^2$$
$$= \tfrac{1}{2}m\omega^2x_0^2$$

The total energy is constant and does not depend on displacement *x*.

From the above energy relationships, we can deduce that

- $E_{k,\max} = \tfrac{1}{2}m\omega^2x_0^2$ when $x = 0$ (equilibrium) since $E_p = 0$.
- $E_{p,\max} = \tfrac{1}{2}m\omega^2x_0^2$ when $x = \pm x_0$ (amplitude positions) since $E_k = 0$.

Fig. 9.17 shows the variation with displacement x of E_k, E_p and E_T with displacement x:

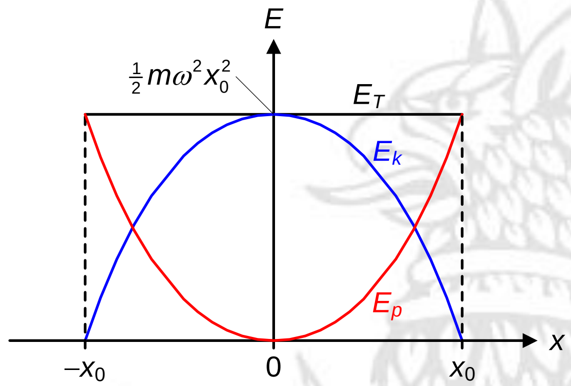


Fig. 9.17

❖ Variation of Energy with Time

Assuming that the body is at B at $t = 0$.

Kinetic energy E_k at time t

$$E_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(-\omega x_0 \sin\omega t\right)^2$$

$$= \tfrac{1}{2}m\omega^2 x_0^2 \sin^2 \omega t$$

Potential energy E_p at time t

$$E_p = \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2\left(x_0 \cos\omega t\right)^2$$

$$= \tfrac{1}{2}m\omega^2 x_0^2 \cos^2 \omega t$$

Since $\omega = \sqrt{k/m}$.

Total energy E_T

$$E_T = E_k + E_p = \tfrac{1}{2}m\omega^2 x_0^2 \sin^2 \omega t + \tfrac{1}{2}m\omega^2 x_0^2 \cos^2 \omega t$$

$$= \tfrac{1}{2}m\omega^2 x_0^2$$

Fig. 9.18 shows the variation with time of E_k, E_p and E_T for the case $x = x_0 \cos\omega t$.

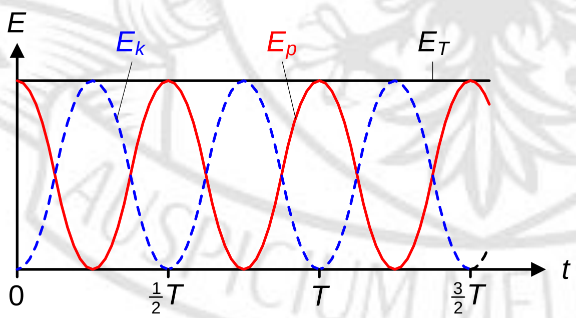


Fig. 9.18

Example 5 (Refer to Fig. 9.1.12)

A mass of 0.50 kg is attached to a light spring which has a force constant of 20 N m^{-1}. The mass is displaced of distance 5.0 cm below the equilibrium position and then released. Calculate

(a) the maximum value of the potential energy of the oscillating system, assuming it is zero at the equilibrium position.

(b) the maximum velocity of the mass,

(c) the distance from the equilibrium position when the kinetic energy is one quarter of its maximum value.

[Solution]

(a) Maximum P.E. occurs at amplitude positions

The potential energy of the system is the sum of the gravitational P.E. and elastic P.E.

(b) K.E. is maximum when the mass passes the equilibrium position where the P.E. is zero. So, maximum K.E. is 0.025 J.

Since $E_{K,\max} = \frac{1}{2}mv_0^2$,

(c) The kinetic energy of the mass is a function of its displacement y from equilibrium position:

Rearranging,

Hence, the distance from equilibrium is 0.043 m.

Note: Refer to tutorial self-practice SP10 for the energy-displacement graphs.

9.4 Damped Oscillations

❖ Damping

The oscillatory motions that have been discussed so far are free oscillations of **ideal** systems.

Real systems experience dissipative forces, such as friction and viscous forces, during oscillations. As a result, the amplitude and total energy of the system decreases with time. Such oscillations are said to be **damped**.

If the resistive force is proportional to the speed of the oscillating body, the decrease in amplitude is **exponential**.

A full mathematical analysis of damped harmonic motion shows that the frequency of oscillations of damped S.H.M is **lower than** its **natural frequency**.

Natural frequency is the frequency of oscillation of free oscillations.

> **Damping**
>
> **Damping** is the process whereby *energy is removed from an oscillating system*.

❖ Degrees of Damping

In practice, the motion of an oscillating system depends on the **degree of damping** which in turn is dependent on the magnitude of the resistive force. In some cases, damping may prevent the system from oscillating and the body simply returns to its equilibrium position.

Light damping occurs when the resistive force is small. This results in oscillations where the amplitude decays exponentially with time. The frequency of oscillations is slightly lower than its natural frequency.

For light damping, we can assume the frequency of oscillation is the same as the natural frequency.

Critical damping occurs when the system experiences a stronger resistive force than light damping which prevents the system from oscillating. The system returns to the equilibrium position in the shortest time after its initial displacement.

Heavy damping occurs when the system experiences an even stronger resistive force than critical damping which prevents the system from oscillating. The system takes a longer time to return to its equilibrium position after its initial displacement.

❖ Variation of Displacement of Damped Oscillations

Fig. 9.19 show how different degrees of damping affect the displacement of an oscillating body.

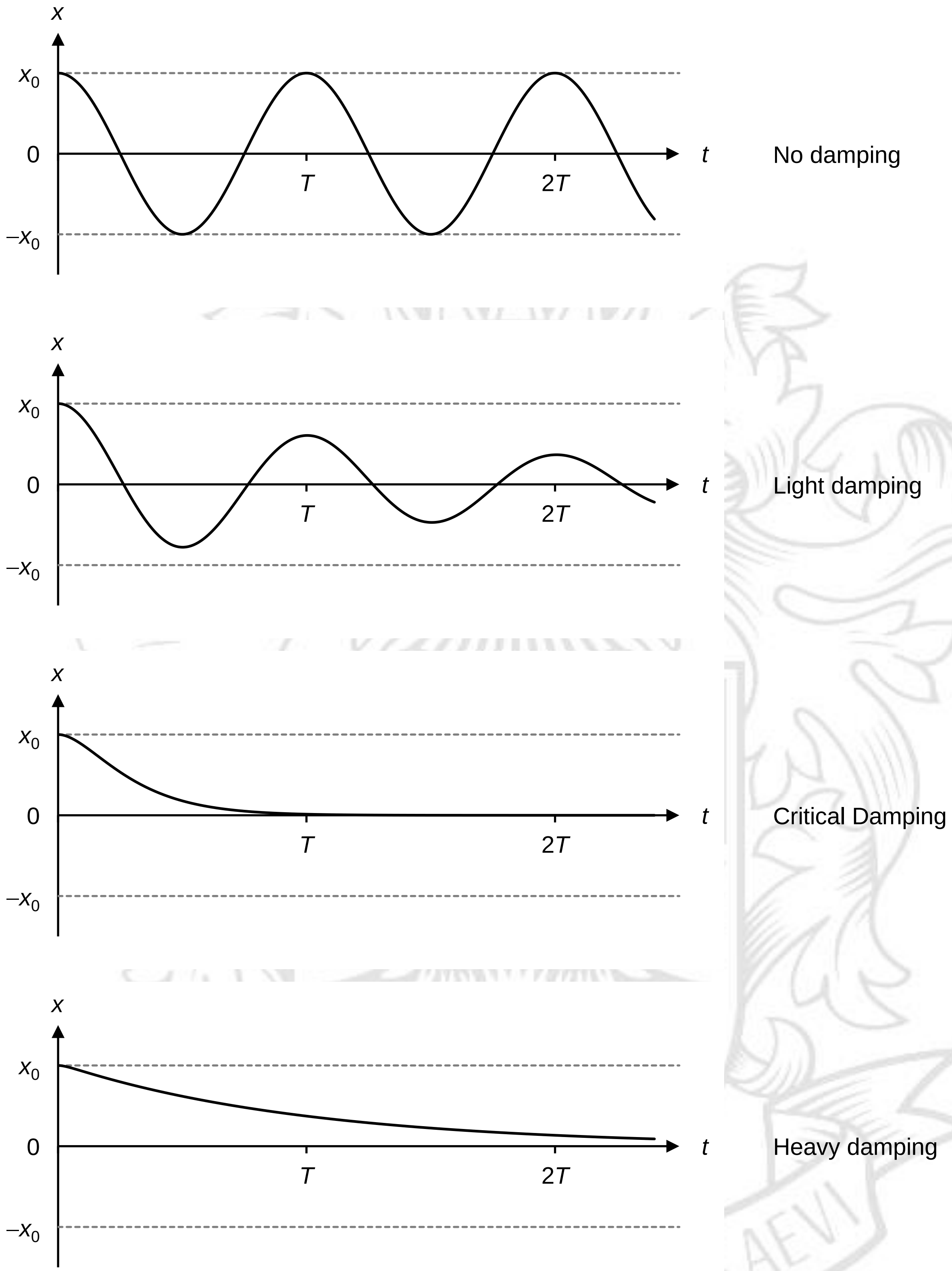


Fig. 9.19

❖ **Variation of Energy of Damped Oscillations**

Fig. 9.20 show how different degrees of damping affect the different forms of energy of an oscillating body.

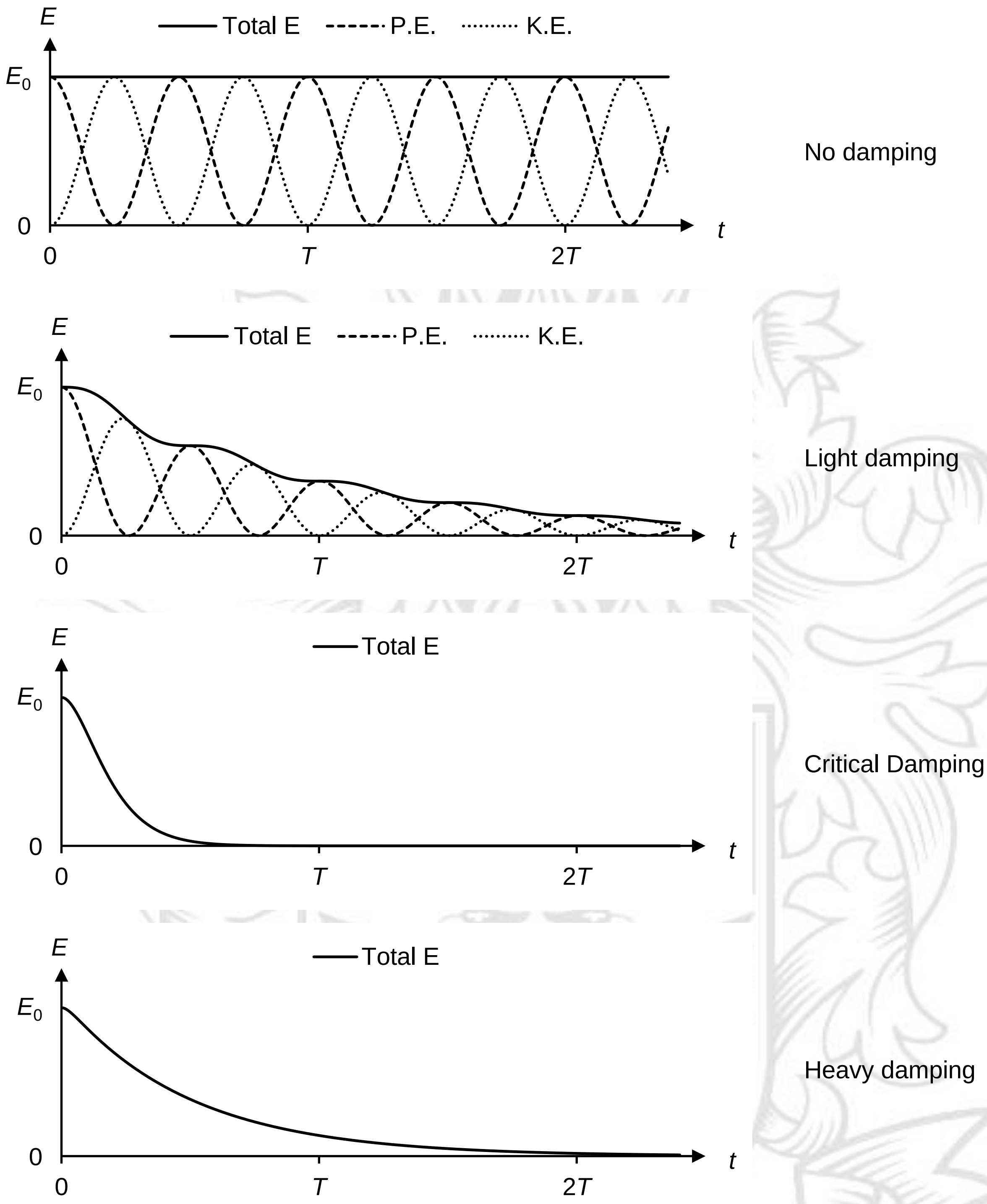


Fig. 9.20

❖ Critical Damping in Car Suspension System

The degree of damping of a mechanical system is important. Too little damping results in a large number of oscillations. Too much damping results in the system taking too long a time to return to its original position and thus cannot respond to further disturbances.

[HowStuffWorks - Car Suspension]

The suspension system of a car performs critical functions as it needs to maximise contact between the tires and the road while ensuring good handling and comfort for the passengers.

The suspension system consists of a spring and a shock absorber. The shock absorber consists of a piston that moves in a cylinder containing a viscous fluid. Holes on the piston allow it to move up and down while the viscous fluid absorbs the energies of any vertically accelerated wheels. The amount of damping can be adjusted.

Without the shock absorber, the spring, after an initial compression, will oscillate at its natural frequency until its energy is completely dissipated. A suspension built on springs alone would make for an extremely bouncy and uncomfortable ride.

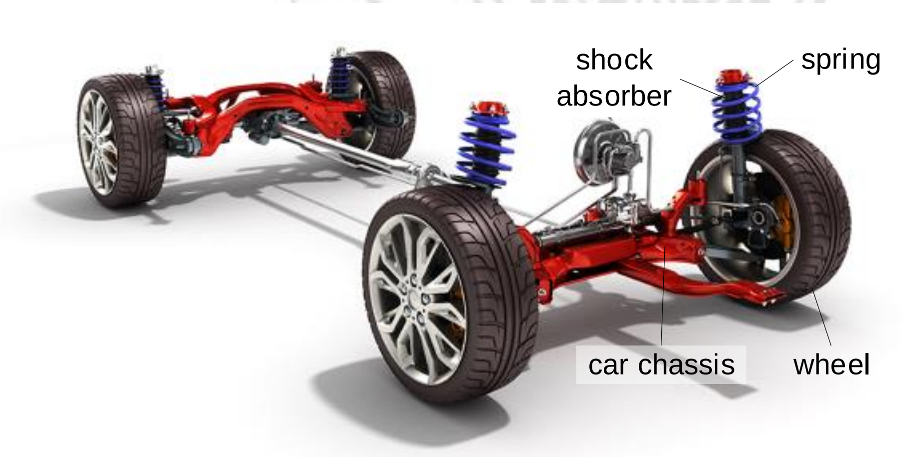


Fig. 9.21

A good suspension system is one in which the damping is critical or slightly under critical as this results in a comfortable ride and also leaves the car ready to respond to further bumps in the road quickly.

Fig. 9.22 shows how the suspension system works when going over a hump. When the car encounters a hump at P, the suspension experiences an initial compression and subsequent expansion. A short distance later at Q, the suspension settles at equilibrium. Similarly, when the car drops to road level at R, the suspension experiences an initial expansion and subsequent compression. A short distance later at S, the suspension settles at equilibrium.

A heavily damped shock absorber would take a long time to reach equilibrium. It would not be ready to respond to the sudden change in road surface.

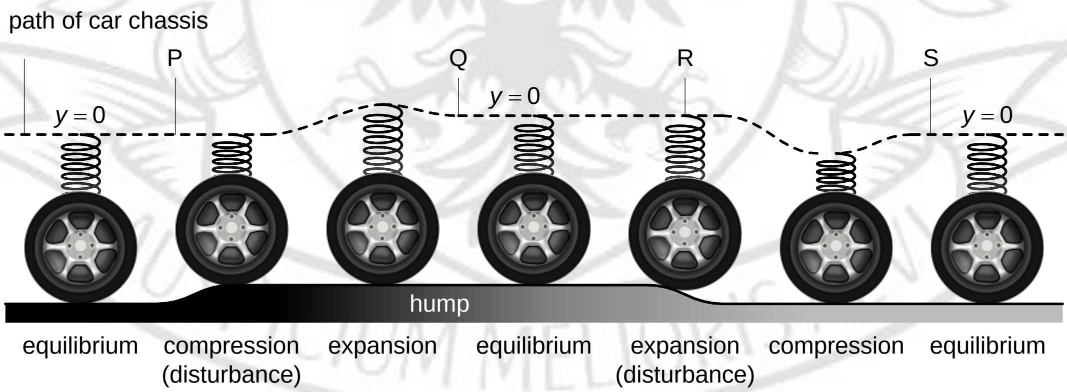


Fig. 9.22

9.5 Forced Oscillations and Resonance

❖ Forced Oscillations

Forced oscillations are produced when a body is subjected to a periodic **external driving force** and is made to oscillate at the frequency of the driving force, which may not be its natural frequency.

The system providing this periodic driving force is known as the **driver**.

❖ Resonance

Fig. 9.23 shows a setup of Barton's pendulums which is used to demonstrate forced oscillations and resonance**.** It consists of several very light pendulums (made from paper cones) of varying length (A, B, C, D and E) and one massive pendulum X. All the pendulums are suspended from the same string.

Barton's Pendulum

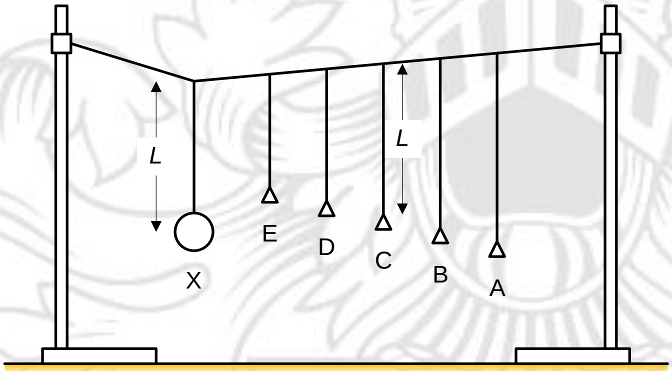


Fig. 9.23

The massive pendulum X (driver) is made to oscillate at its natural frequency and the motion of the other pendulums are observed. Initially, as energy is transferred from the driver pendulum to the driven pendulums, their motions are very chaotic. Eventually, all driven pendulums oscillate at the driver's frequency, which is not equal to the individual natural frequencies.

Every pendulum has its own natural frequencies.

Pendulum C, having the same length and hence the same natural frequency as pendulum X, is observed to oscillate with the largest amplitude. The other driven pendulum oscillates with smaller amplitude than pendulum C. Pendulum C is said to experience **resonance**.

At resonance, the maximum energy is being transferred from the driver (X) to the driven system (C).

> **Resonance**
>
> **Resonance** occurs when the frequency of the driving force is close to or at the natural frequency of the driven system. There is maximum transfer of energy from the driver and the system responds with maximum amplitude.

❖ Variation of Amplitude with Driving Frequency

Fig. 9.24 shows how the amplitude x_0 of forced oscillation varies with the driving frequency *f* and the degree of damping. These curves are also known as **frequency response curve**.

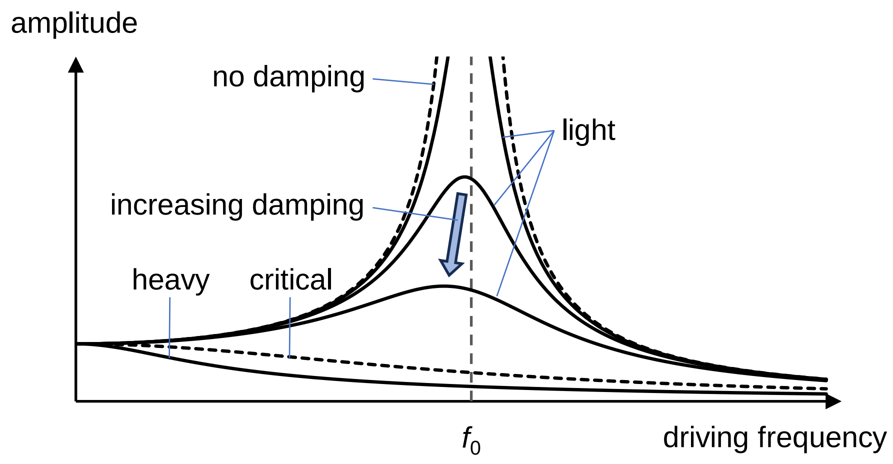


Fig. 9.24

When the oscillations of the driven system have stabilised, the following observations are made:

1. The amplitude of a forced oscillation depends on
   - the damping of the system,
   - the relative values of the driving frequency *f* and the natural frequency f_0 of the driven system.
2. The oscillations with maximum amplitude (resonance) occur when the driving frequency *f* is close to or at the natural frequency f_0 of the system.

The sharpness of the frequency response curves at resonance is dependent on the degree of damping:

1. When there is no damping, resonance occurs at driving frequency $f = f_0$. The amplitude is infinite.
2. When damping is light, resonance occurs at $f \approx f_0$. The amplitude is large but decreases rapidly when the driving frequency deviates from the natural frequency f_0 of the driven system. The resonance peak is sharp.

   Resonance frequency is slightly lower than f_0.
3. With increasing damping (still light damping), resonance occurs at **lower frequency** and with **smaller maximum amplitude**. The resonance peak broadens.
4. When damping approaches critical, the frequency response curve is relatively flat.

❖ Applications where Resonance is Useful

Resonance phenomena occur with all types of vibrations or waves: there is mechanical resonance, electromagnetic resonance, acoustic resonance, nuclear magnetic resonance (NMR), orbital resonance, electron spin resonance (ESR) and resonance of quantum wave functions.

Applications of Resonance

Electromagnetic Resonance

The electrons in a radio receiving aerial are forced to vibrate by the radio wave passing through the aerial. When we tune the receiver, we are making the natural frequency of the electrical circuit equal to the frequency of the signal. Hence the tuning circuitry make use of resonance to isolate and amplify the signal of the required frequency.

How Radio Works

The principle of electromagnetic resonance also applies to microwave cooking. Microwaves are electromagnetic waves with a frequency between radio waves and infrared. In a microwave oven, the energy of microwaves is absorbed by the water molecules in most types of food through resonance. This process heats up the food efficiently and quickly.

How Microwave Cooking Works

Acoustic Resonance

The loudness (intensity) of a note produced by a string instrument (such as guitar) is achieved by coupling the vibrating string to a resonator (body). The air inside the resonator and the surface of resonator vibrate together to produce larger vibrations in the surrounding air than would be produced by the string alone.

Nuclear Magnetic Resonance

Energy from strong oscillating magnetic fields is used to cause the nuclei of atoms to oscillate and emit radio frequency signals. In any given molecule there will be many resonant frequencies, and whenever resonance occurs energy is absorbed. The pattern of energy absorption can be used to detect the presence of particular molecules within any specimen and biochemists are using the technique to study complex molecules and the part they play in biological processes.

How MRI Works

Nuclear magnetic resonance is also being used instead of X-rays as an imaging system (MRI) in the medical field. The radio frequency signals emitted are made to encode position information by varying the magnetic field. The contrast between different tissues is determined by the rate at which excited atoms return to the equilibrium state. One major advantage of magnetic resonance used in this way is that no ionising radiation is involved.

❖ Circumstances in which Resonance is Not Useful

All mechanical structures have one or more natural frequencies, and if a structure is subjected to strong external driving force that matches one of these frequencies, resonance occurs and the resulting oscillations of the structure may rupture it.

Tacoma Bridge

At Angers, France in 1850, a French infantry battalion was marching in a synchronous manner over a suspension bridge when it collapsed, resulting in the deaths of 220 men. Since that time, it has been common practice to order soldiers to break step when crossing a bridge. The soldiers' marching caused sufficient vibration and twisting to break the bridge at resonance.

A more modern bridge disaster occurred in 1940 when wind-induced oscillations caused the collapse of the Tacoma Narrows Bridge in the U.S. state of Washington. The bridge's natural mode of vibration coupled with the wind forces, produced unstable oscillations with increased amplitude that were beyond the strength of the suspender cables.

Resonance was also the cause for the collapse of some buildings during a major earthquake in Mexico in 1985. Many intermediate-height buildings collapsed because their natural frequency matched that of the seismic waves, whereas taller or shorter buildings were unaffected.

A more mundane example of resonance is the way in which the bodywork of a bus can vibrate violently at a particular engine speed.

As such, engineers have to carry out elaborate vibration tests on model structures of, for example, bridges, buildings and aeroplanes before they are satisfied that the design features will prevent extremely large amplitudes from building up in the system.

Summary of S.H.M Equations

If at $t = 0$,	$x = x_0$	$x = 0$
displacement as function of time	$x = x_0 \cos \omega t$	$x = x_0 \sin \omega t$
velocity as function of time	$v = -x_0 \omega \sin \omega t$	$v = x_0 \omega \cos \omega t$
acceleration as function of time	$a = -x_0 \omega^2 \cos \omega t$	$a = -x_0 \omega^2 \sin \omega t$

Velocity as function of displacement	$v = \pm \omega \sqrt{\left(x_o{}^2 - x^2\right)}$
Acceleration as function of displacement (Defining equation for S.H.M.)	$a = -\omega^2 x$
Maximum velocity	$v_0 = \pm \omega x_0$
Maximum acceleration	$a_0 = \pm \omega^2 x_0$
Kinetic energy	$E_k = \frac{1}{2} m\omega^2 \left(x_0^2 - x^2\right)$; $E_k = 0$ (at amplitude positions)
Potential energy (for total P.E. = 0 at equilibrium)	$E_p = \frac{1}{2} m\omega^2 x^2$; $E_p = 0$ (at equilibrium)
Total energy	$E = E_k + E_p = \frac{1}{2} m\omega^2 x_0{}^2$
Period, frequency, angular frequency	$T = \frac{1}{f}$ $\omega = \frac{2\pi}{T} = 2\pi f$
Spring-mass system	$\omega = \sqrt{\frac{k}{m}}$ $T = 2\pi \sqrt{\frac{m}{k}}$
Simple pendulum	$\omega = \sqrt{\frac{g}{L}}$ $T = 2\pi \sqrt{\frac{L}{g}}$

9.6 Appendix

❖ **Relationship between Uniform Circular Motion and S.H.M.**

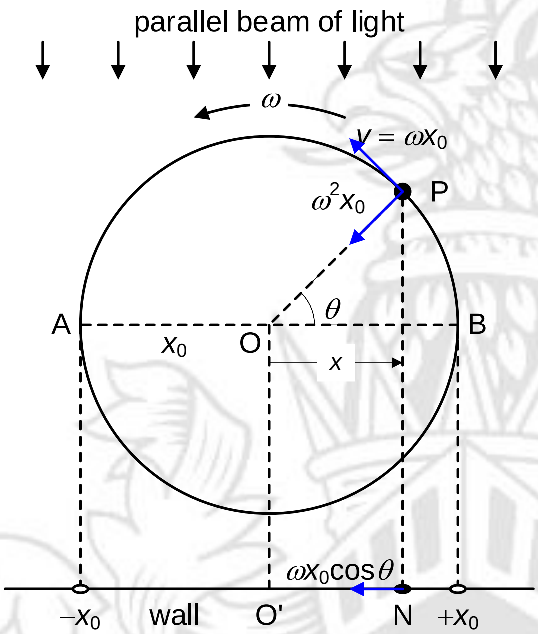


Fig. 9.25

A parallel beam of light is shone on particle P which casts a shadow N along the wall. When particle P moves in a circle of radius x_0 at a steady angular velocity ω, N oscillates to and fro along the wall.

Assuming that $\theta = 0$ ($x = x_0$) when $t = 0$, then

$$x = x_0 \cos\theta = x_0 \cos\omega t$$

The centripetal acceleration of P is $\omega^2 x_0$, directed towards O.

The acceleration of N is the component of the acceleration along the wall:

$$a = -x_0\omega^2 \cos\theta$$

The negative sign indicates that the acceleration of N is directed towards O'.

We can write

$$a = -\omega^2 x_0 \cos\theta = -\omega^2 x_0 \cos\omega t = -\omega^2 x$$

Thus, N is oscillating in S.H.M.

This model shows that when a particle moves in a uniform circular motion, the shadow (projection) of the particle oscillates in S.H.M. with an amplitude equal to the radius of circular motion.

❖ Phase Constant

The general equation of motion for S.H.M is

$$x = x_0 \cos(\omega t + \phi)$$

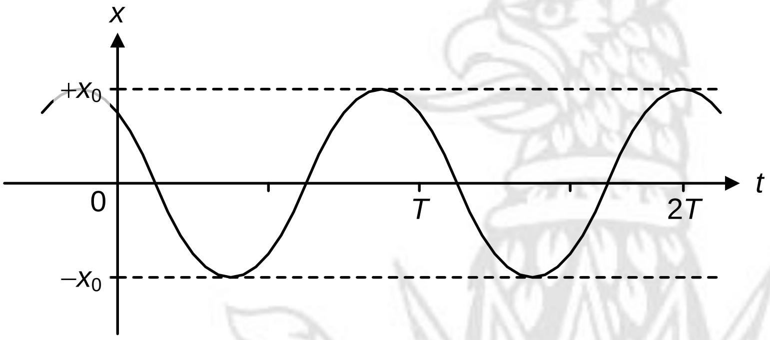


Fig. 9.26

ϕ is the **phase constant** (initial phase angle).

The **phase** of an oscillation is the quantity $\omega t + \phi$.

x_0 and ϕ are uniquely determined by the position and velocity of the particle at $t = 0$. For example, if the particle is at $x = x_0$ when $t = 0$, then $\phi = 0$.

Initial State	$x = 0$ $v = +v_0$	$x = x_0$ $v = 0$	$x = 0$ $v = +v_0$	$x = -x_0$ $v = 0$
Phase Constant / rad	$-\pi/2$	0	$\pi/2$	π or $-\pi$

❖ Phase Difference

The graph of $x = x_0 \cos(\omega t + \phi)$ is the graph of $x = x_0 \cos \omega t$ displaced to the **left** by a time interval of ϕ/ω.

If the motion of a particle P is described by $x = x_0 \cos(\omega t + \phi)$ is out-of-phase by an angle $+\phi$ (equivalent time interval of ϕ/ω) relative to particle Q described by $x = x_0 \cos \omega t$. The $+$ sign implies that P **leads** Q by a time interval of ϕ/ω.

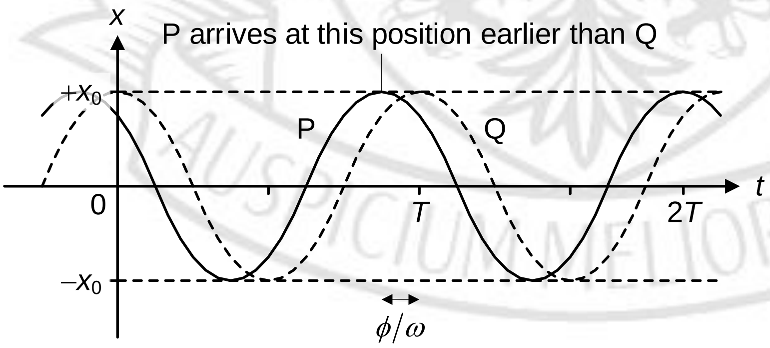


Fig. 9.27

If the motion of a particle R described by $x = x_0 \cos(\omega t - \phi)$ is out-of-phase by an angle $-\phi$, relative to particle Q described by $x = x_0 \cos \omega t$. The – sign implies that R **lags** Q by a time interval of ϕ/ω.

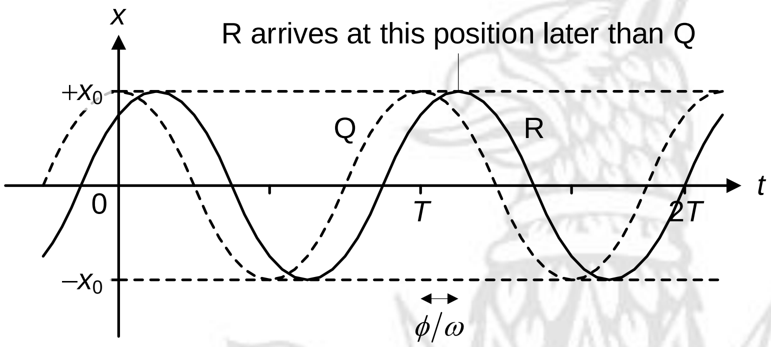


Fig. 9.28

Phase difference between two particles in S.H.M. is the fraction of a complete oscillation one is ahead of the other. It is usually expressed as an angle in radians.