H3 Topic 1 – Frames Of Reference (Non-Relativistic)

Content

- Reference frames
- Inertial frames
- Centre of mass frame

Learning Outcomes

Candidates should be able to:

(a) state that a frame of reference is a set of coordinates that can be used to determine positions and times of events in that frame

(b) show an understanding that Newton's laws of motion are obeyed in all inertial frames of reference

(c) recall and apply the Galilean transformation equations to solve problems relating observations in different inertial frames of reference

(d) show an understanding that the centre of mass frame (or zero momentum frame) is the inertial frame in which the total linear momentum of the system is zero

(e) solve one-dimensional collision problems by considering velocities relative to the centre of mass of the system (i.e. in the zero-momentum frame).

1. The Concept of a Frame of Reference

A **frame of reference** is a specific space or set of coordinates in which an observer makes measurements of physical quantities, such as the position and time of events.

Every velocity or speed measurement is inherently relative to a "point of reference" taken to be stationary.

For example, a car moving at is measured relative to the road's surface. If measured from a moving truck, that speed would differ. Generally, a coordinate system (often with an origin at the point of reference) is established to facilitate quantitative calculations.

1.1 Inertial Frames of Reference

An inertial frame is one in which the point of reference is not undergoing acceleration. In these frames, Newton's laws of motion are valid; specifically, an object with no net external force remains at rest or moves at a constant velocity.

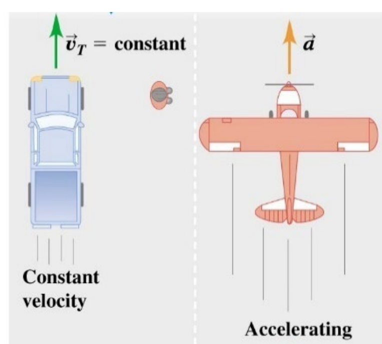
• The Earth is often treated as an approximate inertial frame, despite its small rotational and orbital accelerations.

• Any frame moving at a constant velocity relative to an inertial frame is also an inertial frame.

• Einstein's first postulate suggests that all laws of physics remain the same in every inertial frame.

1.2 Non-Inertial Frames of Reference

A non-inertial frame is one that is accelerating relative to an inertial frame. In such frames, Newton's laws are not obeyed. For instance, an accelerating aeroplane is a non-inertial frame because it does not obey Newton's first law relative to a stationary observer.

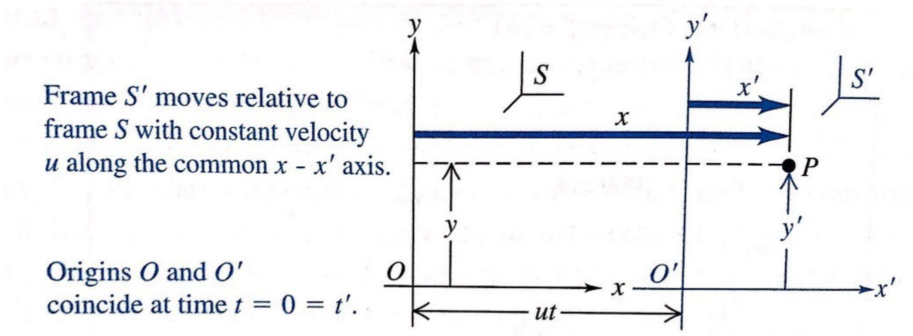


2. Galilean Transformation Equations

2.1 Coordinate Transformations

Consider two inertial frames of reference S and S' respectively. To keep things simple, we have omitted the z-axis. The reference frame S' is moving at a speed of u and the origin of the two reference frames coincide at time $t = t' = 0$ such that their separation at a later time t is ut.

The position of particle P can be described by coordinates x and y in reference frame S or coordinates x' and y' in reference frame S'.



In a 3-dimensional analysis, the position of particle P can be expressed by the Galilean Coordinate Transformation equations:

$$x' = x - ut \quad \cdots(1)$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

2.2 Velocity Transformations

If particle *P* moves in the x-direction, its velocity v_x as measured by an observer stationary in frame *S* is $v_x = \frac{dx}{dt}$ and its velocity v'_x as measured by an observer stationary in frame *S'* is $v'_x = \frac{dx'}{dt}$.

Differentiating (1) with respect to time gives:

$$\frac{dx'}{dt} = \frac{dx}{dt} - u$$

We arrive at the Galilean Velocity Transformation equation:

$$\boxed{v'_x = v_x - u}$$

where v'_x is the horizontal component of velocity with respect to reference frame *S*',
v_x is the horizontal component of velocity with respect to reference frame *S*,
u is the horizontal component of velocity of reference frame *S*' with respect to *S.*

Example 1

A boat crossing a river moves with a speed of 3.0 m s^{-1} relative to the water. The water in the river has a speed of 1.2 m s^{-1} due east relative to the Earth.

(a) If the boat heads due north in the frame of reference of the water, determine the velocity of the boat relative to an observer standing on either bank.

(b) If the boat travels with the same speed of 3.0 m s^{-1} relative to the river and is to travel due north as observed by a stationary on the bank, what direction should it head to?

(c) In which direction, **(a)** or **(b)**, does the boat head to cross the river in the shorter time?

Solution

Example 2

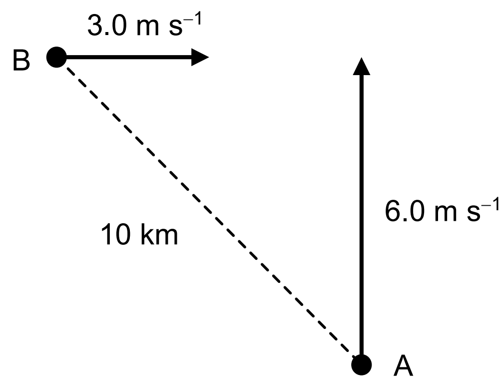
To a cyclist travelling due North along a straight road at 4.0 m s^{-1}, the wind appears to come from the East. If he increases his speed to 9.0 m s^{-1}, it appears to blow from the North-East.

Determine the speed and direction of the wind.

Solution

Example 3

Ship A is travelling due North at 6.0 m s^{-1} and ship B, 10 km North-West of it initially, is travelling due East at 3.0 m s^{-1}, as shown in the figure below.



Determine

(a) the shortest distance between the ships subsequently, and

(b) the time taken for the ships to meet at this distance.

Solution

3. Centre of Mass Frame of Reference

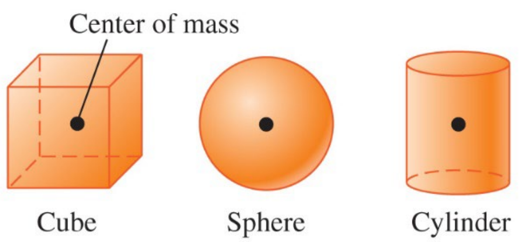
3.1 Location of the Centre of Mass

The centre of mass of a system of masses is the point where the system can be balanced in a uniform gravitational field. It is the point at which total mass of the system is assumed to be concentrated.

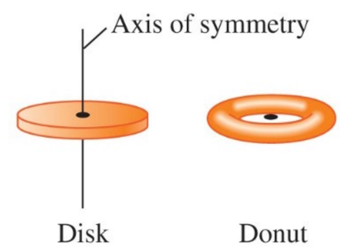
The position of the centre of mass of the system is given by:

$$\boxed{x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots} = \frac{\sum m_i x_i}{\sum m_i} \quad \cdots(2)}$$

In systems with a continuous, uniform distribution of mass, the centre of mass is at the geometric centre of the object. It is common for the centre of mass to be located in a position where no mass exists, as in a donut, where the centre of mass is precisely at the centre of the hole.



If a homogeneous object has a geometric center, that is where the center of mass is located.



If an object has an axis of symmetry, the center of mass lies along it. As in the case of the donut, the center of mass may not be within the object.

Example 4

A system consists of three particles:

m_1 = 1kg located at (1, 0); m_2 = 1kg located at (2, 0); m_3 = 2kg located at (0, 2).

Determine the location of the centre of mass of the system.

Solution

3.2 Centre of Mass Frame of Reference

The **centre of mass frame** (or zero momentum frame) is the inertial frame in which the total linear momentum of the system is zero.

In H2 physics, we studied collisions between particles primarily in the laboratory frame of reference (lab frame), in which all motions of the particles are relative to the lab. The lab frame is intuitive, because that is how we look at a collision event in the lab. We may calculate the total momentum of the system of particles at one instant, and take it to be constant provided that there is no net external force acting on the system. The total momentum of the system is, in general, nonzero.

There is another frame of reference, namely the centre of mass frame of reference (CM frame), that is of special importance. In the CM frame, by definition, the centre of mass is stationary, i.e. v_{CM} is zero. Using Eq. (1.2.3), we see that

$$p_{\text{total}} = \sum_i p_i = Mv_{\text{CM}} = 0 \text{ in the CM frame}.$$

To put it in words, the total momentum of a system of particles is identically zero in the CM frame, by virtue of the definition of the CM frame. Therefore, the CM frame is also called the zero-momentum frame.

Note that, although the total momentum of the system of particles is nonzero in the lab frame but is zero in the CM frame, the principle of conservation of momentum applies to each frame of reference. In the lab frame, the total momentum of the system is a fixed, generally nonzero, value at any instant in time, whereas in the CM frame, the total momentum is always zero. Similarly, the collision is elastic in both frames, so the kinetic energies before and after the collisions are also conserved in each frame, again, with different values for each frame. This is an example of the principle of Galilean relativity, which states that the laws of nature (such as the principle of conservation of momentum) are the same for all inertial frames of reference.

Example 5

A particle of mass 1.0 kg is moving at 2.0 m s^{-1} towards a stationary particle of mass 3.0 kg.

(a) Determine the speed of centre of mass of the system.
(b) Determine the speeds of the two particles in the CM frame.
(c) Show that the total momentum of the system is zero in the CM frame.

Solution

3.3 Collisions in One-Dimension: Inelastic Collisions

In a completely inelastic collision, v_{CM} is equal to the common velocity of the system after collision since v_{CM} remains the same before and after collision.

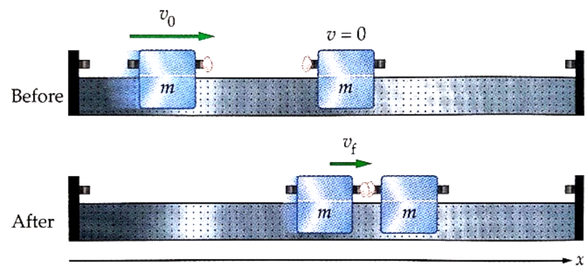
Example 6

An air cart of mass *m* and speed *u* moves towards an identical air cart that is initially at rest. When the carts collide, they stick together and move as one.

Determine the velocity of the centre of mass of this system

(a) before the carts collide and

(b) after the carts collide.

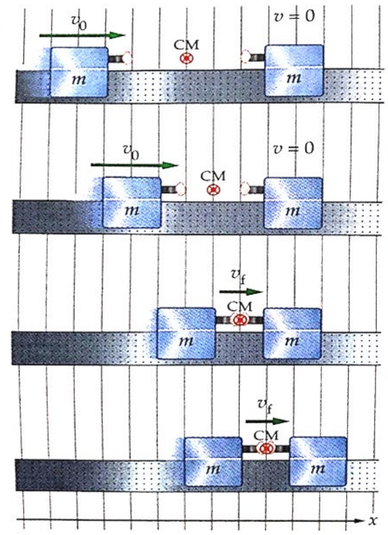


Solution

As expected, the velocity of each cart changes when they collide. However, the velocity of the centre of mass v_{CM} is completely unaffected by the collision.

In the diagram on the right, the incoming cart moves at two units for every time interval until it collides with the second cart. After they coalesce, the carts move together at one unit per time interval.

In contrast, notice how the centre of mass CM progresses uniformly at one unit per time interval before and after the collision.



3.4 Collisions in One-Dimension: Head-on Collisions

Collisions in one-dimension are also called head-on collisions. That is what we studied exclusively in H2 Physics, but only in the lab frame. In the next example, we study the head-on collision in both the lab and the CM frame.

Problem Solving Technique

Using Centre of Mass Frame of Reference in 1-D collision:

Step 1: Determine the velocity of the centre of mass v_{CM}.

Step 2: Determine the velocity of each body relative to the velocity of centre of mass v_{CM}

i.e. velocity of each body in the centre of mass reference frame

Step 3: Apply relevant laws of physics in this frame

Step 4: Convert the values back to the normal frame of reference

Example 7

Show that, when viewed in the CM frame, two particles must move in opposite directions after an elastic collision between them, with the same speeds they had before the collision.

Solution

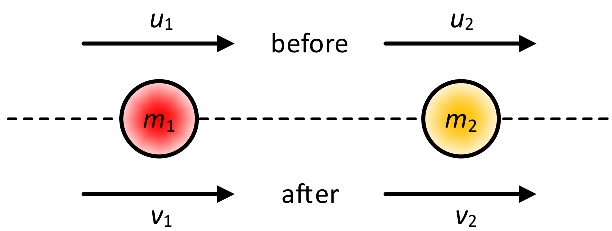
Example 8

Block A of mass 1.0 kg and block B of mass 3.0 kg move towards each other at a speed of 4.0 m s^{-1} and 1.0 m s^{-1} respectively on a frictionless, horizontal surface. They make a head-on elastic collision. By considering their velocities relative to the centre of mass of the system, determine their velocities after the collision.

Solution

Example 9

A mass m_1 with velocity u_1 collides head-on with another mass m_2 with velocity u_2, as shown in the figure below.



Derive the final velocities of the masses if the collision is elastic. Solve this by working in **(a)** the lab frame, and **(b)** the CM frame.

Solution