Title	Secondary School Additional Mathematics Materials Version 15
Author	AprilDolphin – Lim Wang Sheng
Date	12/6/2026

Title	Surds Manipulation
Author	-
Date	31/12/2022

Basic Surds Rules to Understand before Proceeding

Given the following expression can be written in the following form

$g\sqrt{ab^2}$

It can be rewritten as the following

$gb\sqrt{a}$

Example 1.1

$\sqrt{8}$ can be decomposed into the following

$\sqrt{2 \times 4}$

Since $4 = 2^2$, we can rewrite in the following manner: $2\sqrt{2}$

Law of Surds (Only involving square roots)
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
$a\sqrt{b} \times c\sqrt{d} = ac\left(\sqrt{bd}\right)$
$\sqrt{a^2} = a$
$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
$m\sqrt{a} + n\sqrt{a} = \sqrt{a}(m+n)$
$m\sqrt{a} - n\sqrt{a} = \sqrt{a}\,(m-n)$

Rules of rationalizing the denominator in surds calculation and manipulation

Rule 1

If expression is in the following form

$$\frac{a}{g\sqrt{b}}$$

Multiply by the denominator to both numerator and denominator to get the following

$$\frac{a}{g\sqrt{b}} \times \frac{g\sqrt{b}}{g\sqrt{b}}$$

Rule 2.
If the expression is in the following form or show some near resemblance to the following form

$$\frac{a+g\sqrt{b}}{c-d\sqrt{p}}$$

Find the conjugate value of the denominator whereby the sign in between $c-d\sqrt{p}$ is flipped to positive and multiply conjugate value to both numerator and denominator to get the following

$$\frac{a+g\sqrt{b}}{c-d\sqrt{p}}\times\frac{c+d\sqrt{p}}{c+d\sqrt{p}}$$

Rule 3.
If the expression is in the following form or show some near resemblance to the following form

$$\frac{a+g\sqrt{b}}{c+d\sqrt{p}}$$

Find the conjugate value of the denominator whereby the sign in between $c+d\sqrt{p}$ is flipped to negative and multiply conjugate value to both numerator and denominator to get the following

$$\frac{a+g\sqrt{b}}{c+d\sqrt{p}}\times\frac{c-d\sqrt{p}}{c-d\sqrt{p}}$$

Question 1.
1.1 Simplify the following expression
(a) $11\sqrt{7}+6\sqrt{28}-5\sqrt{63}$
(b) $(4\sqrt{3}-\sqrt{2})(\sqrt{3}-5\sqrt{2})$

(a) $11\sqrt{7} + 6\sqrt{28} - 5\sqrt{63}$

Rewrite as $11\sqrt{7} + 6\sqrt{7 \times 4} - 5\sqrt{9 \times 7}$

Once again can be rewritten as $11\sqrt{7} + 6(2)\sqrt{7} - 5(3)\sqrt{7}$

We then proceed to simplify the expression as

$(11 + 12 - 15)\sqrt{7} = 8\sqrt{7}$

(b) $\left(4\sqrt{3} - \sqrt{2}\right)\left(\sqrt{3} - 5\sqrt{2}\right)$

Expand the expression into the following

$4\sqrt{3}\left(\sqrt{3}\right) - \sqrt{2}\,\left(\sqrt{3}\right) - 5\sqrt{(2)}\left(4\sqrt{3}\right) + 5\sqrt{2}\left(\sqrt{2}\right)$

$4(3) - \sqrt{6} - 20\sqrt{6} + 5(2) = 12 + 10 - \sqrt{6} - 20\sqrt{6} = 22 - 21\sqrt{6}$

1.2 Rationalize the denominator of the following

(a) $\frac{12}{\sqrt{3}}$

(b) $\frac{2-\sqrt{7}}{3+4\sqrt{7}}$

(c) $\frac{1}{3-\sqrt{5}}$

Solutions

(a) $\frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

(b) $\frac{2-\sqrt{7}}{3+4\sqrt{7}} \times \frac{3-4\sqrt{7}}{3-4\sqrt{7}} = \frac{(2-\sqrt{7})(3-4\sqrt{7})}{(3+4\sqrt{7})(3-4\sqrt{7})}$

$$= \frac{6 - 3\sqrt{7} - 8\sqrt{7} + 28}{9 - 4^2(7)}$$

$$= \frac{6 + 28 - 3\sqrt{7} - 8\sqrt{7}}{-103} = \frac{\left(34 - 11\sqrt{7}\right)}{-103} = \frac{-34 + 11\sqrt{7}}{103} = \frac{11\sqrt{7}}{103} - \frac{34}{103}$$

(c) $\frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{3+\sqrt{5}}{3^2-5}$

$$= \frac{3 + \sqrt{5}}{4} = \frac{3}{4} + \frac{\sqrt{5}}{4}$$

2.3[Equations Involving Surds]
Step by Step Instructions for solving such equations

1. Rearrange the equation (if necessary) such that all square-roots are on the left-hand-side and all non-square-roots are on the right-hand-side.
2. Square both sides and then use the usual method you learn during Elementary Mathematics to solve equation (i.e. Rearrange and then use Quadratic Formula, etc.)
3. Substitute answers back into the original questions and reject values that don't make sense (i.e. square root of negative values, etc.)
4. Use the remaining values as answers. I typically leave the answers in surd form for Additional Mathematics paper unless the paper specifies something else.

(a)$\sqrt{7x+5} = x+1$
(b) $5\sqrt{5x+9} - 4x - 3 = 0$

(a) $\left(\sqrt{7x+5}\right)^2 = (x+1)^2$
$7x + 5 = x^2 + 2x + 1$
$0 = x^2 + 2x + 1 - 7x - 5$
$0 = x^2 - 5x - 4$

After using the quadratic formula, we get the following
$x = \frac{5}{2} - \frac{\sqrt{41}}{2}\ OR\ x = \frac{5}{2} + \frac{\sqrt{41}}{2}$

(b) $5\sqrt{5x+9} - 4x - 3 = 0$

Rearrange the equation such that square roots are on one side and non-square-roots are on the other side.
$5\sqrt{5x+9} = 4x + 3$
$\left(5\sqrt{5x+9}\right)^2 = (4x+3)^2$
$25(5x+9) = (4x)^2 + 2(3)(4x) + 9$
$125x + 225 = 16x^2 + 24x + 9$
$0 = 16x^2 + 24x - 125x + 9 - 225$
$0 = 16x^2 - 101x - 216$
$x = -\frac{27}{16}\ OR\ x = 8$

After substituting the value $x = -\frac{27}{16}$ into $5\sqrt{5x+9} - 4x - 3 = 0$

$5\sqrt{5\left(-\frac{27}{16}\right) + 9} - 4\left(-\frac{27}{16}\right) - 3 = 7.5$

Literally implies that $7.5 = 0$ (Reason for rejecting $x = -\frac{27}{16}$, substituting the value into the question yield illogical results.)

After substituting value $x = 8$ into $5\sqrt{5x+9} - 4x - 3 = 0$

$5\sqrt{5(8) + 9} - 4(8) - 3 = 0$

When we worked out the left side of the equation, we get $0 = 0$, which is within mathematical logic, therefore, we accept this as the only valid answer and thus $x = 8$

2.4 [Equality of Surds]

(a) Given that $a + b\sqrt{2} = \left(3 - \sqrt{2}\right)^2 - \frac{8}{1-\sqrt{2}}$

Find the value for a and b

$$\left(3 - \sqrt{2}\right)^2 - \frac{8}{1 - \sqrt{2}} \times \frac{1 + \sqrt{2}}{1 + \sqrt{2}}$$

$$= 9 - 2(3)\left(\sqrt{2}\right) + 2 - \frac{\left[8\left(1 + \sqrt{2}\right)\right]}{\left(1 - \sqrt{2}\right)\left(1 + \sqrt{2}\right)}$$

$$= 9 - 6\sqrt{2} + 2 - \frac{\left[8\left(1 + \sqrt{2}\right)\right]}{\left(1 - \sqrt{2}\right)\left(1 + \sqrt{2}\right)}$$

$$= 11 - 6\sqrt{2} - \frac{8+8\sqrt{2}}{1^2-2}$$

$$= 11 - 6\sqrt{2} + 8 + 8\sqrt{2}$$

$$= 19 + 2\sqrt{2}$$

$$19 + 2\sqrt{2} = a + b\sqrt{2}$$

Therefore $b = 2$ and $a = 19$

Title	Quadratic Functions and Quadratic Equation
Date	3/1/2023
Author	-

Finding minimum point or maximum point of a Quadratic Function by completing the square method

Step 1. Determine if the Quadratic Function in question has a minimum point or a maximum point, it can be done after equating the quadratic function to zero and rearranging in the following form

$$ax^2 + bx + c = 0$$

If $a > 0$ the quadratic function in question has a minimum point
If $a < 0$ the quadratic function in question has a maximum point

Step 2. Apply completing the square method

1. Factor the value a out of the equation
   Given an equation $2x^2 - 10x - 18 = 0$
   $2(x^2 - 5x) - 18 = 0$
2. Add $\left(\frac{b}{2a}\right)^2$ into the bracket and deduct $\left(\frac{b^2}{4a}\right)$ from outside the bracket
   $$2\left[x^2 - 5x + \left(\frac{10}{2(2)}\right)^2\right] - 18 - \frac{10^2}{4(2)}$$
   $$2\left(x - \frac{5}{2}\right)^2 - \frac{61}{2}$$
3. In this case, x-coordinate value of the minimum point is $-\frac{b}{2a}$ which is $\frac{5}{2}$
   The y coordinate value in this case is $-\frac{61}{2}$

Basic Concepts of Discriminant to Understand Before Proceeding

Any quadratic equation being rearranged in the following form $ax^2 + bx + c = 0$ has the values has of $D = b^2 - 4ac$ as the discriminant value.

Condition of Discriminant	Consequences and Interpretation
$D > 0$	Quadratic Equation has two real and distinct roots.
$D = 0$	Quadratic Equation has real and equal roots.
$D \geq 0$	Quadratic Equation has real roots in general.
$D < 0$	Quadratic Equation has unreal roots. (No real roots)

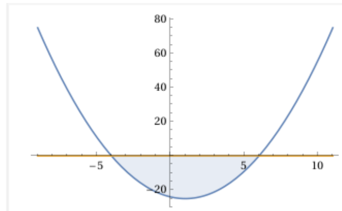
Discriminant Manipulation
Question 1.
Find the range of values of k for which the expression $3x^2 + 6x + k$ is always positive for all real values of x.
[When you see the following keywords: “Always Positive”, “Always Negative”, “No Real Roots”, it implies the quadratic graph in question will not have contact with the x −axis, thus having no real roots and implying $b^2 - 4ac < 0$]

Knowing such information, we can proceed to find values of k that can satisfy the question demands of having no real roots at all.

$b^2 - 4ac = 6^2 - 4(3)(k)$
$6^2 - 4(3)(k)$
$36 - 12k < 0$
$-12k < -36$
$k > 3$

Question 2.
The expression $(k+3)x^2 + 6x + k = 5$ has two distinct solutions for x
(a) Show that k satisfy $k^2 - 2k - 24 < 0$
(b) Find the set of possible value(s) of k
[When you see the following keywords: “pass through x-axis at two distinct points”, “has two distinct solutions” or anything similar, it means the quadratic graph in question has two real and distinct roots and thus, $b^2 - 4ac > 0$.]
2(a)
$(k+3)x^2 + 6x + (k-5) = 0$
$6^2 - 4(k+3)(k-5) > 0$
$36 - 4(k^2 + 3k - 5k - 15) > 0$
$36 - 4k^2 - 12k + 20k + 60 > 0$
$36 - 4k^2 + 8k + 60 > 0$
$96 - 4k^2 + 8k > 0$
$-4k^2 + 8k + 96 > 0$
$-k^2 + 2k + 24 > 0$
$k^2 - 2k - 24 < 0$ (Shown)

2(b)
$k^2 - 2k - 24 < 0$

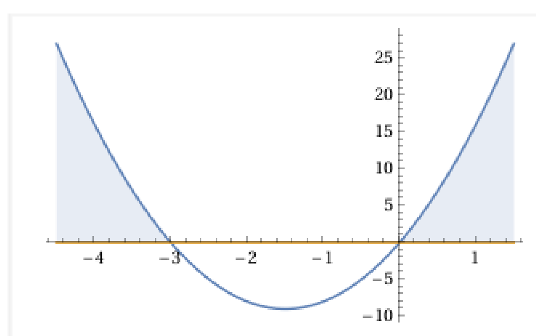


$(k-6)(k+4)<0$
$k>-4\ OR\ k<6$
$-4<k<6$

Question 3
Find the values of p for which $3x^2+2px-p$ has
(a) Real and equal roots
(b) Distinct roots
[When you see the following keywords: “has only 1 solution”, “real and equal roots”, “repeated roots” or anything similar, the quadratic graph have contact with the x −axis one time and thus $b^2-4ac=0$]

3(a)
$(2p)^2-4(3)(-p)=0$
$4p^2+12p=0$
$p^2+3p=0$
$p(p+3)=0$
$p=0\ OR\ p=-3$

3(b)
$(2p)^2-4(3)(-p)>0$
$4p^2+12p>0$
$p^2+3p>0$
$p(p+3)>0$
$P<-3$ OR $p>0$



Title	Polynomials and Partial Fractions Decomposition
Author	AprilDolphin
Date	24/9/2025

A polynomial is any form of algebraic expression that can be expressed in the form as shown below.

$a_n x^n + a_{n-1} x^{n-1} + \; a_{n-2} x^{n-2} \;\ldots\ldots a_2 x^2 + a_1 x + a_0$

Where the degree of polynomial is characterized by the highest power of within the polynomial

In any case, an expression or an algebraic function can only be called a polynomial if

- The expression or function in question only has non-negative integer exponents.

Polynomial Type Example	Name of Polynomial	Degree of polynomial
$3 + 5x^0$	Constant	0
$3x + 6$	Linear	1
$x^2 + 7x + 6$	Quadratic	2
$3x^3 - 9x + 5$	Cubic	3
$x^4 + 3x^3 + 6x - 7$	Quartic	4
$2x^5 - 6x^2 + 7$	Quintic	5

Examples of non-polynomials	Justification on why it is <u>not</u> a polynomial
$9x^2 + 9x + x^{-2}$	The expression contains negative exponents of the unknown x in the third term (x^{-2}).
$6x^{12} - 4x^{\frac{2}{7}}$	The expression contains fractional exponents of unknown x in the second term ($4x^{\frac{2}{7}}$).

Addition and Subtraction of Polynomials

To add and subtract polynomials together, we just simply combine like terms together and add them or subtract them in the following manner: Example 1: Given $P(x) = x^3 - 4x^2 + 3x + 5$ and $Q(x) = 2x^2 - 6x + 7$, evaluate $P(x) + Q(x)$ and $P(x) - Q(x)$. $P(x) + Q(x) = x^3 - 4x^2 + 2x^2 + 3x + (-6x) + 5 + 7 = x^3 - 2x^2 - 3x + 12$ $P(x) - Q(x) = x^3 - 4x^2 - 2x^2 + 3x - (-6x) + 5 - 7 = x^3 - 6x^2 + 9x - 2$

Multiplication of Polynomials

To multiply polynomials, we have to do it as in the following example. Example 2: Expand and simplify $(x + 1)(x^2 - x + 1)$ $x^3 + x^2 - x^2 + x - x + 1 = x^3 + 1$

Division of Polynomials by Long Division

Example 3:
Evaluate $(2x^3 + 5x^2 - 2x + 7) \div (x + 3)$, listing down the dividends and, the remainder if any.

$$
\begin{array}{rl}
 & x^2 - x + 1 \\
x + 3 \,\big) & 2x^3 + 5x^2 - 2x + 7 \\
 & \underline{-(2x^3 + 6x^2)} \\
 & -x^2 - 2x \\
 & \underline{-(-x^2 - 3x)} \\
 & x + 7 \\
 & \underline{-(x + 3)} \\
 & \underline{4}
\end{array}
$$

Hence the dividend is $x^2 - x + 1$ and remainder is 4

Remainder Theorem as Follows:
The remainder theorem states that if a polynomial $P(x)$ is divided by $ax - b$, the remainder of the division is given as $R = P\left(\frac{b}{a}\right)$, where R is the remainder. Specifically, if $P(x)$ is divided by $x - b$, the remainder of the division is given as $R = P(b)$.

Factor Theorem as Follows:
The factor theorem states that if $ax - b$ is indeed a factor of polynomial $P(x)$, the remainder of the division is given as $R = P\left(\frac{b}{a}\right) = 0$. Specifically, if $x - b$ is a factor of $P(x)$, the remainder of the division is given as $R = P(b) = 0$

Let's see how the remainder theorem above works in practice:
Example 4:
Let $P(x) = 8x^3 + 22x^2 + 11x - 7$. Find the remainder when $P(x)$ is divided by
- (a) $x - 1$
- (b) $2x + 3$

For example 4(a), we will substitute $x = 1$ into the polynomial, and get the following results.
$P(1) = 8(1)^3 + 22(1)^2 + 11(1) - 7 = 34$
Hence, remainder $R = 34$
For example 4(b), we will substitute $x = -\frac{3}{2}$ into the polynomial, and get the following results

$$P\left(-\frac{3}{2}\right) = 8\left(-\frac{3}{2}\right)^3 + 22\left(-\frac{3}{2}\right)^2 + 11\left(-\frac{3}{2}\right) - 7 = -1$$

Hence, remainder $R = (-1)$

After having a good look into how the remainder theorem works, we can look into how the factor theorem work in practice as well.

Example 5.

Determine the following if they are factor of $P(x) = 2x^3 + x^2 - 7x - 6$

(a) $x - 1$

(b) $2x + 3$

For example 5(a), we will substitute $x = 1$ into the polynomial $P(x)$

$P(1) = 2(1)^3 + (1)^2 - 7(1) - 6 = -10$

Since the remainder $R \neq 0$, $x - 1$ is not a factor of the polynomial $P(x)$

For example 5(b), we will substitute $x = \frac{-3}{2}$ into the polynomial $P(x)$

$$P\left(\frac{-3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 + \left(-\frac{3}{2}\right)^2 - 7\left(-\frac{3}{2}\right) - 6 = 0$$

Since the remainder $R = 0, 2x + 3$ is a factor of polynomial $P(x)$.

Example 6.

Consider the polynomial $P(x) = ax^3 + bx^2 - 1$, where a and b are constants. When $P(x)$ is divided by $x + 2$, the remainder is -5.

(a) Show that $2a - b = 1$

(b) Given that $2x - 1$ is a factor of $P(x)$, find the values of a and b

Example 6(a)

If $\frac{P(x)}{x+2}$ produces a remainder of -5, remainder theorem would imply that

$P(-2) = a(-2)^3 + b(-2)^2 - 1 = (-5)$

$P(-2) = -8a + 4b - 1 = -5$

$-8a + 4b = -4$

$8a - 4b = 4$

$2a - b = 1$ (Shown) [Equation 1]

Example 6(b)

If $2x - 1$ is a factor of $ax^3 + bx^2 - 1$, factor theorem would imply that

$$P\left(\frac{1}{2}\right) = a\left(\frac{1}{2}\right)^3 + b\left(\frac{1}{2}\right)^2 - 1 = 0$$

$$P\left(\frac{1}{2}\right) = \left(\frac{1}{8}\right)a + \left(\frac{1}{4}\right)b - 1 = 0$$

$\frac{a}{8} + \frac{b}{4} = 1$

$a + 2b = 8$ [Equation 2]
$a = 8 - 2b$ [Equation 2b]

Substituting Equation 2b into Equation 1
$2(8 - 2b) - b = 1$
$16 - 4b - b = 1$
$16 - 5b = 1$
$-5b = -15$
$b = 3$
$a = 8 - 2(3) = 2$

Factoring of Cubic Polynomials and Solving of Cubic Equations
Common identities for dealing with cubic polynomial factoring at secondary level
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

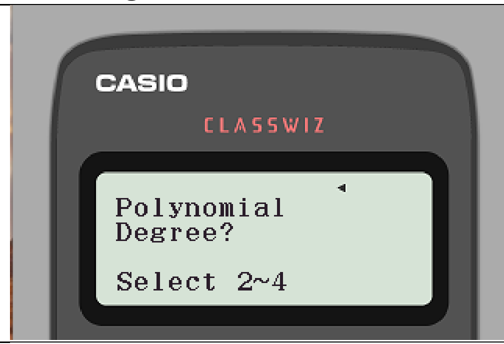
Example 7.
Factorise the following completely
(a) $x^3 + 64$
(b) $64x^3 - 343$

Example 7(a)
$x^3 + 64 = (x + 4)(x^2 - 4x + 4^2) = (x + 4)(x^2 - 4x + 16)$

Example 7(b)
$64x^3 = (4x - 7)[(4x)^2 + 7(4x) + 7^2] = (4x - 7)(16x^2 + 28x + 49)$

Example 8.
Solve the following equations
(a) $9x^3 + 3x^2 - 23x + 4 = 0$
(b) $x^3 + 11x = 6(x^2 + 1)$

Using FX-97 SGX to help find solutions to cubic equations for Example 8(a)

<table>
<tr><th>Action Taken</th><th>Resulting Screenshot</th></tr>
<tr><td>Press [Menu] [5] [2]</td><td>
</td></tr>
<tr><td>When prompted for polynomial degree, press 3 (cubic polynomials have a degree of 3.)</td><td>
</td></tr>
<tr><td>Using $9x^3 + 3x^2 - 23x + 4 = 0$ as an example, we will have to key in 9 into the x^3 field and press [=] button for the cursor to jump to the next field which is the x^2 field.</td><td>
</td></tr>
<tr><td>Press 3 into the x^2 field and press [=] for the cursor to jump to the x field</td><td>
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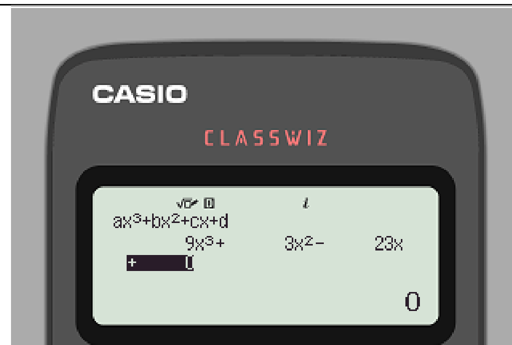
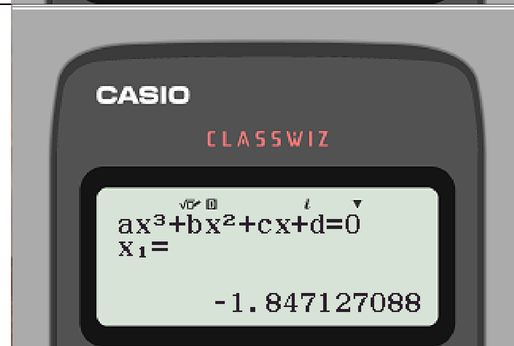
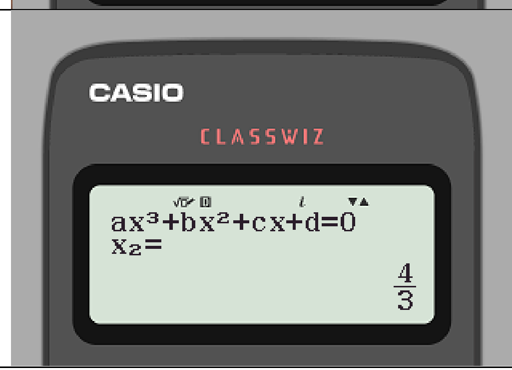
Press -23 into x field and press [=] for the cursor to jump to the field below as seen	
Press 4 into the bottom field and press [=] button and this should appear.	
Keep pressing [=] until you manage to obtain rational roots and, in this case, the rational root is obtained as follows: $x = \frac{4}{3}$	

As mentioned in above, since $\frac{4}{3}$ is a root of the equation, we can deduce that $3x - 4$ is a factor of the cubic polynomial equation $9x^3 + 3x^2 - 23x + 4 = 0$.

After managing to find the polynomial root using a calculator, please prove that the value is a root of the polynomial by demonstration of understanding of factor theorem on your paper in context of the question in any scenario, as failure to demonstrate could result in loss of marks.

For example, since $\frac{4}{3}$ is a root of the polynomial, you must prove to the examiner on why $\frac{4}{3}$ is a root of the polynomial by writing the following as part of your workings.

If $f(x) = 9x^3 + 3x^2 - 23x + 4$

$$f\left(\frac{4}{3}\right) = 9\left(\frac{4}{3}\right)^3 + 3\left(\frac{4}{3}\right)^2 - 23\left(\frac{4}{3}\right) + 4 = 0$$

After being done with this step, we are going to use long division of polynomial to find out the quadratic polynomial which is also a factor of $9x^3 + 3x^2 - 23x + 4$

Description of Steps	**Method as seen**
Long Division	$3x^2 + 5x - 1$ $3x - 4 \overline{)\,9x^3 + 3x^2 - 23x + 4}$ -$(9x^3 - 12x^2)$ $15x^2 - 23x$ -$(15x^2 - 20x)$ $-3x + 4$ $-(-3x + 4)$ 0

Since we found out the quadratic factor which is $3x^2 + 5x - 1$, we can use quadratic formula to solve the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

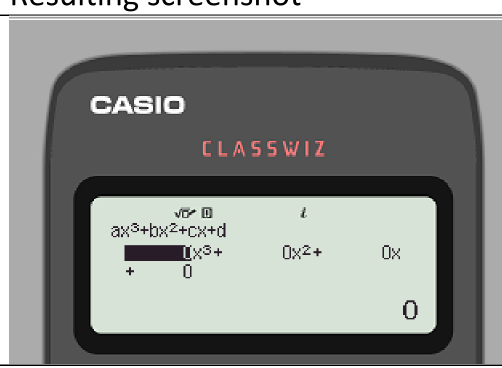
$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(-1)}}{2(3)}$$

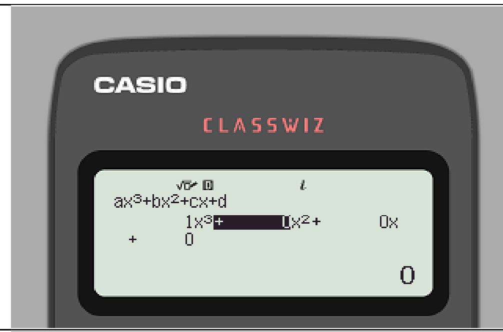
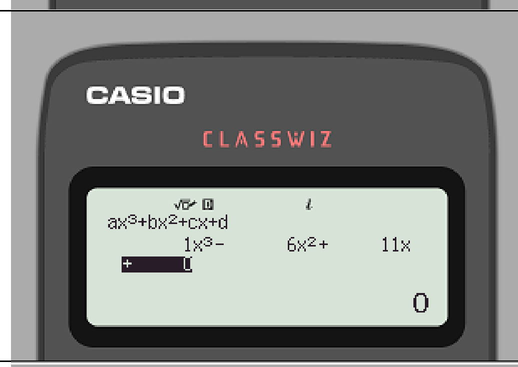
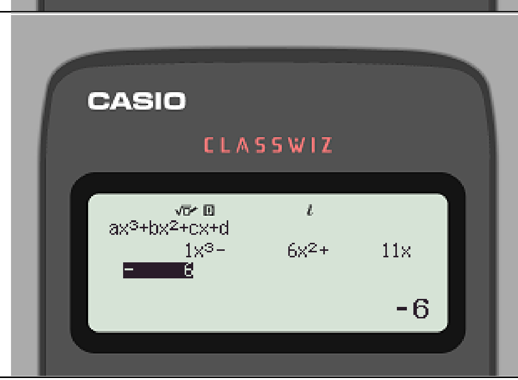
Hence the solutions to the cubic equation are $x = \frac{(-5+\sqrt{37})}{6}$, $x = \frac{(-5-\sqrt{37})}{6}$ OR $x = \frac{4}{3}$

In the case of Example 8(b), we need to first rearrange the cubic equation such that the right-hand-side is exactly 0.

$x^3 + 11x = 6(x^2 + 1)$
$x^3 + 11x - 6(x^2 + 1) = 0$
$x^3 + 11x - 6x^2 - 6 = 0$
$x^3 - 6x^2 + 11x - 6 = 0$

After ensuring the right-hand-side is 0, we proceed to key in the equation into the calculator

Action taken	Resulting screenshot
Press [Menu], [5], [2], [3] in order	

Key in 1 into the field labelled as x^3 and press [=] button	
Key in -6 into the field labelled as x^2 and press [=] button	
Key in 11 into the field labelled as x and press [=] button	
Key in -6 into the bottom field and press [=] button	

Press [=] button again	

In this case, we managed to obtain one of the rational roots of the cubic equation which is $x = 1$ and therefore $x - 1$ is a factor of the cubic polynomial $x^3 + 11x = 6(x^2 + 1)$.

After managing to obtain the rational roots of the cubic equation, please state the following as part of your working:
If $f(x) = x^3 - 6x^2 + 11x - 6$,
$f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0$

After managing to obtain $x - 1$, we have to apply long division method like how we do it in example 8(a)

Long division	$\boldsymbol{x^2 - 5x + 6}$ $\boldsymbol{x - 1\overline{)\,x^3 - 6x^2 + 11x - 6 = 0}}$ $\boldsymbol{-\ (x^3 - x^2)}$ $\boldsymbol{-5x^2 + 11x}$ $\boldsymbol{-(-5x^2 + 5x)}$ $\boldsymbol{6x - 6}$ $\boldsymbol{-(6x - 6)}$ $\boldsymbol{0}$

After obtaining the quadratic factor $x^2 - 5x + 6$, we can then use quadratic formula to further factorise the quadratic factor down into linear factors.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

$x = 3$ or $x = 2$

Hence the solution to the cubic equation is $x = 1, x = 2\ OR\ x = 3$

Partial Fractions Decomposition

While we have learnt about how to combine multiple algebraic fractions, what if we want to reverse the process of combining multiple algebraic fractions together? (i.e. the act of separating already combined algebraic fraction into their original algebraic fractions) Such a technique of separation does exist and is known as partial fraction decomposition.

Applications of Partial Fraction Decomposition

The benefit of learning partial fractions decomposition may not seem immediate but it serves as a rather strong basis for studying calculus at higher levels Mathematics – where you may be required to decompose algebraic fraction into partial fractions to make integrals much easier to evaluate.

Concept of Proper and Improper Algebraic Fractions given $\frac{P(x)}{Q(x)}$

Naming & Characteristics	**Examples**
Proper Algebraic Fractions has: Degree of denominator greater than numerator	$\frac{x^3 + 2x^2}{x^4 + 12x + x^2}, \frac{3x}{(x+1)^2}$
Improper Algebraic Fraction has: Degree of numerator greater than or equal to denominator	$\frac{x^2}{x^2 - 5x + 4}, \frac{5x^3}{(x+5)^2}$

Partial Fraction Common Decomposition Types at O Level Additional Mathematics	**Resulting Decompositions**
$\frac{P(x)}{(ax+b)(cx+d)}$	$\frac{A}{ax+b} + \frac{B}{cx+d}$
$\frac{P(x)}{(ax+b)(cx+d)^2}$	$\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$
$\frac{P(x)}{(ax+b)(x^2+c^2)}$	$\frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$

Example 9:

Express the following in partial fractions

(a) $\frac{5x+3}{x(x+1)}$

(b) $\frac{x+6}{x(x-3)}$

(c) $\frac{7x-1}{(x-1)(x+2)}$

(d) $\frac{x+2}{(2x-3)(x-4)}$

9(a)

Method Used: Substitution Approach

$\frac{5x+3}{x(x+1)}$

$= \frac{A}{x+1} + \frac{B}{x}$

$A(x) + B(x+1) = 5x + 3$

Substituting $x = 0$ would get us the following

$A(0) = B(0+1) = B = 3$

Substituting $x = -1$ would get us the following

$A(-1) + B[(-1)+1)] = 5(-1) + 3$

$-A + B(0) = (-5) + 3 = -2$

$-A = -2$

$A = 2$

∴ The partial fraction is decomposed as $\frac{2}{x+1} + \frac{3}{x}$

9(b)

$\frac{x+6}{x(x-3)} = \frac{A}{x-3} + \frac{B}{x}$

$Ax + B(x-3) = x + 6$

Substituting $x = 0$ would get us the following

$A(0) + B(0-3) = 0 + 6$

$-3B = 6$

$B = -2$

Substituting $x = 3$ would get us the following

$A(3) + B(3-3) = 3A = 3 + 6$

$3A = 9$

$A = 3$

∴ The partial fraction is decomposed as $\frac{3}{x-3} - \frac{2}{x}$

9(c)

$$\frac{7x-1}{(x-1)(x+2)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$A(x-1) + B(x+2) = 7x - 1$

Substitute $x = 1$ would get us the following
$A(1-1) + B(1+2) = 7(1) - 1 = 6$
$3B = 6$
$B = 2$

Substitute $x = -2$ would get us the following
$A[(-2)-1] + B[(-2)+2] = 7(-2) - 1$
$A(-3) + B(0) = -14 - 1$
$-3A = -15$
$A = 5$

∴ The partial fraction is decomposed as $\frac{5}{x+2} + \frac{2}{x-1}$

9(d)

$$\frac{x+2}{(2x-3)(x-4)} = \frac{A}{x-4} + \frac{B}{2x-3}$$

$A(2x-3) + B(x-4) = x + 2$

Substitute $x = \frac{3}{2}$ to get the following:

$$A\left[2\left(\frac{3}{2}\right) - 3\right] + B\left(\frac{3}{2} - 4\right) = \frac{7}{2}$$

$$B\left(-\frac{5}{2}\right) = \frac{7}{2}$$

$$B = -\frac{7}{5}$$

Substitute $x = 4$ to get the following:
$A[2(4)-3] + B(4-4) = 4 + 2$
$5A = 6$
$A = \frac{6}{5}$

$$\frac{\frac{6}{5}}{x-4}-\frac{\frac{7}{5}}{2x-3}$$

$$\frac{6}{5(x-4)}-\frac{7}{5(2x-3)}$$

Example 10
Express the following in partial fractions

(a) $\frac{2x+5}{x^2+2x+1}$

(b) $\frac{x+14}{(2x+3)(x-1)^2}$

10(a) $\frac{2x+5}{x^2+2x+1}=\frac{A}{(x+1)}+\frac{B}{(x+1)^2}$

$\frac{A(x+1)}{(x+1)^2}+\frac{B}{(x+1)^2}=\frac{Ax+A+B}{(x+1)^2}$

$$Ax+A+B=2x+5$$

Substituting $x=-1$ would effectively get us the following:
$A(-1)+A+B=2(-1)+5$
$B=3$

Substituting $B=3$ and $x=0$ would get us the following
$A(0)+A+3=2(0)+5$
$A+3=5$
$A=5-3=2$

The partial fraction decomposition is $\frac{2}{x+1}+\frac{3}{(x+1)^2}$
Q10(b)
$\frac{x+14}{(2x+3)(x-1)^2}=\frac{A}{2x+3}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$

$$\frac{[A(x-1)^2+B(2x+3)(x-1)+C(2x+3)]}{(2x+3)(x-1)^2}=\frac{x+14}{(2x+3)(x-1)^2}$$

$$A(x-1)^2+B(2x+3)(x-1)+C(2x+3)=x+14$$

Substitute $x = 1$ would effectively imply that:
$A(1-1)^2 + B[2(1)+3](1-1) + C[2(1)+3] = 1 + 14$
$A(0) + 5B(0) + 5C = 1 + 14$
$5C = 15$
$C = 3$

Substitute $x = -\frac{3}{2}$ would effectively imply that:
$A\left(-\frac{3}{2} - 1\right)^2 + B\left[2\left(-\frac{3}{2}\right) + 3\right] + C\left[2\left(-\frac{3}{2}\right) + 3\right] = 14 - \frac{3}{2}$

$\dfrac{25}{4}A + 0 + 0 = \dfrac{25}{2}$
$A = \dfrac{\left(\frac{25}{2}\right)}{\left(\frac{25}{4}\right)} = 2$

Substituting $A = 2$, $x = 0$, $C = 3$ would get us the following
$A(x-1)^2 + B(2x+3)(x-1) + C(2x+3) = x + 14$

$2(0-1)^2 + B[2(0)+3][0-1] + 3[2(0)+3] = 0 + 14$
$11 - 3B = 14$
$-3B = 14 - 11 = 3$
$B = -1$

The partial fraction decomposition is $\frac{2}{2x+3} - \frac{1}{x-1} + \frac{3}{(x-1)^2}$

Example 11
Express the following in partial fractions

(a) $\frac{8x^2+x+20}{x^3+4x}$

$$\frac{8x^2 + x + 20}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + c} = \frac{A(x^2+4) + (Bx+C)x}{x(x^2+4)}$$

$Ax^2 + 4A + Bx^2 + Cx = 8x^2 + Cx + 4A = 8x^2 + x + 20$
$(A+B)x^2 + Cx + 4A = 8x^2 + x + 20$

Substitute $x = 0$ would get us the following
$A(0)^2 + 4A + B(0)^2 + C(0) = 8(0)^2 + C(0) + 4A = 8(0)^2 + 0 + 20$
$4A = 20$
$A = 5$

By comparison of coefficients of x , we get the following
$Cx = x$
$C = 1$

By comparison of coefficient of x^2,
$A + B = 8$
$B = 8 - 5 = 3$

The partial fraction decomposition is $\frac{5}{x} + \frac{3x+1}{x^2+4}$

Partial Fraction Decomposition of Improper Algebraic Fraction

Example 12

Express the following in partial fractions

$\frac{6x^3+29x^2-26x+31}{2x^2+11x-6}$

Long Division	$$\begin{array}{r} 3x - 2 \\ 2x^2 + 11x - 6 \overline{) 6x^3 + 29x^2 - 26x + 31} \\ -(6x^3 + 33x^2 - 18x) \\ -4x^2 - 8x + 31 \\ -(4x^2 - 22x + 12) \\ 14x + 19 \end{array}$$

The long division steps above would yield us the following result:

$6x^3 + 29x^2 - 26x + 31 = (2x^2 + 11x - 6)(3x - 2) + (14x + 19)$

$$\frac{6x^3 + 29x^2 - 26x + 31}{3x - 2} = 3x - 2 + \frac{14x + 19}{2x^2 + 11x - 6}$$

In this case, we will start decomposing the algebraic fraction portion of the result and leave anything that isn't/aren't algebraic fractions intact, we will decompose the fraction $\frac{14x+19}{2x^2+11x-6}$ only

$$\frac{14x + 19}{2x^2 + 11x - 6} = \frac{A}{(2x - 1)} + \frac{B}{(x + 6)}$$

$A(x + 6) + B(2x - 1) = 14x + 19$

Substituting $x = -6$ will get us the following

$A(-6 + 6) + (2B(-6) - B) = 14(-6) + 19 = -65$

$-13B = -65$
$B = 5$

Substituting $x = \frac{1}{2}$ will get us the following

$$A\left(\frac{1}{2} + 6\right) + B\left[2\left(\frac{1}{2}\right) - 1\right] = 14\left(\frac{1}{2}\right) + 19$$

$6.5A + 0 = 7 + 19 = 26$
$A = 4$

∴ The partial fraction decomposition is $3x - 2 + \frac{4}{2x-1} + \frac{5}{x+6}$

Title	Exponents and Logarithms
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	23/9/2018

**This topic assumes you already understand your Elementary Mathematics Concepts.

Applicable to the following courses/students.

- 'O' Level Additional Mathematics
- Nanyang Polytechnic – School of Chemical and Life Sciences – Mathematics for Life Sciences Module
- Nanyang Polytechnic – School of Engineering – Engineering Mathematics 1A Module
- Junior College/ A Levels – H1/H2 Mathematics
- Institute of Technical Education (ITE Colleges) – Technical Mathematics Module (Specific Courses and Levels Only)

Relationship Between Exponents and Logarithm as Follows	
An expression written in Exponential Form	$3^{12} = 531441$
If written as Logarithmic Format, it looks like this	$\log_3 531441 = 12$

Generally Speaking	
If	$a^b = c$ In this case, b is called the exponent or power a is called the base value.
It can be rewritten as	$\log_a c = b$ In this case b is known as the logarithm a is called the base value

Laws of Logarithm	
$\log_a(xy) = \log_a x + \log_a y$	
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	
$\log_a(x)^y = y\log_a(x)$	

$\log_a x = \frac{\log_b(x)}{\log_b(a)}$	Calculator Input [Calculator assumes the input is in base value is 10 by default] If $a^x = y$ You literally type in the following into the calculator to find x $\frac{\log(y)}{\log(a)} = x$

Relationship Between Logarithm and Exponents as follows
If $\boldsymbol{a^x = y}$
$x = \log_a(y)$

If $\boldsymbol{\log_a x = y}$
$x = a^y$

If you are thinking about calculator input, the following property will be useful if needed. (As said before, calculator assumes the $\boldsymbol{log(x)}$, as logarithm of which base is 10.)
$a^x = y$

$$x = \frac{\log(y)}{\log(a)}$$

$\log(x) = y$
$x = 10^{\log(x)} = 10^y$

Motivations for introducing logarithm and modern use of logarithm includes the following

- Historical use of logarithm includes multiplication of large numbers, which can be converted to adding of their logarithm under a certain base value and getting the final value
- Modern use of logarithm includes, astronomy (studying brightness of stars), computation (performing data encryption) and more.

Introducing concept of base e, also known as Euler's Number.

- e is approximately 2.71828 when rounded of to 6 significant figures.
- e can be approximated using the formula as shown below

  $$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \dots \dots \frac{1}{n!}$$

  $$n! = n(n-1)(n-2)(n-3) \dots \dots (3)(2)(1)$$
- The approximation gets better as more terms are added.
- Used in modelling continuous growth or continuous decay in Mathematics and Sciences.
- Modern scientific calculators have this functionality, just locate button $e^x \; or \; e$ on your scientific calculator.

Introducing Natural Logarithm ($\mathbf{ln}(\boldsymbol{x})$)

- Natural Logarithm assumes that the base of a certain exponent is exactly e
- It finds the logarithm where the value has base value is e

The relationship between $\mathbf{ln}(\mathbf{x})$ and $\mathbf{e^x}$ goes as follows
If, $\boldsymbol{e^x = y}$ and you are being asked to find the value of $\boldsymbol{x}$
$\ln(e^x) = \ln(y)$
$x = \ln(y)$

If, $\mathbf{ln}(\boldsymbol{x}) = \boldsymbol{y}$ and you are being asked to find the value of $\boldsymbol{x}$
$e^{\ln(x)} = e^y$
$x = e^y$

(For students studying higher level Mathematics, your lecturers may tell you Logarithm an inverse function of Exponent and vice-versa is true as well, this is because, inverse function sends a function back to where it originally started.)

Solving equations involving Logarithm and Exponents
(The following questions are provided by my friend in Nanyang Polytechnic who is studying Electrical Engineering related courses, he explained they are taken from his Engineering Mathematics Notes.)
If questions specify $\ln(x)$ or something similar, the base is exactly e. If question specify $\log(x)$ without base value information, the base value is to be assumed to be 10.

Example 1.
(a) $2^x = 16$
(b) $5^x = 0.3$
(c) $e^{-x} = 17.54$

(d) $\log(x) = 2.3$
(e) $\log(2x) + \log(x) = 9$

Solutions
1(a)
$2^x = 16$
$\log_2 16 = x = 4$

1(b)
$5^x = 0.3$
$\log_{0.3} 5 = -0.748$

1(c)
$e^{-x} = 17.54$
$\ln(e^{-x}) = \ln(17.54)$
$-x = \ln(17.54) = 2.86$
$x = -2.86$

1(d)
$\log(x) = 2.3$
$10^{\log(x)} = 10^{2.3}$
$x = 10^{2.3} = 200$

1(e)
$\log(2x) + \log(x) = 9$

Applying law of logarithm, we get
$\log(2x) + \log(x) = \log(2x^2) = 9$

$10^{\log(2x^2)} = 2x^2 = 10^9$
$2x^2 = 10^9$
$$x^2 = \frac{10^9}{2}$$

$$x = \sqrt{\frac{10^9}{2}}$$

Example 2

(a) $2^{2x} - 8(2^x) + 15 = 0$
(b) $e^{2x} - 7e^x + 12 = 0$

(c) $e^{e^x} = 3$

Depending on the situation, you sometimes may have to reject values, you should always substitute back into the original equation or check with the original question if the values make sense in the question's context.

[I always use u to perform substitution, you can use any letter you like but make sure you don't get yourself confused afterwards.]

Solutions

2(a)

$$2^{2x} - 8(2^x) + 15 = 0$$

Substitute $u^x = 2^x$ (In this case, I pick the smallest power component within the equation to substitute.)

The equation effectively becomes the following after I substituted the component values.

$u^2 - 8u + 15 = 0$

Using Quadratic Formula, which I won't elaborate further on.

I get the following roots for u

$u = 3\ OR\ u = 5$

The roots can be written as the following values since $u = 2^x$ as defined earlier

$2^x = 3\ OR\ 2^x = 5$

$x = \log_2(3)\ \ OR\ x = \log_2(5)$

$x = 1.584962\ OR\ x = 2.321928$

2(b) $e^{2x} - 7e^{x} + 12 = 0$
Substitute $u = e^{x}$
Equation becomes the following
$u^2 - 7u + 12 = 0$

Using quadratic formula, we can get the following answers
$u = 3\ OR\ u = 4$

Which can be rewritten as
$e^{x} = 3\ OR\ e^{x} = 4$

Taking natural logarithm on both sides of each solution to arrive at value of x
$\ln(e^{x}) = \ln(3)\ \ OR\ \ln(e^{x}) = \ln(4)$
$x = \ln(3) = 1.098612\ OR\ x = \ln(4) = 1.386294$

2(c) $e^{e^{x}} = 3$
Take natural logarithm on both sides,
$\ln\left(e^{e^{x}}\right) = \ln(3)$

We get the following equation
$e^{x} = \ln(3)$

Take natural logarithm on both sides once again the get the following,
$\ln(e^{x}) = \ln(\ln(3))$
$x = \ln(\ln(3))$
$x = 0.094048$

Title	Binomial Expansion and Binomial Theorem
Author	AprilDolphin – Lim Wang Sheng
Date	12/6/2026

Before even browsing through this section of the Additional Mathematics notes, you may have been familiar with the idea of using algebraic identity to perform expansion of simple expressions like $(a+b)^2 = a^2 + 2ab + b^2$ as taught in Elementary Mathematics classes, while such methods are easy to execute at lower powers, what if you encountered a situation where you are required to perform expansion such as $(a+b)^n$ where $n \geq 3$?

Here comes the concept of Binomial Expansion where people can use a single concept to expand binomial usually for higher values of n beyond the limit of what one would typically encounter in Elementary Mathematics.

Common Notations for Binomial Expansion	
Name of Notation/Symbol	**Purpose, Definition and Calculator Input Matter**
$\mathrm{n}!$ - Also called "n factorial"	**$\mathbf{n}!$ is defined as the following** • $\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)\ldots(3)(2)(1)$ **Calculator Input (For CASIO FX-97 SGX Calculator)** Example if I want to input 6! (6 factorial) I press the number 6 followed by the button [x!] located just right below the "ON" button. **Common Error Messages of CASIO FX-97 SGX Calculator concerning the use of factorial button and ways to troubleshoot them:** If you get "Math Error" after input, consider the following possible hints to get to the culprit of the error: • Did you input negative values followed by factorial into the calculator? (Factorials of negative values are usually undefined at secondary level Additional Math) • Did you input fractions or decimal followed by factorial into the calculator? (Factorials of fractions or decimal <u>cannot</u> be calculated on usual scientific calculator such as the FX-97 SGX) • Did you input factorials that exceeded the calculation limits of typical scientific calculators?

	Typical scientific calculators are unable to compute factorials of 70 and above ($\geq 70!$) If you get "Syntax Error" after input, consider the following possible issues as the culprit of the error • Examine the input closely for anything like common input errors and correct them accordingly [Examples can sometimes include forgetting to input the close bracket or accidentally pressing the wrong button]
$\binom{n}{r}$ The notation just right above literally means n choose r. Important notes: Calculators as well as other education resources may use nCr to represent the idea of n choose r instead.	$\binom{\boldsymbol{n}}{\boldsymbol{r}} = \frac{\boldsymbol{n}!}{\boldsymbol{r}!\,(\boldsymbol{n}-\boldsymbol{r})!}$ Which is the total number of ways to choose r objects out of n objects, <u>assuming the order doesn't matter</u>. **Calculator Input (For CASIO FX-97 SGX Calculator) (Combinations Calculator Functionality)** Let's say you have n number of objects and you want to determine how many ways are there to select r objects. For example, given 30 students in a class and you want to choose 3 students to represent the class in a school event in which the <u>order doesn't matter</u> and calculate the number of ways to choose any 3 students in the class: In that case, that will be $\binom{30}{3}$ which on the FX-97 SGX Calculator, you press the following buttons: Step 1. Press "30" on the calculator Step 2. Press [Shift] button followed by [÷] button and you should see the Letter "C" appearing after "30" Step 3. Press "3" on your calculator and press [=] button and you should see 4060 appearing on the screen. The number of ways to choose 3 students in a 30-students-class is 4060.

General formula for performing binomial expansion:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 \ldots \binom{n}{r} a^{n-r} b^r \ldots + \binom{n}{n} a^{n-n} b^n$$

Formula for the $r+1$th term of binomial expansion:

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Question 1:

Expand each of the following

(a) $(3p+2q)^4$
(b) $(x-y)^5$

For question 1(a), we simply apply the general formula of binomial expansion given above as follows:

$$(3p+2q)^4$$
$$= \binom{4}{0}(3p)^4(2q)^0 + \binom{4}{1}(3p)^3(2q)^1 + \binom{4}{2}(3p)^2(2q)^2 + \binom{4}{3}(3p)^1(2q)^3 + \binom{4}{4}(3p)^0(2q)^4$$

$$= 1(3p)^4(2q)^0 + 4(3p)^3(2q)^1 + 6(3p)^2(2q)^2 + 4(3p)^1(2q)^3 + 1(3p)^0(2q)^4$$
$$= 81p^4 + 216p^3q + 216p^2q^2 + 96pq^3 + 16q^4$$

For question 1(b), we simply apply the general formula of binomial expansion given above as follows (⚠<u>Please be cautious when dealing with negative signs.</u>)

$$(x-y)^5$$
$$= \binom{5}{0}x^5(-y)^0 + \binom{5}{1}x^4(-y)^1 + \binom{5}{2}x^3(-y)^2 + \binom{5}{3}x^2(-y)^3 + \binom{5}{4}x^1(-y)^4 + \binom{5}{5}x^0(-y)^5$$

$$= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

Question 2

Write down and simplify the fourth term in the following expansion

(a) $\left(x^2+\frac{1}{x}\right)^7$

(b) $\left(\frac{x}{2}-2y\right)^8$

We shall use the formula for the $r+1$th term of the expansion to find the 4th term

2(a)

$$T_{3+1}=\binom{7}{3}[(x^2)^4]\left(\frac{1}{x}\right)^3=35x^8\left(\frac{1}{x}\right)^3=\frac{35x^8}{x^3}=35x^5$$

2(b)

$$T_{3+1}=\binom{8}{3}\left[\frac{1}{2}x\right]^{8-3}(-2y)^3=56\left(\frac{x}{2}\right)^5(-8y^3)=\frac{56x^5}{32}(-8y^3)=-\frac{1}{4}(56x^5)(y^3)$$

$$=-14x^5y^3$$

Question 3

Expand the following and write down the term independent of x

$$\left(x^5-\frac{4}{x}\right)^6$$

$$\binom{6}{0}(x^5)^6\left(-\frac{4}{x}\right)^0+\binom{6}{1}(x^5)^5\left(-\frac{4}{x}\right)^1+\binom{6}{2}(x^5)^4\left(-\frac{4}{x}\right)^2+\binom{6}{3}(x^5)^3\left(-\frac{4}{x}\right)^3$$
$$+\binom{6}{4}(x^5)^2\left(-\frac{4}{x}\right)^4+\binom{6}{5}(x^5)^1\left(-\frac{4}{x}\right)^5+\binom{6}{6}(x^5)^0\left(-\frac{4}{x}\right)^6=$$

$$x^{30}+6x^{25}\left(-\frac{4}{x}\right)+15x^{20}\left(-\frac{4}{x}\right)^2+20x^{15}\left(-\frac{4}{x}\right)^3+15x^{10}\left(-\frac{4}{x}\right)^4+6x^5\left(-\frac{4}{x}\right)^5$$
$$+(1)\left(-\frac{4}{x}\right)^6=$$

$$x^{30}-24x^{24}+15x^{20}\left(\frac{16}{x^2}\right)-20x^{15}\left(\frac{64}{x^3}\right)+15x^{10}\left(\frac{256}{x^4}\right)-6x^5\left(\frac{1024}{x^5}\right)+\left(\frac{4096}{x^6}\right)=$$
$$=x^{30}-24x^{24}+240x^{18}-1280x^{12}+3840x^6-6144+\frac{4096}{x^6}$$

In this case, we were being asked to write down the term independent of x after expanding, the term independent of x is the term that doesn't include x after simplifying, in this case, we are talking about the constant term which is -6144.

Question 4

Find, in descending powers of x, the first four terms in the expansion $\left(2x - \frac{1}{2}\right)^9$.

Hence obtain the coefficient of x^7 in the expansion of $(1 - x)\left(2x - \frac{1}{2}\right)^9$.

Two things to note: the question mentioned "descending power of x", which implies that the power of x has to decrease from the highest possible value to lower values.

The question also mentioned, hence obtain the coefficient, which implies we must make use of the first four terms to find the coefficient of x^7 term in the expansion of $(1 - x)\left(2x - \frac{1}{2}\right)^9$.

Step 1. Obtain the first four terms of the expansion $\left(2x - \frac{1}{2}\right)^9$

$$\binom{9}{0}(2x)^9\left(-\frac{1}{2}\right)^0 + \binom{9}{1}(2x)^8\left(-\frac{1}{2}\right)^1 + \binom{9}{2}(2x)^7\left(-\frac{1}{2}\right)^2 + \binom{9}{3}(2x)^6\left(-\frac{1}{2}\right)^3$$

$$512x^9 - 1152x^8 + 1152x^7 - 672x^6 \ldots$$

Step 2. Multiplying first four terms of the expansion by $(1 - x)$

$(512x^9 - 1152x^8 + 1152x^7 - 672x^6 \ldots)(1 - x)$ will get us the following

First term $(512x^9)(1 - x) = 512x^9 - 512x^{10}$
Second term $(-1152x^8)(1 - x) = -1152x^8 + 1152x^9$
Third term $(1152x^7)(1 - x) = 1152x^7 - 1152x^8$
Fourth term $(-672x^6)(1 - x) = -672x^6 + 672x^7$

Adding up only the third term and fourth term (which actually indeed contain the term x^7) we get the following as a result:
$1152x^7 - 1152x^8 - 672x^6 + 672x^7 = -1152x^8 + 1824x^7 - 672x^6$

Hence the coefficient of the x^7 is 1824

Question 5:
Write down and simplify, in ascending powers of x, the first three terms of the expansion of $(1+2x)^7$

One thing to note here, as the question mentioned "ascending powers of x", the power of x shall increase from the lowest possible value (zero included) to higher possible values.

$$(1+2x)^7 =$$

$$\binom{7}{0}(1^7)(2x)^0 + \binom{7}{1}(1^6)(2x)^1 + \binom{7}{2}(1^5)(2x)^2 \dots$$

$$= 1 + 14x + 84x^2$$

Question 6:
The term independent of x in the binomial expansion of $\left(x^3 + \frac{a}{x^2}\right)^{10}$ is 210. Find the value of a.

Let $r + 1^{th}$ term be the term independent of x

$$T_{r+1} = \binom{10}{r}[(x^3)^{10-r}]\left(\frac{a}{x^2}\right)^r = 210$$

$$T_{r+1} = \binom{10}{r}(x^{30-3r})\left(\frac{a^r}{x^{2r}}\right) = \binom{10}{r}\left[\frac{(a^r)(x^{30-3r})}{x^{2r}}\right] = \binom{10}{r}\left[\frac{(a^r)(x^{30-3r})}{x^{2r}}\right]$$

$$T_{r+1} = \binom{10}{r}a^r(x^{30-3r-2r}) = \binom{10}{r}a^r x^{30-5r} = 210$$

As $T_{r+1} = \binom{10}{r}a^r x^{30-5r} = 210$ is indeed independent of x, the following has to be true:
$30 - 5r = 0 \because$ the power of x term in this case has to be zero.
$30 = 5r$
$r = 6$

$$\binom{10}{6}a^6 x^0 = 210$$

$$a^6 x^0 = \frac{210}{\binom{10}{6}} = 1$$

$a^6 = 1$
Since the power of a is even, $a = \pm 1$

Title	Coordinate Geometry of Circles
Date	28/11/2023
Note	Questions all taken from Additional Mathematics Textbook

Warnings and Assumptions: This article assumes that you already have a good foundation in the following:

- Basic Coordinate Geometry (Such as the ability to use formula to calculate the length of a line given coordinate of both ends of the line)
- Use and understand substitution method to find solutions of simultaneous equations in Elementary Mathematics.
- Able to solve quadratic equations, use completing the square method extensively, understand how to solve quadratic inequalities efficiently and understand how discriminant affect the behavior of a quadratic function

Standard Form of a Circle
$(x-a)^2+(y-b)^2=r^2$ Coordinates $C(a,b)$ is located at the center of the circle. r refers to the radius of the circle
General Form of a Circle
$x^2+y^2+2gx+2fy+c=0$ In this case, the coordinates of the center of the circle are $C(-g,-f)$, the radius of the circle will be $r=\sqrt{g^2+f^2-c}$
Note: Make sure the coefficient of both x^2 and y^2 are 1 before proceeding, if the coefficient is not 1, you need to divide the whole equation by the coefficient until you get 1 as the coefficient of both x^2 and y^2.

Conditions Relating to Points Being on the circle, inside the circle or outside of the circle (Let d be the distance between the center of the circle and the point in question)	
Condition	Interpretation & Consequences
$d>r$	Point is outside of the circle
$d=r$	Point is on the circle
$d<r$	Point is inside the circle

Given the following equation of circles, derive the coordinates of the center of the circle along with the radius of the circle.

1(a). $(x-4)^2+(y+5)^2=36$
1(b). $3(x-7)^2+3y^2-48=0$

1(a) Since the equation of the circle is already written in the standard form, we can deduce the center of the circle is $C(4,-5)$ and the radius $r=6$ units

1(b).
Step 1: Divide the entire equation by 3
Which would get you $(x-7)^2+y^2-\frac{48}{3}=0$
$(x-7)^2+(y-0)^2=16$
Coordinates of center of the circle = $C(7,0)$ and the radius $r=4$ units

Given the following equation of circles, derive the coordinates of the center of the circle along with the radius of the circle.

2(a). $x^2+y^2+4x-6y-3=0$
Method 1: Comparing with Generic Equation and Apply Formula Directly
Comparing the above equation with the generic equation for a circle,
$x^2+y^2+2gx+2fy+c=0$

Therefore, $2gx=4x$ and $2fy=6y$
$g=-2$ and $f=3$
Coordinates of center of the circle is $C(-2,3)$

Radius of circle = $\sqrt{g^2+f^2-c}=\sqrt{(-2)^2+3^2-(-3)}=\sqrt{16}$
Radius of circle = $r=4$

Method 2: Complete the Square
$x^2+y^2+4x-6y-3=0$
Step 1. Rearrange the equation in a way it looks like the following
$x^2+4x+y^2-6y=3$

Step 2. Complete the square for both x^2 and y^2

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 + y^2 - 6y + \left(\frac{6}{2}\right)^2 = 3 + \left(\frac{4}{2}\right)^2 + \left(\frac{6}{2}\right)^2$$

$$(x + 2)^2 + (y - 3)^2 = 16$$

Comparing with the standard form of a circle which is
$(x - a)^2 + (y - b)^2 = r^2$

Center $C(-2{,}3)$, $r = 4$

3(a)

(i) Find the equation of the circle which passes through point (1,3) and has its center at $C(2{,}5)$.

(ii) Determine whether $A(3{,}4)$ and $B(0{,}6)$ lies outside, inside or on the circle.

(i) Set (x_1, y_1) as the point at center and set (x_2, y_2) as the point which the circle passes through

Distance from center to the point which circle pass through is precisely the radius of the circle, therefore, we can use the formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to deduce the radius of the circle.

$r = \sqrt{(1 - 2)^2 + (3 - 5)^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$
$r^2 = 5$

Center of Circle = (2,5), therefore the equation for the circle in question is
$(x - 2)^2 + (y - 5)^2 = 5$

After expanding the terms, we get the following
$x^2 - 2(2x) + 4 + y^2 - 2(5y) + 25 = 5$
$x^2 - 4x + 4 + y^2 - 10y + 25 = 5$
$x^2 + y^2 - 4x - 10y + 24 = 0$

(ii) Length of Line AC is $d = \sqrt{(3 - 2)^2 + (4 - 5)^2} = \sqrt{2}$
Since $\sqrt{2} < \sqrt{5}$, Point A lies inside the circle

Length of Line BC is d = $\sqrt{(0 - 2)^2 + (6 - 5)^2} = \sqrt{4 + 1} = \sqrt{5}$
Since $\sqrt{5} = \sqrt{5}$ Point B Lies on the circle.

Solving simultaneous equations involving a straight-line intersecting a circle.

4. Solve the following simultaneous equation.
$x - 4y + 16 = 0$ (Equation 1)
$x^2 + y^2 - 4x - 6y + 5 = 0$ (Equation 2)

$x = 4y - 16$ (Equation 1A)

Substitute Equation 1A into Equation 2

$(4y - 16)^2 + y^2 - 4x - 6y + 5 = 0$
$16y^2 - 2(16)(4y) + 16^2 \mp y^2 - 4(4y - 16) - 6y + 5 = 0$
$16y^2 - 128y + 256 + y^2 - 16y + 64 - 6y + 5 = 0$
$17y^2 - 150y + 325 = 0$

Using quadratic formula, we get the following values for y

$y = \frac{65}{17}$ OR $y = 5$

Substitute both values into Equation 1A, we get the following values for x

$x = 4\left(\frac{65}{17}\right) - 16$ OR $x = 4(5) - 16$

$x = -\frac{12}{17}$ OR $x = 4$

5. Finding a range of value for which a line intersect, is a tangent to or doesn't intersect a circle.

The line $y = 2x + k$, where k is a constant, is a tangent to the circle $x^2 + y^2 + 4x - 6y + 8 = 0$. Find the possible values of k and the corresponding points of contact for each value of k.

Equation 1: $y = 2x + k$
Equation 2: $x^2 + y^2 + 4x - 6y + 8 = 0$

Substitute Equation 1 into Equation 2
$x^2 + (2x + k)^2 + 4x - 6(2x + k) + 8 = 0$
$x^2 + 4x^2 + 2(2x)(k) + k^2 + 4x - 12x - 6k + 8 = 0$

Rearrange the equation and we get the following
$5x^2 + 4kx + k^2 + 4x - 12 - 6k + 8 = 0$
$5x^2 + 4kx - 8x + k^2 - 6k + 8 = 0$

Note that the general form of a quadratic equation is $ax^2 + bx + c = 0$

$5x^2 + (4k - 8)x + (k^2 - 6k + 8) = 0$

Since the question demands a set of lines that is tangent to the circle, we can use the rule of discriminant we learnt previously in the quadratic equation topic where a line tangent to a circle would also produce a discriminant of $b^2 - 4ac = 0$

$b^2 - 4ac = (4k - 8)^2 - 4(5)(k^2 - 6k + 8)$
$= 16k^2 - 2(4)(8)k + 64 - 20(k^2 - 6k + 8)$
$= 16k^2 - 64k + 64 - 20k^2 + 120k - 160$
$= -4k^2 + 56k - 96$

$-4k^2 + 56k - 96 = 0$

Using the quadratic formula, we get the following values for k

$k = 2$ OR $k = 12$

Equation for both lines that are tangent to the circle are therefore
$y = 2x + 12$ and $y = 2x + 2$

Title	Trigonometric Identities & Formula
Author	
Date	12/9/2023

Assumptions: This article assumes you have already studied basic trigonometry in Elementary Mathematics.

Converting Between Degrees and Radians
Degree → Radian: Multiply Angle by $\frac{\pi}{180}$ Radian → Degree: Multiply Angle by $\frac{180}{\pi}$

Let r be the hypotenuse side of the right-angled triangle, x be the adjacent side and y be the opposite side.

Basic Trigonometric Functions and Ratio	Reciprocal Trigonometric Functions
$\sin(\theta) = \frac{Opposite}{Hypotenuse} = \frac{y}{r}$ $\cos(\theta) = \frac{Adjacent}{Hypotenuse} = \frac{x}{r}$ $\tan(\theta) = \frac{Opposite}{Adjacent} = \frac{y}{x}$	$cosec(\theta) = \frac{1}{\sin(\theta)} = \frac{r}{y}$ $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{r}{x}$ $\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{x}{y}$

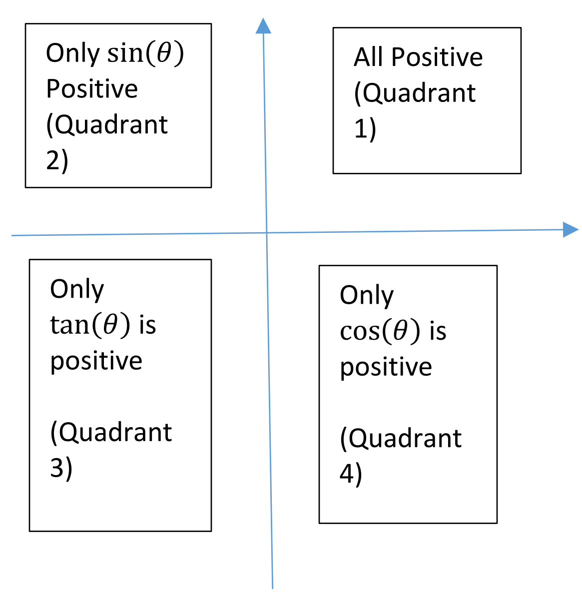
Trigonometric Identities Involving division of one trigonometric function by another function
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Pythagorean Identities
$\sin^2(\theta) + \cos^2(\theta) = 1$
$\tan^2(\theta) + 1 = \sec^2(\theta)$
$\cot^2(\theta) + 1 = cosec^2\,(\theta)$

Compound Angle Formulae
$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
$\tan(A \pm B) = \dfrac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$

Double Angle Formulae
$\sin(2A) = 2\sin(A)\cos(A)$
$\cos(2A) = 1 - 2\sin^2(A)$
$\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

R-Formulae
$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$
$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$
$R = \sqrt{a^2 + b^2}$ $\alpha = \tan^{-1}\frac{b}{a}$ where $a, b > 0$ α is an acute angle



The above diagram is also called the ASTC Diagram.

Concept of Reference Angle

Reference angle refers to trigonometric functional value of any arbitrary angle expressed as an acute angle, The reference angle of a given angle θ is the positive acute angle between the x-axis and the terminal side of θ.

Solutions involving trigonometric functions can be found using the following steps:
1. Use ASTC diagram to determine the quadrant in which angle θ is in
2. Find the reference angle θ_{ref}
3. The value of θ is as follows:

Quadrant	In Degrees	In Radians
1	$\theta = \theta_{ref}$	$\theta = \theta_{ref}$
2	$\theta = 180° - \theta_{ref}$	$\theta = \pi - \theta_{ref}$
3	$\theta = 180° + \theta_{ref}$	$\theta = \pi + \theta_{ref}$
4	$\theta = 360° - \theta_{ref}$	$\theta = 2\pi - \theta_{ref}$

Typical approach to dealing with trigonometric equations.

Step 1. (If necessary) Rewrite the equation so to ensure the trigonometric equation is expressed in terms of a single trigonometric functions using trigonometric identities.

Step 2. (If necessary): If there is a need, substitute the trigonometric function using a letter and obtain the value of trigonometric functions through algebraic methods. i.e. (Quadratic Formula)

Step 3. Use inverse trigonometric function to obtain value of θ after all trigonometric functions are combined together into a single function and use ASTC diagram to determine other possible value of θ.

Question 1.

Solve $2\sin^2 x + \sin x - 1 = 0$ for $0° \leq x \leq 360°$

Substitute $u = \sin x$

And we get the following equation

$2u^2 + u - 1 = 0$

Using quadratic formula, which we will not elaborate here,
We get the following

$u = \frac{1}{2}\ OR\ u = -1$
$x = 30°$ or $x = -90°$

Using ASTC, we apply the formula for finding angle in quadrant 2

$x = 180° - 30 = 150$ or $x = 180 - (-90) = 270°$

Question 2.
Solve $2\cos^2 x + \cos x = 3$ for between $0°$ to $360°$ (both inclusive)

Substitute $u = \cos x$
$2u^2 + u = 3$
$2u^2 + u - 3 = 0$

$u = 1$ Or $u = -1.5$ (Rejected, invalid input)

$x = 0°$

Using ASTC, we deduce cosine is positive at the 4th quadrant and therefore
$x = 360° - 0° = 360°$

Question 3.
$\sec^2 x + 13 = 9\tan x$

Using identity: $1 + \tan^2 x = \sec^2 x$

$1 + \tan^2 x + 13 = 9\tan x$
$14 + \tan^2 x = 9\tan x$

Rearranging we get the following
$\tan^2 x - 9\tan x + 14 = 0.$

Substitute $u = \tan x$
$u^2 - 9u + 14 = 0$
$u = 7$ or $u = 2$

$x = 81.8699°$ or $x = 63.4349°$

Since $\tan\theta$ is positive on the 3rd quadrant, we apply the formula for quadrant 3 and get the following:
$x = 180° + 81.8699° = 261.8699°$ OR $x = 180 + 63.435° = 243.4349°$

Question 4
Solve $12\cos^2 x + 5\sin x - 10 = 0$ for $0° \leq x \leq 360°$

Using identity: $\sin^2\theta + \cos^2\theta = 1$
The above identity can be rewritten as: $1 - \sin^2\theta = \cos^2\theta$

Substitute the rewritten identity into the equation above, we get:
$12(1 - \sin^2 x) + 5\sin x - 10 = 0$
$12 - 12\sin^2 x + 5\sin x - 10 = 0$
$2 - 12\sin^2 x + 5\sin x = 0$
$-12\sin^2 x + 5\sin x + 2 = 0$

Substitute $u = \sin x$ to get the following equation
$-12u^2 + 5u + 2 = 0$
$u = \frac{2}{3}\ OR\ u = -\frac{1}{4}$

$x = 41.8103°$ OR $x = -14.4775°$

$x = -14.478$ is located at the fourth quadrant and is to be rewritten as $360 - 14.4775° = 345.5225°$

Since $\sin\theta$ is positive on the 2nd quadrant, we use the formula for the second quadrant to deduce the value of x

$x = 180° - 41.810° = 138.190°$

$x = 180° - (-14.478°) = 194.478°$

Proving of trigonometric identities
In this section, we will be talking about how to prove a trigonometric identity.
Typically, students will be given an identity and as a student, you have to prove the left-hand-side of a trigonometric identity is exactly equal to the right-hand-side of that same identity, while the idea is rather simple, there is no standard procedures that applies to all and each trigonometric identity is to be dealt with on a case-by-case basis.

Question 5
Prove that the following trigonometric identity is true
$\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$
Using identity: $1 + \tan^2\theta = \sec^2\theta$ and substituting in the left-hand side of the identity, we get the following:

$(1 + \tan^2 A)^2 - (1 + \tan^2 A) = \tan^4 A + \tan^2 A$
$(1^2 + 2(1)\tan^2 A + \tan^4 A) - (1 + \tan^2 A) = \tan^4 A + \tan^2 A$
$1 + 2\tan^2 A + \tan^4 A - 1 - \tan^2 A = \tan^4 A + \tan^2 A$
$2\tan^2 A - \tan^2 A + \tan^4 A = \tan^4 A + \tan^2 A$
$\tan^2 A + \tan^4 A = \tan^4 A + \tan^2 A$
(Since LHS = RHS, the identity is proven)

Question 6
Prove the following trigonometric identity
$$1 + \frac{1}{\tan^2\theta} = cosec^2\theta$$

Using identity: $1 + \cot^2\theta = cosec^2\theta$

To be rewritten as $1 + \frac{1}{\tan^2\theta} = cosec^2\theta$

$$1 + \frac{1}{(\tan^2\theta)} = 1 + \frac{1}{(\tan^2\theta)}$$

Since LHS = RHS, the identity is proven

Use of Compound Angle Formula
Question 7
It is given that A and B are acute angles such that $\tan A = \frac{3}{4}$ and $\tan B = \frac{1}{7}$. Without using a calculator, find the values of angle $A + B$.
Using formula as follows:
$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

Substituting $\tan A$ *and* $\tan B$ with specified values in the question, we get the following

$$\tan(A + B) = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}\left(\frac{1}{7}\right)} = 1$$

$A + B = 45°$

Question 8

Given that $\sin A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$ and that A and B are in the first quadrant.
Find, without solving for the angles

(a) $\sin(A + B)$ (b) $\cos(A + B)$

8(a)

Using formula

$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$ while we substitute the values in the formula with values mentioned in the question, we get the following:

$$\sin(A + B) = \frac{3}{5}\left(\frac{12}{13}\right) + \cos A \sin B$$

To find the value of $\cos A$, we can apply Pythagoras Theorem and obtain the following value, which is $\frac{4}{5}$

To find the value of $\sin B$, we can apply Pythagoras Theorem and obtain the following value, which is $\frac{5}{13}$

Substituting the above value into $\cos A \ \sin B$, we get the following

$$\sin(A + B) = \frac{3}{5}\left(\frac{12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right) = \frac{56}{65}$$

8(b)

Using formula

$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

$$\cos(A + B) = \frac{4}{5}\left(\frac{12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right) = \frac{33}{65}$$

Question 9

If $\sin A = \frac{7}{25}$ and $\cos B = -\frac{3}{5}$, where A and B are obtuse angles, find the value of $\sin(A+B)$

Illustration of A, as obtuse angle in second quadrant

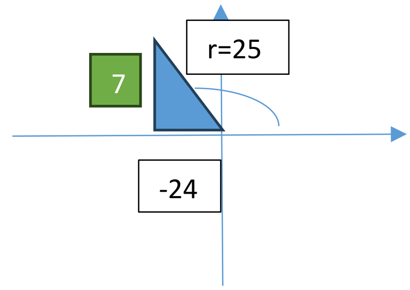
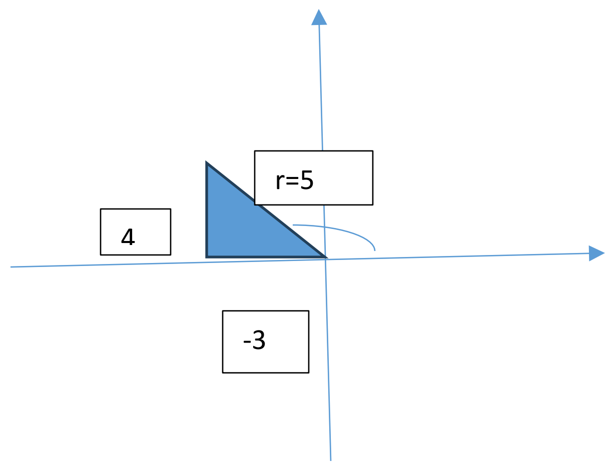


Illustration of B, as obtuse angle in second quadrant



Using formula

$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

$$\sin A = \frac{7}{25}$$

$$\cos B = -\frac{3}{5}$$

$$\cos A = -\frac{24}{25}$$

$$\sin B = \frac{4}{5}$$

Multiply according to the formula, we get the following

$$\frac{7}{25}\left(-\frac{3}{5}\right) + \left(-\frac{24}{25}\right)\left(\frac{4}{5}\right) = -\frac{117}{125}$$

Question 10

If $\cos\alpha = -\frac{3}{5}$ and $\sin\beta = \frac{5}{8}$ and α is a third quadrant angle and β is a first quadrant angle. Find, without solving for the angles:

(a) $\sin(\alpha - \beta)$
(b) $\cos(\alpha + \beta)$
(c) $\tan(\alpha + \beta)$

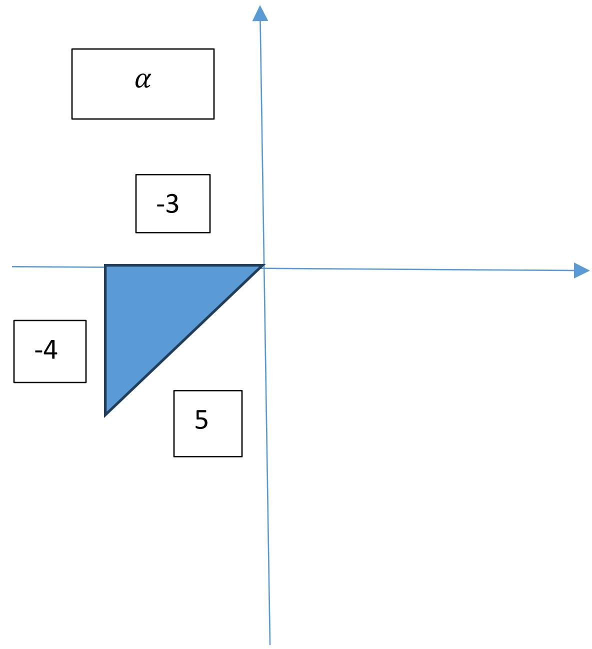


Illustration of angle α in the 3rd quadrant

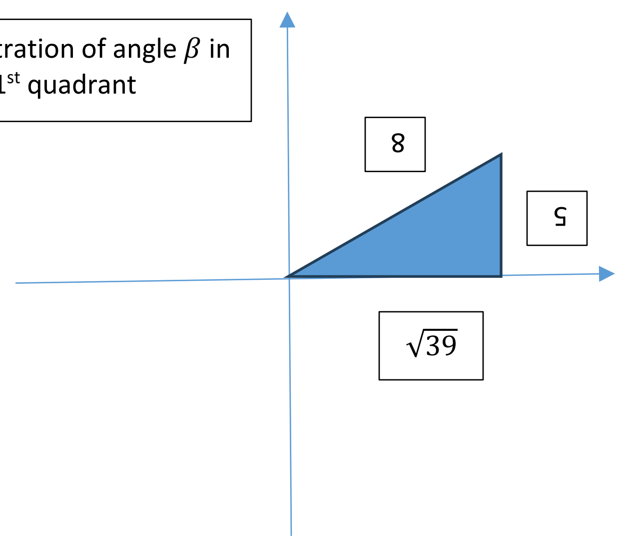


Illustration of angle β in the 1st quadrant

10(a)
Using identity
$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
Which rewritten in terms of α and β looks like the following in context of this question
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \left(-\frac{4}{5}\right)\left(\frac{\sqrt{39}}{8}\right) - \left(-\frac{3}{5}\right)\left(\frac{5}{8}\right) = -0.249$$

10(b)
Using identity $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
Which rewritten in terms of α and β looks like the following in context of this question
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{3}{5}\right)\left(\frac{\sqrt{39}}{8}\right) - \left(-\frac{4}{5}\right)\left(\frac{5}{8}\right) = -0.0316$$

10(c)
Using identity

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

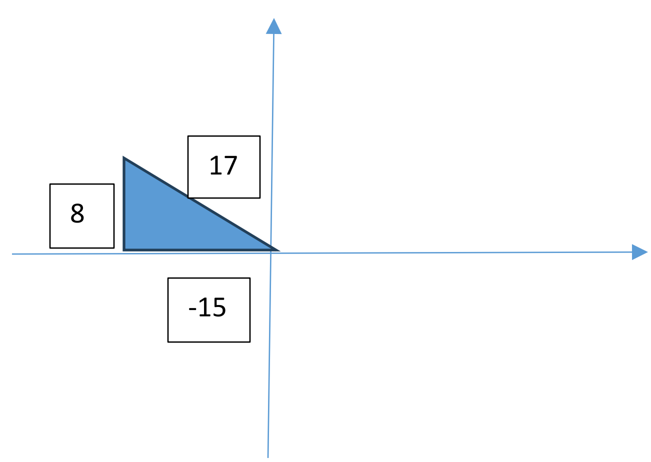
Which rewritten in terms of α and β looks like the following in context of this question

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\left(\frac{-4}{-3}\right) + \left(\frac{5}{\sqrt{39}}\right)}{1 - \left(\frac{-4}{-3}\right)\left(\frac{5}{\sqrt{39}}\right)} = -31.6$$

Use of double angle formula
Question 11
If $sin\,\theta = \frac{8}{17}$ and θ is an obtuse angle, find the values of the following using the double angle formula.
(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$

θ is an obtuse angle and therefore shown in the diagram sketch on the right to be in quadrant 2.



11(a)
Using identity $\sin(2A) = 2\sin(A)\cos(A)$

$\sin 2\theta = 2\sin\theta\; cos\theta$

$$= 2\left(\frac{8}{17}\right)\left(-\frac{15}{17}\right) = -\frac{240}{289}$$

11(b)
Using identity $\cos(2A) = 1 - 2\sin^2(A)$
$\cos 2\theta = 1 - 2\sin^2\theta$

$$1 - 2\left(\frac{8}{17}\right)^2 = \frac{161}{289}$$

11(c)
Using identity
$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2(\theta)}$$

$$= \frac{2\left(\frac{8}{-15}\right)}{1 - \left(\frac{8}{-15}\right)^2}$$

$$= -\frac{240}{161}$$

Use of R-Formula
Question 12
Solve $7\cos x - 2\sin x = 1$ for between $0°$ and $360°$
Using identity $a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$

$R = \sqrt{7^2 + 2^2} = \sqrt{53}$ $\tan\alpha = \frac{2}{7}$
$\alpha = \tan^{-1}\left(\frac{2}{7}\right) = 15.9454$

$7\cos x - 2\sin x = 1$
$\sqrt{53}\cos(x + 15.9454°) = 1$
$\cos(x + 15.9454°) = \dfrac{1}{\sqrt{53}}$

Condition: $0° \leq x \leq 360°$

$\therefore 15.9454° \leq x \leq 375.9454°$

Since $\cos(x) > 0$, it exists in quadrant 1 as well as quadrant 4.

Let $\cos\alpha = \frac{1}{\sqrt{53}}$

$\alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{53}}\right) = 82.1049°$

$x + \tan^{-1}\left(\frac{2}{7}\right) = \cos^{-1}\left(\frac{1}{\sqrt{53}}\right)$ OR $x + \tan^{-1}\left(\frac{2}{7}\right) = 360 - \cos^{-1}\left(\frac{1}{\sqrt{53}}\right)$

$x + 15.9454° = 82.1049$ OR $x + 15.9454° = 277.8951°$

$x = 66.1595°$ OR $x = 261.9497°$

Title	Differentiation of Algebraic Functions
Editor	
Date	1/4/2023

This topic is about basic differentiation on algebraic functions specifically. The following concepts are covered in this topic:

Basic Overview of Differentiation

- Reason to study this topic
- Notation guide

Basic Rules of Differentiation on Algebraic Functions

- Power Rule
- Sum Rule
- Difference Rule
- Product Rule
- Quotient Rule
- Chain Rule

Brief Overview of Differentiation (And Calculus in General)

Without going into too much detail about the history of calculus, calculus is a branch of Mathematics introduced to study about change. Differentiation is a sub-branch that deals with the gradient of a given function at every point on a curve.

Calculus is very important in the following field for the following reasons

- Computing – To study time complexity of programming algorithms and do various data analytics and estimation
- Physics – Majority of physical laws are derived with help of calculus (Newton used calculus to research how physics works and used that in his later works)
- Engineering – Engineering is closely related to physics as Engineering students study physics as their basic module before they carry on with intermediate modules
- Chemistry – Study speed of chemical reactions

And the list can go on and on

Notation	**Meaning**	**Explanation**	**Notes**
$F(x)$	Function of x		I also use $f(x), g(x)$ in certain situations
$F'(x)$	First Derivative of $f(x)$		I also use $f'(x)$, $g'(x)$ in certain situations
$F''(x)$	Second Derivative of $f(x)$	First derivative of $f'(x)$	I also use $f''(x), g''(x)$ in certain situations
$\frac{dy}{dx}$	First derivative of y with respect to x		
$\frac{d^2y}{dx^2}$	Second derivative of y with respect to x	First derivative of $\frac{dy}{dx}$	

Follow the question with regards to notation matters:

If the question mentions a function in the following form:

$$f(x) = ax^n + ax^{n-1} \dots + c$$

When answering the question, use $f'(x)$ for first derivative and $f''(x)$ for second derivative.

If the question mentions an equation in the following form:

$$y = ax^n + ax^{n-1} \dots + c$$

When answering the question use $\frac{dy}{dx}$ for first derivative and $\frac{d^2y}{dx^2}$ for second derivative

Power Rule
Given Function $f(x) = ax^n$
The derivative of the function is $f'(x) = anx^{n-1}$

Question 1.
Differentiate the following with respect to x
(a) $y = x^5$
(b) $y = \frac{1}{x^3}$
(c) $y = 3x^4$
(d) $y = \pi$

1(a)
$y = x^5$
Differentiate using power rule $\frac{dy}{dx} x^5 = 5x^4$

1(b)
$$y = \frac{1}{x^3}$$

Convert to index notation: $y = x^{-3}$
Differentiate using power rule: $\frac{dy}{dx}\ x^{-3} = -3x^{-4}$

1(c)
$y = 3x^4$
Differentiate using power rule: $\frac{dy}{dx}\ 3x^4 = 12x^3$

1(d)
$y = \pi$
Derivative of any constant value is 0: $\frac{dy}{dx}\ \pi = 0$

Sum Rule

Given Function $F(x) = f(x) + g(x)$

The derivative is $F'(x) = f'(x) + g'(x)$

I know in notation form it is rather complicated so I am going to explain it in relatively simple terms.

Imagine you have the following function

$f(x) = ax^n + bx^m$

To find the derivative, you must first split the function into two sections:

First Section	Second Section
ax^n	bx^m

Differentiate both separately to get

Derivative of first section	Derivative of second section
anx^{n-1}	bmx^{m-1}

Combine them back together to get your "overall" derivative

$f'(x) = anx^{n-1} + bmx^{m-1}$

If you still don't get it, take note of the following examples, often seeing how a real question work out clears most of your doubts.

Question 2

Differentiate the following with respect to x

(a) $y = 6x^7 + 2x^3$

(b) $y = 15x^2 + 4x^{-2} + \frac{1}{x}$

2(a)

Split the equation into two sections:

$\frac{dy}{dx} 6x^7 + 2x^3 = \frac{d}{dx} 6x^7 + \frac{d}{dx} 2x^3$

Differentiate the components one by one to get

$\frac{d}{dx} 6x^7 = 42x^6$	$\frac{d}{dx} 2x^3 = 6x^2$

Combine them back together to get the following

$\frac{dy}{dx} = 42x^6 + 6x^2$

2(b)

Convert all to index notation

$15x^2 + 4x^{-2} + x^{-1}$

Split equation into three sections

$\frac{dy}{dx} 15x^2 + 4x^{-2} + x^{-1} = \frac{d}{dx} 15x^2 + \frac{d}{dx} 4x^{-2} + \frac{d}{dx} x^{-1}$

Differentiate the components one by one to get

$\frac{d}{dx} 15x^2 = 30x$	$\frac{d}{dx} 4x^{-2} = -8x^{-3}$	$\frac{d}{dx} x^{-1} = -1(x)^{-2} = -x^{-2}$

Combine them to get

$\frac{dy}{dx} = 30x - 8x^{-3} - x^{-2}$

Difference Rule
Given Function $F(x) = f(x) - g(x)$
The derivative is $F'(x) = f'(x) - g'(x)$

Imagine the following function:

$$f(x) = ax^n - bx^m$$

To find the derivative split the function into 2 sections

First Section	Second Section
ax^n	$-bx^m$

Differentiate both section separately to get

anx^{n-1}	$-bmx^{m-1}$

Combine them back together to get your "overall" derivative

$f'(x) = anx^{n-1} - bmx^{m-1}$

Question 3:
Differentiate the following with respect to x
(a) $y = 5x^7 - 2x^3 - 7$
(b) $y = 3x^4 - 4x^2 + 5x^{-4}$

3(a)
Split equation into 3 sections to get
$\frac{dy}{dx}5x^7 - 2x^3 - 7 = \frac{d}{dx}5x^7 - \frac{d}{dx}2x^3 - \frac{d}{dx}7$

Differentiate the components one by one to get

$\frac{d}{dx}5x^7 = 35x^6$	$\frac{d}{dx} - 2x^3 = -6x^2$	$\frac{d}{dx}7 = 0$

Combine them back and you should get the following
$\frac{dy}{dx} = 35x^6 - 6x^2$

3(b)

Split equation into 3 sections to get

$\frac{dy}{dx}3x^4 - 4x^2 + 5x^{-4} = \frac{d}{dx}3x^4 - \frac{d}{dx}4x^2 + \frac{d}{dx}5x^{-4}$

Differentiate components one by one to get

$\frac{d}{dx}3x^4 = 12x^3$	$\frac{d}{dx} - 4x^2 = -8x$	$\frac{d}{dx}5x^{-4} = -20x^{-5}$

Combine them back to get the following:

$$\frac{dy}{dx} = 12x^3 - 8x - 20x^{-5}$$

Product Rule
Given Function $F(x) = f(x) \cdot g(x)$
The derivative is $F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Imagine the following function
$f(x) = (ax^n)(bx^m)$
$f'(x) = anx^{n-1}(bx^m) + ax^n(bmx^{m-1})$

Question 4
Differentiate the following with respect to x
(a) $y = (x^2 + 15)(6x^4 + 9)$
(b) $y = (5x^9 + 6x - 5)(2x^2 + 6)$

4(a)
$y = (x^2 + 15)(6x^4 + 9)$
$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 15) \cdot (6x^4 + 9) + \frac{d}{dx}(6x^4 + 9) \cdot (x^2 + 15)$$

$$\frac{dy}{dx} = 2x(6x^4 + 9) + 24x^3(x^2 + 15)$$
$$= 12x^5 + 18x + 24x^5 + 360x^3$$
$$= 36x^5 + 360x^3 + 18x$$

4(b)
$y = (5x^9 + 6x - 5)(2x^2 + 6)$
$$\frac{dy}{dx} = \frac{d}{dx}(5x^9 + 6x - 5) \cdot (2x^2 + 6) + \frac{d}{dx}(2x^2 + 6) \cdot (5x^9 + 6x - 5)$$

$$\frac{dy}{dx} = (45x^8 + 6)(2x^2 + 6) + (4x)(5x^9 + 6x - 5)$$

$$= (90x^{10} + 12x^2 + 270x^8 + 36) + (20x^{10} + 24x^2 - 20x)$$
$$= 110x^{10} + 270x^8 + 36x^2 - 20x + 36$$

Quotient Rule

Given function $F(x) = \frac{f(x)}{g(x)}$

The derivative is $F'(x) = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2}$

Imagine you have the following function

$f(x) = \frac{ax^n}{bx^n}$

$$f'(x) = \frac{anx^{n-1}\,(bx^n) - bnx^{n-1}(ax^n)}{(bx^n)^2}$$

Question 5

Differentiate the following with respect to x

(a) $y = \frac{2x^5+3x^2-7}{(2x+6)}$

$$y = \frac{2x^5 + 3x^2 - 7}{(2x + 6)}$$

$$\frac{dy}{dx} = \frac{\left[\frac{d}{dx}\left(2x^5+3x^2-7\right) \cdot (2x+6) - \frac{d}{dx}(2x+6) \cdot \left(2x^5+3x^2-7\right)\right]}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{\left[(10x^4+6x)(2x+6) - 2(2x^5+3x^2-7)\right]}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{20x^5+12x^2+60x^4+36x-4x^5-6x^2+14}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{16x^5+6x^2+60x^4+36x+14}{(2x+6)^2}$$

$$\frac{dy}{dx} = \frac{16x^5+60x^4+6x^2+36x+14}{\left(2(x+3)\right)^2}$$

$$\frac{dy}{dx} = \frac{16x^5+60x^4+6x^2+36x+14}{4(x+3)^2}$$

$$\frac{dy}{dx} = \frac{8x^5+30x^4+3x^2+18x+7}{2(x+3)^2}$$

Chain Rule
Given function
$F(x) = [f(x) + g(x)]^c$ {where c is a constant.}

The derivative is
$F'(x) = c[f(x) + g(x)]^{c-1} \cdot [f'(x) + g'(x)]$

Imagine you have the following function
$f(x) = (ax^n + bx^m)^c$
The derivative is
$f'(x) = c(ax^n + bx^m)^{c-1}\ [anx^{n-1} + bmx^{n-1}]$

Question 6
Differentiate the following with respect to x
(a) $y = (2x^3 - 5x^2 + 2x)^7$

$y = (2x^3 - 5x^2 + 2x)^7$

$\frac{dy}{dx} = 7(2x^3 - 5x^2 + 2x)^{7-1} \cdot (6x^2 - 10x + 2)$

$\frac{dy}{dx} = 7(2x^3 - 5x^2 + 2x)^6\ \ (6x^2 - 10x + 2)$

Tips and Tricks Below
Question 7
Differentiate the following with respect to x

$$y = \frac{1}{(x+9)^5}$$

Question: Should I use chain rule or quotient rule for this question?
Answer: Chain rule, it is easier to use chain rule as demonstrated below.

Reason Below:
By Chain Rule (3 Steps)

$$y = \frac{1}{(x+9)^5} = (x+9)^{-5}$$

$$\frac{dy}{dx} = -5(x+9)^{-6}(1)$$

$$\frac{dy}{dx} = -5(x+9)^{-6}$$

By Quotient Rule (4 Steps)**:**

$$y = \frac{1}{(x+9)^5}$$

$$\frac{dy}{dx} = \frac{\left[0(x+9)^5 - 5(x+9)^{5-1}(1)\right]}{(x+9)^{5(2)}}$$

$$\frac{dy}{dx} = \frac{-5(x+9)^4}{(x+9)^{10}}$$

$$\frac{dy}{dx} = -5(x+9)^{4-10}$$

$$\frac{dy}{dx} = -5(x+9)^{-6}$$

Conclusion: If the numerator $=$ 1 and the denominator is in the form of $(\boldsymbol{ax^n} + \boldsymbol{bx^m} \ldots + \boldsymbol{c})$, you just have to convert the expression or equation into index notation and use chain rule to differentiate. (This only applies when the numerator = 1.)

Title	Differentiation of Exponential and Logarithmic Function
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	20/12/2018

This article assumes you know and understand the notation for differentiation.
Applicable to

- Singapore-Cambridge Secondary School GCE 'O' Level Additional Mathematics
- Nanyang Polytechnic, School of Chemical and Life Sciences – Mathematics for Life Science Module
- Nanyang Polytechnic, School of Engineering – Engineering Mathematics 1A Module
- Singapore-Cambridge Junior College GCE 'A' Level H1 Mathematics
- Singapore-Cambridge Junior College GCE 'A' Level H2 Mathematics (Bridging Only)

Chain Rule (In General)
Given $f(x) = g(h(x))$ The derivative is given as $f'(x) = [g'(x)][h'(x)]$

General Cases for Derivatives of Exponential and Logarithmic Functions [Derived by Applying Chain Rule to the Specific Cases Mentioned Further Below.]	
Function	Derivatives
$f(x) = a^{g(x)}$	$f'(x) = [a^{g(x)} \ln(a)]g'(x)$
$f(x) = ce^{g(x)}$ Where c is a coefficient of $e^{g(x)}$, with e representing Euler's Number.	$f'(x) = \left[ce^{g(x)}\right]g'(x)$
$f(x) = \ln\left(g(x)\right)$	$f'(x) = \frac{1}{g(x)}[g'(x)] = \frac{g'(x)}{g(x)}$

Specific Case	Derivatives
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = ce^x$	$f'(x) = ce^x$
$f(x) = a^x$	$f'(x) = a^x \ln(a)$
$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$

Question 1.

Differentiate the following with respect to x

(a) e^{6x}
(b) $7e^{2x+6}$
(c) 5^{7x}
(d) 3^{2x+6}
(e) $\ln(6x^2 + 5x)$

Question 1

(a) $\frac{d}{dx}e^{6x}$

Rewrite as

$f(x) = e^{6x}$

$g(x) = e^{6x}$	$g'(x) = e^{6x}$
$h(x) = 6x$	$h'(x) = 6$
$f'(x) = 6e^{6x}$ $\frac{d}{dx} = 6e^{6x}$	

(b) $\frac{d}{dx}7e^{2x+6}$

Rewrite as

$f(x) = 7e^{2x+6}$

$g(x) = 7e^{2x+6}$	$g'(x) = 7e^{2x+6}$
$h(x) = 2x + 6$	$h'(x) = 2$
$f'(x) = 14e^{2x+6}$ $\frac{d}{dx} = 14e^{2x+6}$	

(c) $\frac{d}{dx} 5^{7x}$

Rewrite as
$f(x) = 5^{7x}$

$g(x) = 5^{7x}$	$g'(x) = 5^{7x} \ln(5)$
$h(x) = 7x$	$h'(x) = 7$
$f'(x) = 7\ (5^{7x}) \ln(5)$ $\frac{d}{dx} = 7(5^{7x}) \ln(5)$	

(d)
$\frac{d}{dx} 3^{2x+6}$

Rewritten as $f(x) = 3^{2x+6}$	
$g(x) = 3^{2x+6}$	$g'(x) = 3^{2x+6} \ln(3)$
$h(x) = 2x + 6$	$h'(x) = 2$

$\frac{d}{dx} = 2(3^{2x+6} \ln(3))$

(e)

$\frac{\mathrm{d}}{\mathrm{dx}}\ln(6x^2 + 5x)$

For students who can remember the formula for differentiation of $\ln[g(x)]$, go ahead and use the formula, unfortunately, this might not be the case for all students and I still want to demonstrate the more "tedious" approach. However, please note that the more tedious method requires less memory work.

Rewritten as:

$f(x) = \ln(6x^2 + 5x)$

$f'(x) = \frac{1}{6x^2+5x}(12x + 5)$

$f'(x) = \frac{(12x+5)}{6x^2+5x} \qquad \frac{d}{dx} = \frac{12x+5}{6x^2+5x}$

Title	Differentiation of Trigonometric Functions
Author	AprilDolphin
Date	15/9/2024
Notice	Questions all taken from books

Chain Rule in General
Given a function $F(x) = f(g(x))$ The derivative is given by $F'(x) = f'(x)\ g'(x)$

Rules of differentiation of trigonometric functions	
Function	Derivatives
$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$f(x) = \tan(x)$	$f'(x) = \sec^2(x)$

Application of Chain Rule to the above trigonometric functions	
Function	Derivatives
$f(x) = \sin[g(x)]$	$f'(x) = \cos[f(x))]\ [g'(x)]$
$f(x) = \cos[g(x)]$	$f'(x) = -\sin[f(x)]\ [g'(x)]$
$f(x) = \tan\ [g(x)]$	$f'(x) = \sec^2[f(x)][g'(x)]$

Example

Given the following functions, write down the derivative

1. $f(x) = x\tan(3x)$

   Using product rule of differentiation, given function of $f(x) = g(x)h(x)$, the derivative is given by $f'(x) = g(x)h'(x) + h(x)g'(x)$

   Using chain rule of differentiation, given function of $f(x) = g[h(x)]$, the derivative is given by $f'(x) = g'(x)\ h'(x)$

   Given $g(x) = \tan(3x)$
   The derivative is $g'(x) = 3\sec^2(3x)$

   Given $f(x) = x\tan(3x)$
   $f'(x) = \tan(3x)\ [1] + x[3\sec^2(3x)]$
   $f'(x) = \tan(3x) + 3x\sec^2(3x)$

2. $f(x) = \frac{1+\cos(x)}{\sin(x)}$

Using quotient rule of differentiation

$$F(x) = \frac{f(x)}{g(x)}$$

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[(g(x)]^2}$$

$$f'(x) = \frac{\sin(x)\,[-\sin(x)] - [1+\cos(x)][\cos(x)]}{\sin^2(x)}$$

$$f'(x) = \frac{[-\sin^2(x) - \cos^2(x) - \cos(x)]}{\sin^2(x)}$$

Using trigonometric identity $\sin^2 x + \cos^2 x = 1$,
Therefore $-\sin^2 x - \cos^2 x = -1$

$$f'(x) = \frac{-1 - \cos(x)}{\sin^2(x)}$$

3. $y = \sin\,(2x^2 + 5)$

Using chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{du} = \cos(2x^2 + 5)$$

$$\frac{du}{dx} = 4x$$

$$\frac{dy}{dx} = 4x\,[(\cos(2x^2 + 5)]$$

Differentiation of Trigonometric Functions Involving Numbered Exponents (Also called Powers)

Function	Derivatives
$k\sin^n(ax+b)$	$akn\sin^{n-1}(ax+b)\cos(ax+b)$
$k\cos^n(ax+b)$	$-akn\,cos^{n-1}(ax+b)\sin(ax+b)$
$k\tan^n(ax+b)$	$akn\tan^{n-1}(ax+b)\sec^2(ax+b)$

Given the following functions, differentiate them with respect to x

Q1. $3\sin^5(2x)$

Q2. $4\cos^3(5x-3\pi)$

Q3. $3\tan^4\left(\frac{\pi}{2}-\frac{\pi}{3}x\right)$

Question 1.

$$\frac{d}{dx}[3\sin^5(2x)]$$
$$= 3(5)(2)\sin^{5-1}(2x)\cos(2x)$$
$$= 30\sin^4(2x)\cos(2x)$$

Question 2.

$$\frac{d}{dx}[4\cos^3(5x-3\pi)]$$
$$= -4(5)(3)\cos^2(5x-3\pi)\sin(5x-3\pi)$$
$$= -60\cos^2(5x-3\pi)\sin(5x-3\pi)$$

Question 3.

Rewriting the question, we get the following

$$3\tan^4\left(\frac{\pi}{2}-\frac{\pi}{3}x\right) = 3\tan^4(-\frac{\pi}{3}x+\frac{\pi}{2})$$

$$\frac{d}{dx}[3\tan^4(-\frac{\pi}{3}x+\frac{\pi}{2})]$$

$$= -4\pi\,[\tan^3\left(\frac{\pi}{2}-\frac{\pi}{3}x\right)]\sec^2\left(\frac{\pi}{2}-\frac{\pi}{3}x\right)$$

Title	Differentiation – Applications of Differentiation to Finding Gradient, Tangent Line of Curves and Normal Line of Curves
Author	AprilDolphin
Date	18/11/2024

Properties of Normal Lines and Tangent Lines of Curve

Given that 2 lines are

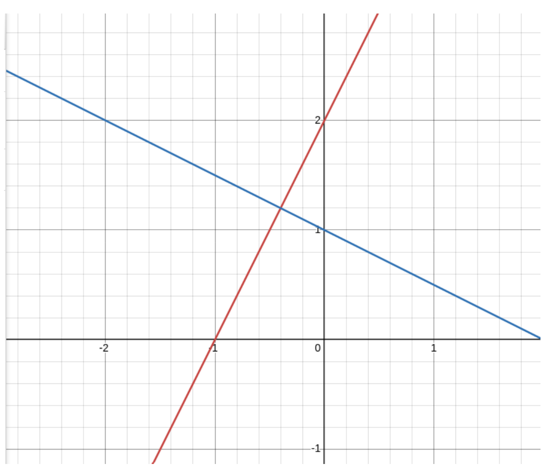
- Perpendicular to each other in a 2-Dimensional Cartesian Coordinate System
- One line is not perfectly vertical (By perfectly vertical, we imply the gradient is undefined resulting from division by zero.)
- One line is not perfectly horizontal (By perfectly horizontal, we imply the gradient is zero.)

And

M_1 is the gradient of the first line and M_2 is the gradient of the second line.

$(M_1)(M_2) = -1$

Example as shown below, the red line represents $y = 2x + 2$ and the blue line represents $y = -\frac{1}{2}x + 1$, multiplying the gradient of both lines will produce -1 as a result as both lines are perpendicular to each other.



Instructions for finding gradient and therefore the tangent line of the curve at the particular point and normal line intersecting the particular point.
Step 1. Differentiate the function and obtaining the first derivative
Step 2. Substitute the x-coordinate points into the function's first derivative to obtain the gradient of the function at the particular point (the gradient for which we shall represent it with m).
Step 3. Given the coordinate at the particular point is (x_1, y_1), the equation of the tangent line is $y - y_1 = m(x - x_1)$
Step 4. Given the coordinate at the particular point is (x_1, y_1), the equation of the normal line is $y - y_1 = -\frac{1}{m}(x - x_1)$
Example 1. Find the equations of tangent and normal to the curve $y = x - 2x^2 + 3$ at the point $(2, -3)$. Step 1. $\frac{dy}{dx} = 1 - 4x$ Step 2. $\frac{dy}{dx}\|_{x=2} = 1 - 4(2) = -7$ Gradient $= -7$ Step 3. Since gradient = -7, the value of m in $y = mx + c$ is -7 To find the value of c , we use the formula $y - y_1 = m(x - x_1)$ $y - (-3) = -7(x - 2)$ $y + 3 = -7x + 14$ $y = -7x + 11$ Step 4. Since the product of gradient of 2 perpendicular line is -1, the gradient of the normal line in question is $-\frac{1}{m}$. Therefore, equation of normal line is, $y - y_1 = -\frac{1}{m}(x - x_1)$ $y - (-3) = -\frac{1}{-7}(x - 2)$

$y + 3 = \frac{1}{7}x - \frac{2}{7}$ $y = \frac{1}{7}x - 3\frac{2}{7}$ $7y = x - 23y$
Example 2. Find the equation of the normal to the curve $y = \frac{3x-1}{x^2}$ at the point where $x = -1$.
Using quotient rule of differentiation, we find the gradient function, $\frac{dy}{dx}$ as seen below. $y = \frac{u}{v}$ $\frac{dy}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$ $\frac{dy}{dx} = \frac{x^2(3) - 2x(3x-1)}{x^4}$ $\frac{dy}{dx} = \frac{3x^2 - 6x^2 + 2x}{x^4}$ $\frac{dy}{dx} = \frac{-3x^2 + 2x}{x^4}$ $\frac{dy}{dx} = \frac{-3x + 2}{x^3}$ $\frac{dy}{dx}\vert_{x=-1} = \frac{-3(-1) + 2}{(-1)^3} = -5$ Find y −coordinate of the point where $x = -1$, using $y = \frac{3x-1}{x^2}$ $\frac{3(-1) - 1}{(-1)^2} = -4$

Finding equation of normal line at $(-1, -4)$

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - (-4) = -\frac{1}{-5}\big(x - (-1)\big)$$

$$y + 4 = \frac{1}{5}(x + 1)$$

$$y = \frac{1}{5}x + \frac{1}{5} - 4$$

$$y = \frac{1}{5}x - \frac{19}{5}$$

$$5y = x - 19$$

Title	Application of Differentiation – Rates of Change
Author	AprilDolphin & EpicMath5621A
Date	14/5/2026

Formula Required to Understand This Topic Well
Let's say we have two variables, where x and y are related so that $y = f(x)$ **We can relate the rate of change of y, $\frac{dy}{dx}$ to the rate of change of x, $\frac{dx}{dt}$ by means of the equation as a further application of the chain rule:** $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$
The following property of derivatives may be useful for students regardless of level Given two variables y and x , the **change in x with respect to y** can also be derived from the change in **y with respect to x** as follows: $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$

Question 1: (Taken from Mastering Additional Mathematics by Fairfield Book Publishers)
A viscous liquid is poured onto a flat surface. It forms a circular patch which grows at a steady rate of $3\pi\ cm^2$ per second. Find the following

(i) the radius of the patch 27 seconds after pouring the liquid

(ii) the rate of increase of the radius at that instant

Q1(i)
In order to be able to find the radius of the patch 27 seconds after pouring the liquid, we must make use of pre-existing information available in the question to determine the area of the patch at that timing.

Since the question mentioned the area increases at 3π per second, the area of the patch must be $3(27)\pi = 81\pi$ after 27 seconds has elapsed after the pouring action is done.

Let A be the area of the circular patch and r be radius of the circular patch.
Both can be related by $A = \pi r^2$
$81\pi = \pi r^2$
$r^2 = 81$
$r = 9$ cm

Q1(ii)
Since $A = \pi r^2$

The change in A with respect to r can be written as follows:
$\frac{dA}{dr} = 2\pi r$ as a consequences of the power rule

Which is $2\pi(9) = 18\pi$

$$\frac{dA}{dr} = \frac{dA}{dt} \times \frac{dt}{dr}$$
(Keep this in mind as it will be needed later in this question)

The change in radius with respect to time can be written as follows
Since, $18\pi = 3\pi \times \frac{dt}{dr}$

$$\frac{dt}{dr} = \frac{18\pi}{3\pi} = 6$$

Since $\frac{1}{\left(\frac{dt}{dr}\right)} = \frac{dr}{dt}$
$\frac{dr}{dt} = \frac{1}{6}$ cm/s

Question 2. (Taken from Study Smart Calculus Additional Mathematics by Singapore Asia Publisher Private LTD & Henry Loh)

Water is pumped into an elastic spherical ball and the rate of increase of the volume is $86.4\pi\ cm^3/s$. The initial volume of water in the spherical ball is $100\ cm^3$. Find the rate of increase of the radius when

- (i) the radius is 6 cm
- (ii) the volume is 972$\pi\ cm^3$
- (iii) the surface area is $36\pi\ cm^2$

Let V = volume, r = radius, t = time and A = surface area

(i)

Since $V = \frac{4}{3}\pi r^3$,

$\frac{dV}{dr} = \frac{4}{3}(3)\pi r^2 = 4\pi r^2$

Substitute $r = 6$ into the above will result in

$4(\pi)(6^2) = 144\pi$

$\therefore \frac{dV}{dr} = 144\pi$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$,

$86.4\pi = 144\pi \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{86.4\pi}{144\pi} = 0.6$ cm/s

(ii)

$$V = \frac{4}{3}\pi r^3 = 972\,\pi\ cm^3$$

$$r^3 - \frac{972\pi}{\left(\frac{4}{3}\right)\pi} = 729\ cm^3$$

$$r^3 = \sqrt[3]{729} = 9\ cm$$

Since as established earlier that $\frac{dV}{dr} = 4\pi r^2$, substituting $r = 9$ results in the following:

$4\pi(9^2) = 324\,\pi$

$$\frac{dV}{dr} = 4\pi r^2 = 324\,\pi$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$86.4\pi = 324\pi \times \frac{dr}{dt}$$

$\frac{dr}{dt} = \frac{86.4\pi}{324\pi} = 0.267$ cm/s

(iii)
Since the surface area of the spherical ball is $36\pi\ cm^2$
$4\pi r^2 = 36\pi$
$r^2 = 9$
$r = 3$
$V = \frac{4}{3}\pi(3)^3 = 36\pi$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$86.4\ \pi = 36\ \pi \times \frac{dr}{dt}$$

$\frac{dr}{dt} = \frac{86.4}{36} = 2.4\ cm/$s

Q3 (Taken from Mastering Additional Mathematics by Fairfield Book Publishers)
A sphere is increasing in volume at a rate of $10\pi\ cm^3$/s. Given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, calculate the radius of the sphere at the instant when the surface area is increasing at $8\pi\ cm^2$/s.

Let V be the volume of the sphere
Let A be the surface area of the sphere
Let r be the radius
Let t be the time in seconds
Volume of Sphere: $V = \frac{4}{3}\pi r^3$
Surface Area of Sphere = $4\pi r^2$

$$\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 = 4\pi r^2 = 10\pi\ cm^3/\text{s}$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times 10\pi = \frac{5}{2r^2}$$

$$\frac{dA}{dt} = 2(4)(\pi)(r) = 8\pi r = 8\pi\ cm^2/\text{s}$$

$$\frac{dA}{dr} = 8\pi r$$

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{8\pi r} \times 8\pi = \frac{1}{r}$$

$$\frac{1}{r} = \frac{5}{2r^2}$$

$$\frac{2r^2}{r} = 5$$
$$r = \frac{5}{2} = 2.5\ cm$$

Q4 (Taken from Mastering Additional Mathematics by Fairfield Book Publishers)
A circular metal plate expands by heating. If its radius changes at a constant rate of 0.06 cm/s, find the rate of change of its area when the radius is 5 cm.

Let A be the area of the metal plate, r be the radius and t be the time in seconds.
$\frac{dr}{dt} = 0.06$ cm/s.

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r = 2\pi(5) = 10\pi$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 10\pi \times 0.06 = 0.6\pi\ cm^2$/s

Q5 (Taken from Additional Mathematics 360 by Marshall Cavendish)

The radius r cm, of a melting spherical snowball is decreasing at a constant rate of p cm/h.

(i) Find an expression for the rate of change of the volume of the snowball, in terms of p.

(ii) Given that the initial rate of change of the volume of the snowball is M, find, in terms of M, the rate of change of the volume of the snowball when the radius is one-third of its initial value.

(i)

Let V be the volume of the snowball and r be the radius of the spherical snowball.

Volume of spherical ball: $V = \frac{4}{3}\pi r^3$

Rate of change of volume of spherical ball in relation to the radius: $\frac{dV}{dr} = 3\left(\frac{4}{3}\right)\pi r^2 = 4\pi r^2$

Rate of change of radius of spherical ball in relation to time: $\frac{dr}{dt} = -p$ cm/h.

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = -4p\pi r^2\ cm^3$/h

(ii)

Given M as the initial rate of volume change of snowball

$M = -4\pi r^2$

Let N be the rate of volume change of snowball when the radius is one third of initial value

$$N = -4\pi\left(\frac{r}{3}\right)^2$$

$$\frac{N}{M} = \frac{1}{9}$$

$$9N = M$$

$N = \frac{M}{9}\ cm^3$/h

Topic	Differentiation – Application of Differentiation – First and Second Derivatives (Stationary Points)
Author	AprilDolphin – Lim Wang Sheng
Date	4/6/2026

Notation (If function given in the form of $f(x)$)	
$f'(x)$	First derivative of function $f(x)$
$f''(x)$	Second derivative of function $f(x)$

Notation (If function is given in form of $y = f(x)$)	
$\frac{dy}{dx}$	First derivative of y with respect to x
$\frac{d^2y}{dx^2}$ $\frac{d}{dx}\left[\frac{dy}{dx}\right]$	Second derivative of y with respect to x

In any form of calculus, when we are asked to describe the "order" of derivative, we are actually describing them as follows

Order	Meaning
First Derivative	First Derivative of Function (Differentiate the function)
Second Derivative	Second Derivative of Function (Differentiate the derivative of the function)
Third Derivative	Third Derivative of Function (Differentiate the derivative of the derivative of the function)

While we are just focusing on the first and second derivative of functions in this topic, the list can keep going on and on as we approach more and more advanced calculus topics beyond the scope of this document.

Question 1:
Find the first and second derivative of the following functions
(a) $y = x^3 - 12x$
(b) $y = x^4 - 8x^2 + 2$

1(a)
First Derivative: $\frac{dy}{dx}[x^3 - 12x] = 3x^2 - 12$

Second Derivative: $\frac{d^2y}{dx^2}[x^3 - 12] = \frac{d}{dx}[3x^2 - 12] = 6x$

1(b)
First derivative $\frac{dy}{dx}[x^4 - 8x^2 + 2] = 4x^3 - 16x$

Second derivative: $\frac{d^2y}{dx^2}[x^4 - 8x^2 + 2] = \frac{d}{dx}[4x^3 - 16x] = 12x^2 - 16$

After understanding core basic concepts of first and second derivatives, the first and second derivative of a function can be used to determine how a function behaves, particularly in the context of determining if the function has a minimum point, maximum point or stationary point of inflexion.

Application of First Derivative Test, given the stationary point $\boldsymbol{P}$ at $\boldsymbol{x = a}$	**Consequence**
If the value of $\frac{dy}{dx}$ changes from negative to zero to positive as x increases through a	The point P is said to be a minimum point.
If the value of $\frac{dy}{dx}$ changes from positive to zero to negative as x increases through a	The point P is said to be a maximum point.
If the value of $\frac{dy}{dx}$ does not change sign as x increases through a (i.e. positive to zero to positive or negative to zero to negative)	The point P is said to be a stationary point of inflexion.

Application of Second Derivative Test, given the stationary point $\boldsymbol{P}$ at $\boldsymbol{x = a}$	**Consequence**
If $\frac{d^2y}{dx^2} < 0$ at $x = a$	P is a maximum point
If $\frac{d^2y}{dx^2} > 0$ at $x = a$	P is a minimum point
If $\frac{d^2y}{dx^2} = 0$ at $x = a$	The characteristic of point P cannot be concluded using Second Derivative Test and has to be concluded with First Derivative Test

General guidelines for using first derivative or second derivative test are as follows:
First derivative test is preferred if

- The second derivative at the point produces a value of zero
- The second derivative at the point cannot be obtained within reasonable time in assignments or exam conditions.

(Important note here: if the question restricts you to a specific method, just use that method as specified by the question.)

Question 2

Given that $y = x\sqrt{x+2}$

(i) Find $\frac{dy}{dx}$

(ii) Explain why the curve has only one stationary point and determine the nature of the stationary point.

Using product rule of differentiation, we can find the first derivative of $y = x\sqrt{x+2}$, which is $\frac{dy}{dx}$.

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + g'(x)f(x)$$

Which gives us the following steps required to differentiate.

$$y = x\ (x+2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 1(x+2)^{\frac{1}{2}} + \frac{1}{2}(x+2)^{-\frac{1}{2}}(1)(x) = (x+2)^{\frac{1}{2}} + \frac{1}{2}x(x+2)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \sqrt{x+2} + \frac{x}{2}\left(\frac{1}{\sqrt{x+2}}\right)$$

$$\frac{dy}{dx} = \sqrt{x+2} + \frac{x}{2\sqrt{x+2}}$$

Using some basic surds manipulation, we can simplify the derivative as follows

$$\frac{dy}{dx} = \sqrt{x+2} + \frac{x}{2\sqrt{x+2}} \times \frac{\sqrt{x+2}}{\sqrt{x+2}}$$

$$\frac{dy}{dx} = \frac{2(x+2)\sqrt{x+2}}{2(x+2)} + \frac{x\sqrt{x+2}}{2(x+2)}$$

$$\frac{dy}{dx} = \frac{(2x+4+x)\sqrt{x+2}}{2(x+2)}$$

$$\frac{dy}{dx} = \frac{\left[(3x+4)\sqrt{x+2}\right]}{2(x+2)}$$

Using indices law, we apply rule where $a^m(a^n) + a^{m+n}$

$$\frac{dy}{dx} = \frac{(3x+4)(x+2)^{\frac{1}{2}}(x+2)^{-1}}{2} = \frac{3x+4}{2}(x+2)^{\frac{1}{2}-1} = \frac{3x+4}{2\sqrt{x+2}}$$

(ii)
In order to be able to find the stationary point, we need to equate the first derivative $\frac{dy}{dx} = 0$ in the following manner.

$$\frac{3x + 4}{2\sqrt{x + 2}} = 0$$

$$3x + 4 = 0\left(2\sqrt{x + 2}\right)$$
$$3x + 4 = 0$$
$$3x = -4$$
$$x = -\frac{4}{3}$$

Since $\sqrt{x + 2}$ is defined for $x + 2 \geq 0$
$y = x\sqrt{x + 2}$ is well defined for when $x \geq -2$
Since $x = -\frac{4}{3}$ is the only solution of $\frac{dy}{dx} = 0$ that can satisfy the inequality above, the curve only has one stationary point.

To find the y-coordinates of the stationary point in question, we substitute $x = -\frac{4}{3}$ back into the original given function
So $-\frac{4}{3}\sqrt{\left(-\frac{4}{3}\right) + 2} = -1.09$ (3 significant figures)

So, the stationary point lies at $\left(-\frac{4}{3}, -1.09\right)$

Finding the second derivative of $y = x\sqrt{x + 2}$, we will apparently have to differentiate the first derivative as follows
Using quotient rule of differentiation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{3x + 4}{2\sqrt{x + 2}}\right]$$

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{x + 2}(3) - (3x + 4)\left(2 \cdot \frac{1}{2\sqrt{x + 2}}\right)}{\left[2\sqrt{x + 2}\right]^2}$$

$$\frac{d^2y}{dx^2} = \frac{6\sqrt{x+2} - \frac{3x+4}{\sqrt{x+2}}}{4(x+2)}$$

Using law of surds, we simply multiply the numerator and denominator by $\sqrt{x+2}$

$$\frac{d^2y}{dx^2} = \frac{\left(6\sqrt{x+2} - \frac{3x+4}{\sqrt{x+2}}\right)}{4(x+2)} \cdot \frac{\sqrt{x+2}}{\sqrt{x+2}}$$

$$\frac{d^2y}{dx^2} = \frac{6(x+2) - \frac{3x+4}{1}}{4(x+2)^{\frac{3}{2}}}$$

$$\frac{d^2y}{dx^2} = \frac{(6x+12-3x-4)}{4(x+2)^{\frac{3}{2}}}$$

$$\frac{d^2y}{dx^2} = \frac{3x+8}{4(x+2)^{\frac{3}{2}}}$$

By substituting the x-coordinate value of minimum point into the second derivative, we can obtain the second derivative value which is

$\frac{d^2y}{dx^2}$ at $x = -\frac{4}{3}$ is $\frac{3\left(-\frac{4}{3}\right)+8}{4\left[\left(-\frac{4}{3}\right)+2\right]^{\frac{3}{2}}} = 1.84$ (3 significant figures)

Since $\frac{d^2y}{dx^2}$ at $x = -\frac{4}{3} = 1.84 > 0$, the point is a minimum point.

Question 3.
Find the coordinates of the stationary points for the curve $y = 2x^3 + 6x^2 - 18x + 30$.
Determine the nature of all stationary point.

In the case of question 3, we are going to locate all stationary points first.
Finding the first derivative of the mentioned function, we have the following as a result:

$$\frac{dy}{dx} = 6x^2 + 12x - 18$$

Equating the first derivative as zero, we get:

$$\frac{dy}{dx} = 6x^2 + 12x - 18 = 0$$

$x = -3$ OR $x = 1$
Substituting the obtained value back into the original function would get us the y −coordinates of the stationary points.

$$y = 2(-3)^3 + 6(-3)^2 - 18(-3) + 30 = 84$$

$$y = 2(1)^3 + 6(1)^2 - 18(1) + 30 = 20$$

∴ The stationary points are established to be (-3.84) and $(1{,}20)$

Using the first derivative test, we can determine whether those coordinates are minimum point(s) or maximum point(s) through the following method:
To determine the gradient of the function near the established stationary point $(-3, 84)$, we pick a value slightly smaller than -3 as well as a value slightly larger than -3 and substitute into the first derivative obtained.
Example

$$\frac{dy}{dx} = 6x^2 + 12x - 18 = 6(-3.001)^2 + 12(3.001) - 18 = 0.024006$$

Test Value	Gradient Obtained	Result
−3.001	0.024006	Positive
−3	0	Zero
−2.998	- 0.047976	Negative

Since the graph of the function near $(-3{,}84)$ has a positive gradient to zero to negative as the value of x increases through the point, we can establish the point as a maximum point.

To determine the gradient of the function near the established stationary point $(1,20)$, we pick a value slightly smaller than 1 and a value slightly larger than 1 and substitute into the first derivative obtained.

After some calculation, we get the following:

Test Value	Gradient Obtained	Result
0.998	- 0.047976	Negative
1	0	Zero
1.001	0.024006	Positive

Since the gradient of the function as x increases through the point goes from negative to zero to positive, the point $(1,20)$ is a minimum point.

Using the second derivative test, we can also determine the characteristics of stationary points using the following method.

Since $\frac{dy}{dx} = 6x^2 + 12x - 18 = 0$

$x = -3$ or $x = 1$

The second derivative of the function can be obtained by differentiating the first derivative, which gives us $\frac{d^2y}{dx^2} = 12x + 12$.

Substituting $x = -3$ into the second derivative would give us $12(-3) + 12 = -24$, which is less than 0, therefore establishing $(-3, 84)$ as a maximum point

Substituting $x = 1$ into the second derivative would give us $12(1) + 12 = 24$, which is more than 0, therefore establishing $(1,20)$ as a minimum point.

Q4. The equation of a curve is $y = \frac{3x+1}{x-2}$ where $x \neq 2$.

(a) Find $\frac{dy}{dx}$.

(b) Explain whether this curve has a stationary point.

4(a)

Using quotient rule to find $\frac{dy}{dx}$, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\left(v\frac{du}{dx} - u\frac{dv}{dx}\right)}{v^2}$

$$\frac{[(x-2)(3) - (3x+1)(1)]}{(x-2)^2} =$$

$$\frac{3x - 6 - 3x - 1}{(x-2)^2} = -\frac{7}{(x-2)^2}$$

$$\frac{dy}{dx} = -\frac{7}{(x-2)^2}$$

4(b)

Since $\frac{dy}{dx} = -\frac{7}{(x-2)^2} < 0$ for all real value of x (therefore demonstrating the function is always decreasing), the curve has no stationary points.

Q5. For each curve, find the turning points and use the Second Derivative Test to determine the nature of each turning point.

(a) $y = 2x + \frac{18}{x}$

(b) $y = \frac{x^2}{x+1}$

5(a)

$$\frac{dy}{dx}(2x + 18x^{-1}) = 2 + (-1)18x^{-2} = 2 - 18x^{-2}$$

Equating first derivative as 0, the following is obtained:

$$2 - \frac{18}{x^2} = 0$$

$$-\frac{18}{x^2} = -2$$

$$-18 = -2(x^2)$$
$$0 = -2x^2 + 18$$

And hence, two turning points are established for the function $y = 2x + \frac{18}{x}$

$x = 3$ OR $x = -3$

Substituting $x = 3$ as well as $x = -3$ into the original function will get us the y-coordinates of the function:

$y = 12$ or $y = -12$

Hence, the turning points are as follows $(3,12)$ and $(-3,-12)$

Using the second derivative test to determine the nature of turning points, we find the second derivative and substitute the x-coordinates values of turning points back into second derivative function.

Obtaining the second derivative as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[2 - 18x^{-2}] = -2(-18)x^{-3} = 36x^{-3}$$

For point $(3,12)$, the second derivative value is $\frac{4}{3} > 0$ therefore establishing the point as a minimum point

For point $(-3,-12)$, the second derivative value is $-\frac{4}{3} < 0$, therefore establishing the point as a maximum point

5(b)

Equating first derivative as zero, we get the following steps

$$y = \frac{x^2}{x+1}$$

$$\frac{dy}{dx}\left[\frac{x^2}{x+1}\right] = \frac{(x+1)(2x) - (x^2)(1)}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}$$

$$\frac{x^2 + 2x}{(x+1)^2} = 0$$

$x^2 + 2x = 0$

$x = 0$ or $x = -2$

Substituting $x = 0$ and $x = -2$ back into the original equation will get us the y coordinate values of the turning points.

$y = 0$ or $y = \frac{(-2)^2}{(-2)+1} = \frac{4}{-1} = -4$

Therefore, coordinates of turning points of the curve are $(0,0)$ and $(-2,-4)$

Using the second derivative test to determine the nature of turning points, we find the second derivative and substitute the x-coordinates values of turning points back into second derivative obtained.

Obtaining second derivative as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{x^2 + 2x}{(x+1)^2}\right] = \frac{(x+1)^2(2x+2) - (x^2+2x)[2(x+1)(1)]}{(x+1)^4}$$

For turning point $(0,0)$

$$= \frac{(0+1)^2(2(0)+2) - (0^2 + 2(0))[2(0+1)(1)]}{(0+1)^4}$$

$= 2 > 0$ therefore, the turning point $(0,0)$ is a minimum point.

For turning point $(-2,-4)$

$$= \frac{(-2+1)^2(2)(-2) - [(-2)^2 + 2(-2)][2(-2)+1]}{[(-2)+1]^4}$$

$= -4 < 0$ therefore, the turning point $(-2,-4)$ is a maximum point.

Title	Integration of Algebraic Functions
Author	AprilDolphin – Lim Wang Sheng
Date	27/5/2026

The following concepts shall be covered in this topic

- Integration as the reverse process of differentiation
- Notation of Indefinite Integral
- Basic Concepts of Indefinite Integrals

Rules of Integration for Algebraic Functions

- Constant Multiple Rule
- Power Rule
- Sum Rule & Difference Rule

Integration as the reverse process of differentiation

As covered in earlier guide, we discussed how differentiation works, the reason why we study integration is the following reasons:

- There will be situations where finding the "anti-derivative" (AKA integral) is useful, if you are only given the gradient and ask to find out what is the equation of the function.
- Further study of integration will also be applied to finding the area under the graph of the function, using definite integral, which is useful for solving certain problems in various fields like Physics, Engineering, Chemistry and Information Technology.

Notation of Indefinite Integral

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

"Indefinite Integral of Function" Symbol

Arbitrary Constant

Explanation

Arbitrary Constant: The reason why we must have a "+C" to any indefinite integral is

- Reversing the integration process by differentiating the integral will result in many possible values of the constants that satisfy the integral, thus a "$+C$" is added, to represent "Constant"

Rules of Integration

Integration of Constant Values

$$\int k\ dx = kx + C$$

Integration of Functions with Power (Power Rule) [Provided $n \neq -1$]

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$

$$\int kx^n\ dx = k\left(\frac{x^{n+1}}{n+1}\right) + C = \frac{kx^{n+1}}{n+1} + C$$

$$\int (ax+b)^n\ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

Generalisation of Sum Rule

$$\int k \cdot f(x) + h \cdot g(x)\ \ dx\ = k\int f(x) + h\int g(x) + C$$

Generalisation of Difference Rule

$$\int k \cdot f(x) - h \cdot g(x)\ \ dx\ = k\int f(x) - h\int g(x) + C$$

Generalisation of Sum Rule, Difference Rule and Power Rule of Integration

(Provided $n \neq -1$ and $m \neq -1$)

$$\int kx^n \pm gx^m\ dx = k\frac{(x^{n+1})}{n+1} \pm g\frac{(x^{m+1})}{m+1} + C$$

Indefinite Integral

Example 1:

Find the following integrals [Power Function Integration Rule]

(a) $\int x^6 \; dx$

(b) $\int \frac{1}{x^8} \; dx$

1(a) $\int x^6 \; dx = \frac{x^{6+1}}{6+1} = \frac{x^7}{7} + C$

1(b) $\int \frac{1}{x^8} \; dx = \int x^{-8} = \frac{x^{-8+1}}{-8+1} = \frac{x^{-7}}{-7} + C = -\frac{1}{7}x^{-7} + C$

Example 2:

Find the following integrals [Integration of Constant Values]

(a) $\int 15 \, dx$

(b) $\int 25 \, dx$

2(a) $\int 15 \, dx \; = 15x + C$

2(b) $\int 25 \, dx = 25x + C$

Example 3:

Find the following integrals [Constant-Multiple (Power Function Integration) Rule]

(a) $\int 2x^5 \, dx$

(b) $\int 16x^7 \, dx$

3(a) $\int 2x^5 \, dx$

$$\int 2x^5 \, dx = 2 \int x^5 \, dx + C = 2\left(\frac{x^{5+1}}{5+1}\right) + C = \frac{2x^{5+1}}{6} = \frac{2x^6}{6} + C = \frac{1}{3}x^6 + C$$

3(b) $\int 16x^7 \, dx$

$$\int 16x^7 \, dx = 16 \int x^7 \, dx = 16\left(\frac{x^{7+1}}{7+1}\right) + C = \frac{16x^8}{8} + C = 2x^8 + C$$

Example 4:

Find the following integrals [Generalized Sum Rule]

(a) $\int 2x^9 + 5x^3\, dx$

(b) $\int 9x^7 + 6x^{-8} + 12x + 7\, dx$

4(a) $\int 2x^9 + 5x^3\, dx$

$$= 2\int x^9 + 5\int x^3$$

$$= 2\left(\frac{x^{9+1}}{9+1}\right) + 5\left(\frac{x^{3+1}}{3+1}\right) + C$$

$$= 2\left(\frac{x^{10}}{10}\right) + 5\left(\frac{x^4}{4}\right) + C$$

$$= \frac{1}{5}x^{10} + \frac{5}{4}x^4 + C$$

4(b) $\int 9x^7 + 6x^{-8} + 12x + 7\, dx$

$$= 9\int x^7 + 6\int x^{-8} + 12\int x + \int 7$$

$$= 9\left(\frac{x^{7+1}}{7+1}\right) + 6\left[\frac{x^{(-8)+1}}{(-8)+1}\right] + 12\left(\frac{x^{1+1}}{1+1}\right) + 7x + C$$

$$= \frac{9}{8}x^8 + 6\left(\frac{x^{-7}}{-7}\right) + \frac{12x^2}{2} + 7x + C$$

$$= \frac{9}{8}x^8 - \frac{6}{7}x^{-7} + 6x^2 + 7x + C$$

Example 5

Find the following integrals [Generalized Difference Rule]

(a) $\int 2x^2 - 9x - \frac{2}{x^9} - 8 \; dx$

5(a)

$$\int 2x^2 - 9x - \frac{2}{x^9} - 8 \, dx$$

$$= 2\int x^2 - 9\int x - 2\int x^{-9} - \int 8$$

$$= 2\left(\frac{x^{2+1}}{2+1}\right) - 9\left(\frac{x^{1+1}}{1+1}\right) - 2\left(\frac{x^{-9+1}}{-9+1}\right) - 8(x) + C$$

$$= 2\left(\frac{x^3}{3}\right) - 9\left(\frac{x^2}{2}\right) - 2\left(\frac{x^{-8}}{-8}\right) - 8(x) + C$$

$$= 2\left(\frac{x^3}{3}\right) - 9\left(\frac{x^2}{2}\right) + 2\left(\frac{x^{-8}}{8}\right) - 8x + c$$

$$= \tfrac{2}{3}x^3 - \tfrac{9}{2}x^2 + \tfrac{2}{8}x^{-8} - 8x + C$$

$$\frac{2}{3}x^3 - \frac{9}{2}x^2 + \frac{1}{4}x^{-8} - 8x + C$$

Example 6

Find the following integrals [Generalization of all Rules Mentioned Combined]

(a) $\int 3x^5 - 6x^4 + 11x - 16 + 9x^{-7}\ \ dx$

$$3\int x^5 - 6\int x^4 + 11\int x - \int 16 + 9\int x^{-7}$$

$$3\left(\frac{x^{5+1}}{5+1}\right) - 6\left(\frac{x^{4+1}}{4+1}\right) + 11\left(\frac{x^{1+1}}{1+1}\right) - 16(x) + 9\left(\frac{x^{-7+1}}{-7+1}\right) + C =$$

$$3\left(\frac{x^6}{6}\right) - 6\left(\frac{x^5}{5}\right) + 11\left(\frac{x^2}{2}\right) - 16x + 9\left(\frac{x^{-7+1}}{-7+1}\right) + C =$$

$$\frac{3}{6}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x + \frac{9x^{-6}}{(-6)} + C =$$

$$\frac{1}{2}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x - \frac{9}{6}x^{-6} + C =$$

$$\frac{1}{2}x^6 - \frac{6}{5}x^5 + \frac{11}{2}x^2 - 16x - \frac{3}{2}x^{-6} + C$$

Title	Integration Leading to Logarithmic Functions and Integration of Exponential Functions
Author	Lim Wang Sheng, School of Information Technology, Nanyang Polytechnic [CCA: NYP Mentoring Club]
Date	21/12/2018

Applicable to

- JC H2 'A' Level Mathematics (Bridging only) and H1 'A' Level Mathematics
- Nanyang Polytechnic School of Chemical and Life Sciences – Mathematics for Life Sciences
- Nanyang Polytechnic School of Engineering – Engineering Mathematics 1B
- Institute of Technical Education – Calculus/Mathematics Modules
- Secondary School GCE 'O' Level Additional Mathematics

Function	Corresponding Integrals
$f(x) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$f(x) = e^x$	$\int e^x\,dx = e^x + C$
$f(x) = ae^x$	$\int ae^x\,dx = ae^x + C$

(Generic Cases) Functions	Corresponding Integrals
$f(x) = \frac{1}{ax+b}$	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$
$f(x) = \frac{a}{x}$	$\int \frac{a}{x} = a(\ln(x)) + C$
$f(x) = e^{ax+b}$	$\int e^{ax+b}\,dx = \frac{1}{a}\,(e^{ax+b}) + C$

Find the following integrals

a) $\int \frac{3}{x}\, dx$

b) $\int e^{6x+5}\, dx$

c) $\int 5e^{x}\ dx$

d) $\int \frac{1}{9x+5} dx$

e) $\int e^{8x+6}$

Solutions

a) $\int \frac{3}{x}\, dx = 3(\ln(x)) = 3\ln(x) + C$

b) $\int e^{6x+5}\, dx = \frac{1}{6}(e^{6x+5}) + C$

c) $\int 5e^{x}\ dx = 5e^{x} + C$

d) $\int \frac{1}{9x+5} dx\ = \frac{1}{9}(\ln(9x+5)) + C$

e) $\int e^{8x+6}\ dx = \frac{1}{8}(e^{8x+6}) + C$

Title	Integration of Trigonometric Functions
Author	AprilDolphin
Date	16/9/2024
Notice	Questions all taken from books

Functions	Corresponding Integrals
$F(x) = \sin(x)$	$f(x) = -\cos(x) + c$
$F(x) = \cos(x)$	$f(x) = \sin(x) + c$
$F(x) = \sec^2(x)$	$f(x) = \tan(x) + c$

Functions	Corresponding Integrals
$F(x) = \sin(ax + b)$	$f(x) = \dfrac{-\cos(ax + b)}{a} + c$
$F(x) = \cos(ax + b)$	$f(x) = \dfrac{\sin(ax + b)}{a} + c$
$F(x) = \sec^2(ax + b)$	$f(x) = \dfrac{\tan\,(ax + b)}{a} + c$

Example:

Given the following derivatives, find the corresponding integrals

1. $\cos(2x) + \sin\,(5x - 1)$

$$\int [\cos(2x) + \sin(5x - 1)]\, dx \; =$$

$$\frac{\sin(2x)}{2} + \frac{-\cos(5x - 1)}{5} =$$

$$\frac{\sin(2x)}{2} - \frac{\cos(5x - 1)}{5} + c$$

2. $\sec^2(3x + 5) - \cos\,(5x + 5)$

$$\int [\sec^2(3x + 5) - \cos(5x + 5)]\, dx \; =$$

$$\frac{\tan(3x + 5)}{3} - \frac{\sin(5x + 5)}{5} + c$$

Title	Definite Integrals – Area Between Curve and x-axis
Author	AprilDolphin – Lim Wang Sheng
Date	5/3/2026

In this topic, we have to cover three cases of finding definite integral, namely

- Case 1: Region above x-axis and under the curve
- Case 2: Region below x-axis and above the curve
- Case 3: Region for which some parts of function is above x-axis and some parts of function below x-axis.

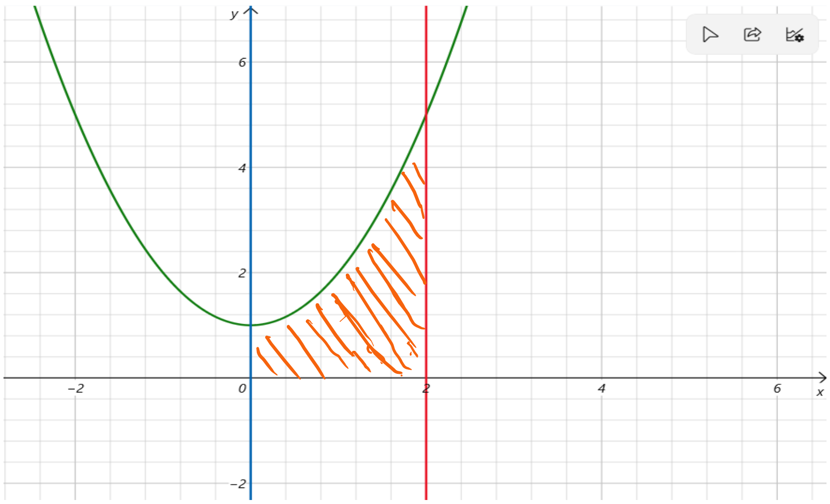
Notation used in definite integrals	
$\int_b^a f(x) = F(b) - F(a)$	
Explanation of Notation: $F(b) - F(a)$	a and b in this case represents the limit of the definite integral.

Case 1: Region above x-axis and under the curve

1.

Find the following definite integral

$$\int_0^2 x^2 + 1\ dx$$

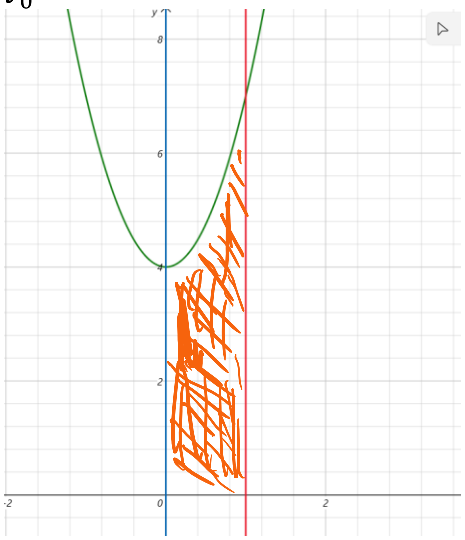


Step 1	Find indefinite integral of the function mentioned $\int x^2 + 1\, dx = \frac{x^3}{3} + x$
Step 2	Rewrite the indefinite integral in definite form as mentioned by the question. $\left[\frac{x^3}{3} + x\right]_0^2$
Step 3	Substitute upper and lower limit into definite integral, deducting the upper limit by lower limit to find area under curve and above x-axis. $\left[\frac{x^3}{3} + x\right]_0^2$ $\left\lvert\left(\frac{2^3}{3} + 2\right) - \left(\frac{0^3}{3} + 0\right)\right\rvert = \frac{14}{3}\ \text{units}^2$

2.

Find the following definite integral

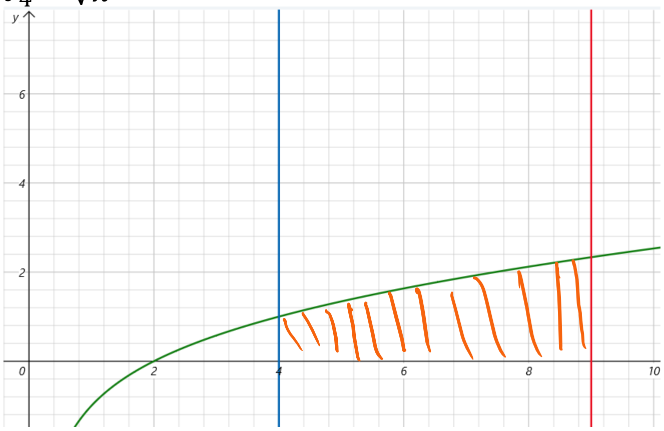
$$\int_0^1 4 + 3x^2 \, dx$$



Step 1	Find indefinite integral of the function mentioned $\int 4 + 3x^2 \, dx = 4x + \frac{3x^3}{3} = 4x + x^3$
Step 2	Rewrite the indefinite integral in definite form as mentioned by the question. $\left[4x + \frac{x^3}{1}\right]_0^1$
Step 3	Substitute upper and lower limit into definite integral, deducting the upper limit by lower limit to find area under curve and above x-axis. $\left\|\left[4(1) + \frac{1^3}{1} - \left(4(0) + \frac{0^3}{1}\right)\right]\right\| = 5 \text{ units}^2$

3. Find the following definite integral

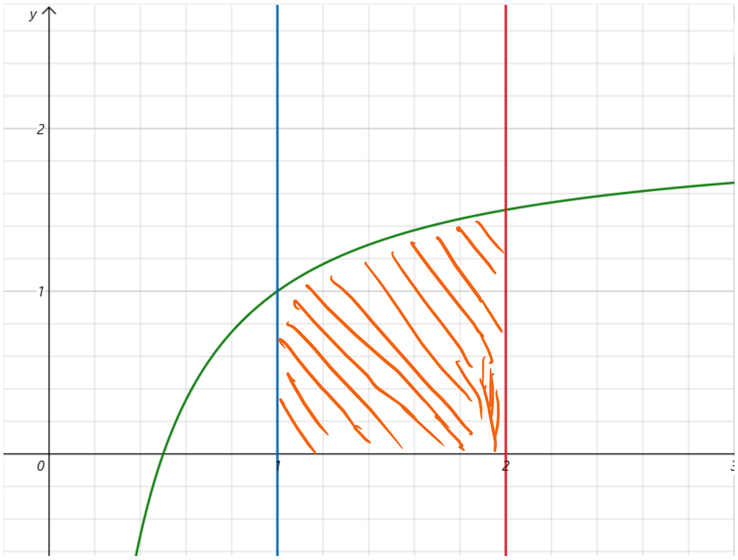
$$\int_4^9 \frac{x-2}{\sqrt{x}}\,dx$$



Step 1	Rewrite the function in index form $\int_4^9 \frac{x-2}{x^{\frac{1}{2}}}dx = \int_4^9 x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$
Step 2	Find the indefinite integral of the function mentioned $\int x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 2\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) = \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$
Step 3	Rewrite the indefinite integral of function in definite form as mentioned by the question $\left[\frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}}\right]_4^9$
Step 4	Substitute upper and lower limit into definite integral, deducting the upper limit by lower limit to find area under curve and above x-axis. $\left\|\left[\frac{2}{3}(9)^{\frac{3}{2}} - 4(9)^{\frac{1}{2}} - \left[\frac{2}{3}(4)^{\frac{3}{2}} - 4(4)^{\frac{1}{2}}\right]\right\|$ $= 6 - \left(-\frac{8}{3}\right)$ $= \frac{26}{3}$ $= 8\frac{2}{3}$ units^2

4. Find the following definite integral

$$\int_1^2 \frac{2x-1}{x}\,dx$$



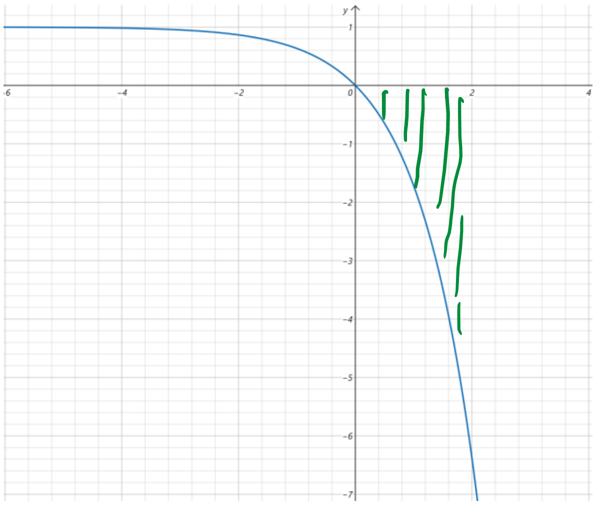
Step 1	Rewrite the function in index form $\int_1^2 \left(2-\frac{1}{x}\right)\,dx$
Step 2	Find the indefinite integral of the function. $\int \left(2-\frac{1}{x}\right)dx = 2x - \ln(x)$
Step 3	Rewrite the indefinite integral of function in definite form as mentioned by the question $[2x-\ln(x)]_1^2$
Step 4.	Substitute upper and lower limit into definite integral, deducting the upper limit by lower limit to find area under curve and above x-axis. $2(2)-\ln(2)-[2(1)-\ln(1)] =$ $4-\ln(2)-2+0$ $= [2-\ln(2)]\text{ units}^2$

Case 2: Region below x-axis and above the curve

5. Find the following definite integral

$$\int_2^0 (1 - e^x)dx$$

In this case, we sketched the graph using a graphing application to determine the behavior of the function within the specified limit and roughly shaded the area above curve and below the x-axis.

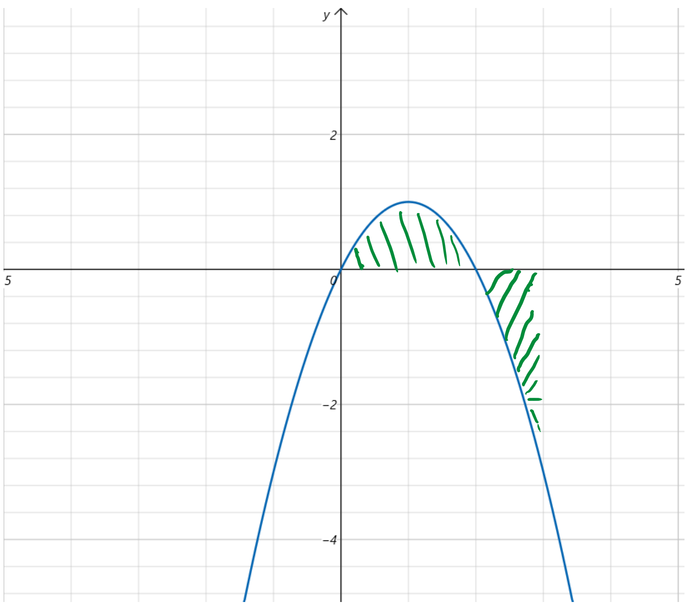


Step 1.	Find indefinite integral of function $\int (1 - e^x)\,dx = \frac{x^{0+1}}{0+1} - e^x = x - e^x$
Step 2.	Rewrite the indefinite integral of function in definite form as mentioned by the question $[x - e^x]_2^0 =$
Step 3.	Substitute upper and lower limit into definite integral, deducting the upper limit by lower limit to find area under curve and above x-axis. $2 - e^2 - (0 - e^0) = 2 - e^2 - (-1) = 3 - e^2$
Step 4.	If the definite integral has a negative value, negate the answer by applying negative signs to the answer to obtain area under curve. $-(3 - e^2) = (e^2 - 3)\ \text{units}^2$

Case 3: Region for which some parts of function is above x-axis and some parts of function below x-axis.

6. Evaluate $\int_3^0 (2x - x^2)\,dx$

In this case, a sketch of the graph is created to determine the behavior of the function within the specified limits.



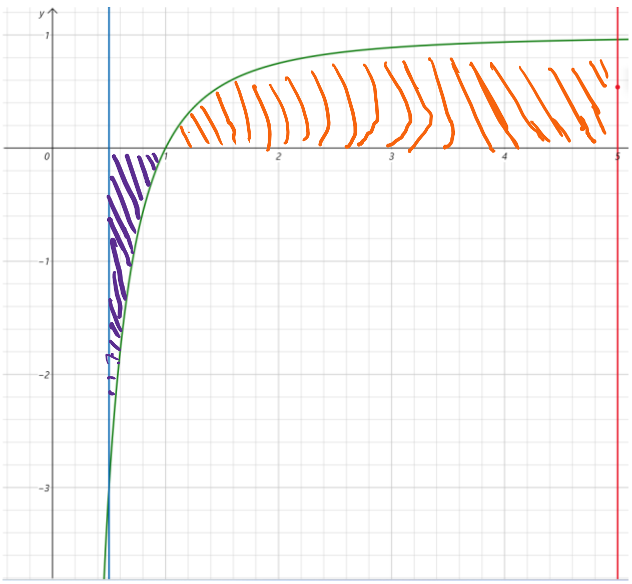
Step 1	Find the indefinite integral of the function $\int (2x - x^2)dx = \frac{2x^{1+1}}{1+1} - \frac{x^3}{2+1} = x^2 - \frac{x^3}{3}$
Step 2	Determine the x-coordinates for which the function is zero. In this case, the x-coordinates can be determined using the calculator which give us $x = 0$ and $x = 2$
Step 3	Split up the integral into two sections by working out the area at between 0 and 2 and between 2 and 3 For definite integral between 0 and 2 $\left[x^2 - \frac{x^3}{3}\right]_2^0 = 1\frac{1}{3}$ units^2 For definite integral between 2 and 3 $\left[x^2 - \frac{x^3}{3}\right]_3^2 = 3^2 - \frac{3^3}{3} - \left(2^2 - \frac{2^3}{3}\right) = -1\frac{1}{3}$ units^2

Step 4.	Add up definite integrals from both sides to get $1\frac{1}{3} + \left(-1\frac{1}{3}\right) = 0 \text{ units}^2$

7. The diagram shows a part of the curve $y = 1 - \frac{1}{x^2}$, and the lines $x = 0.5$ and $x = 5$. Find the area bounded by the curve $y = 1 - \frac{1}{x^2}$, the x-axis and the line

(i) $x = 0.5$

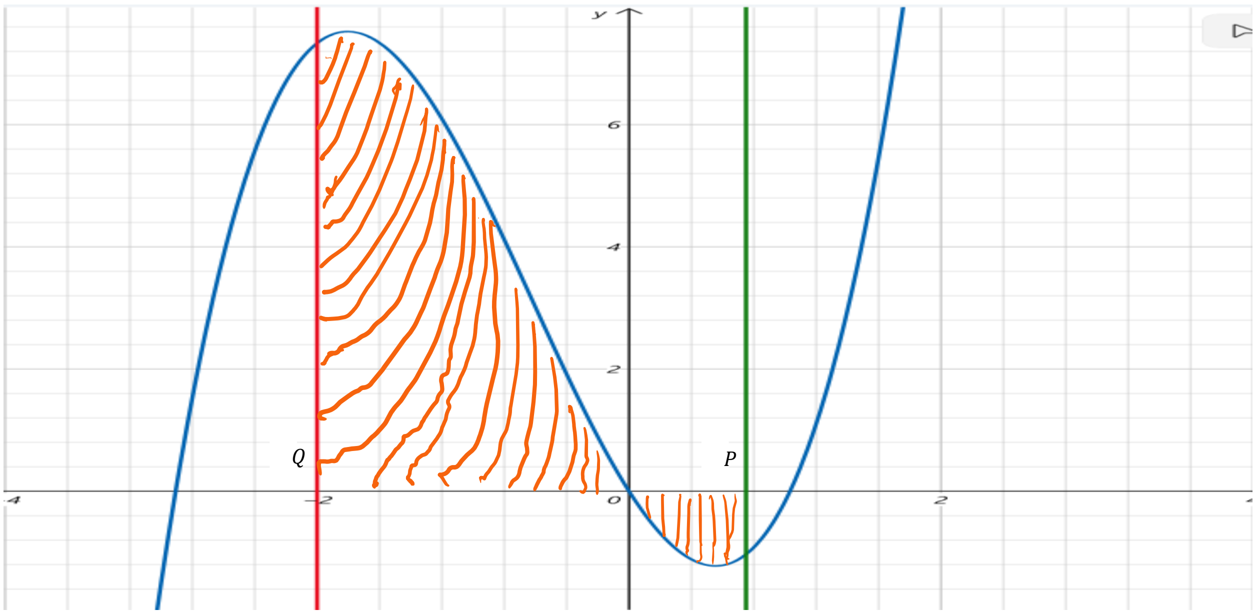
(ii) $x = 5$



Step 1:	Find indefinite integral of function $\int (1 - x^{-2})\ dx = x + \frac{1}{x}$
Step 2.	Since from the diagram we determined that the function is zero at $x = 1$, the integral can be evaluated as split up. (i) $\left[x + \frac{1}{x}\right]_{1}^{0.5} = \left(1 + \frac{1}{1}\right) - \left(0.5 + \frac{1}{0.5}\right) = 0.5 \text{ units}^2$ (ii) $\left[x + \frac{1}{x}\right]_{5}^{1} = \left(5 + \frac{1}{5}\right) - \left(1 + \frac{1}{1}\right) = 3.2 \text{ units}^2$

8. The diagram shows part of the graph of $y = \frac{4}{3}x^3 + \frac{5}{2}x^2 - 4x$.

The x-coordinates of the point P is $\frac{3}{4}$. The x-coordinate of point Q is -2. Find the total area of the shaded region bounded by the curve, the x-axis and the lines from P and Q perpendicular to the x-axis.



Step 1	Find indefinite integral of the function $\int \frac{4}{3}x^3 + \frac{5}{2}x^2 - 4x = \frac{\frac{4}{3}x^{3+1}}{3+1} + \frac{\frac{5}{2}x^{2+1}}{2+1} - \frac{4x^{1+1}}{1+1}$ $= \frac{x^4}{3} + \frac{5x^3}{6} - 2x^2$
Step 2	Determine points for which the function is zero. In this case, the diagram showed the function within the specified limits is zero when the function has x being exactly zero so the integral must be evaluated as split up.
Step 3	Evaluate definite integral from $x = -2$ to $x = 0$ $\left[\frac{x^4}{3} + \frac{5x^3}{6} - 2x^2\right]_{-2}^{0} = 0 - \left[\frac{(-2)^4}{3} + \frac{5(-2)^3}{6} - 2(-2)^2\right] = \frac{28}{3}$

	Evaluate definite integral from $x = 0$ to $x = \frac{3}{4}$ $-\left[\frac{\left(\frac{3}{4}\right)^4}{3} + \frac{5\left(\frac{3}{4}\right)^3}{6} - 2\left(\frac{3}{4}\right)^2\right] - 0 = \frac{171}{256}$
Step 4	Add up definite integral from both limits to obtain total area under curve. $\frac{171}{256} + \frac{28}{3} = \frac{7681}{768}$ units^2