H3 Topic 3: Special Relativity

Content

- Michelson-Morley experiment
- Inertial frames and universal light speed
- Lorentz transformations
- Length contraction and time dilation
- Velocity addition
- Energy–momentum relation

Learning Outcomes

Candidates should be able to:

(a) discuss qualitatively the results of the Michelson–Morley interferometer experiment and its implications on the ether theory (knowledge of the details of the experiment is not required)

(b) state the postulates of the special theory of relativity, that in all inertial frames, the laws of physics are the same and the speed of light in free space is the same regardless of the motion of the light source or observer

(c) appreciate the failure of Galilean transformation equations when applied to a moving source of light

(d) discuss the concept of simultaneity

(e) show an understanding of the terms proper time and proper length

(f) apply the Lorentz transformation equations to solve one-dimensional problems

(g) derive the time dilation formula and length contraction formula, making use of the Lorentz factor

(h) apply the time dilation formula and the length contraction formula in related situations (e.g. the lifetime of fast-moving muons) or to solve problems

(i) use the one-dimensional relativistic velocity addition formula to calculate velocities in different inertial frames or to solve problems

(j) use the relativistic energy–momentum relation $E^2 = (pc)^2 + (mc^2)^2$ to solve problems, and show that it reduces, in the appropriate limits, to:

1. $E = pc$ (for massless particles); or

2. $E = mc^2 + \frac{1}{2}mv^2$ (for particles moving at low speeds $v << c$).

1 Speed of Light and the Michelson-Morley's Experiment

1.1 Speed of Light

1.1.1 The Measurement of the Speed of Light

The speed of light (exactly 299,792,458 m s^{-1}, usually denoted *c*) is an important fundamental constant in physics that is known to great precision today. In fact, one of the base units, the metre, is defined based on the speed of light.

Up until late 1600s, however, the speed of light was believed to be infinite. Galileo Galilei (1564 – 1642) was one of the first to question that. On a clear night in the year 1638 (or thereabout), in the Italian countryside, Galileo conducted the first experiment to measure the speed of light with the help of an assistant. Each carrying a lantern supplied with shutters, Galileo and his assistant placed themselves as far apart as they could stand yet still able to see each other (about 1.5 km from each other). Galileo's instruction to his assistant was to flash his lantern as soon as he saw Galileo's lantern flash. Knowing their physical separation, Galileo had hoped that the time delay between his lantern flash and his observation of the lantern flash of his assistant would allow him to calculate the speed of light. The result of this experiment was, however, quite inconclusive since we now know that the expected delay of about one hundred-thousandth of a second is way below the reaction time of human beings.

More than two centuries later, a French Physicist – Armand Hippolyte Fizeau – repeated a much-improved version of Galileo's experiment. His experimental set-up is shown in Fig. 1.1.

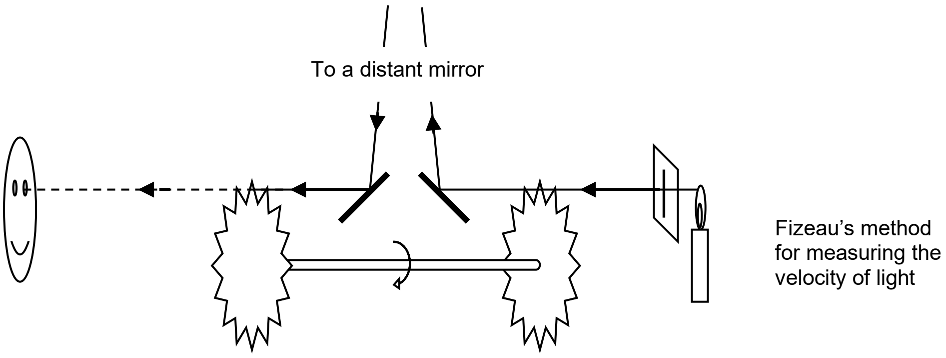


Fig. 1.1

The setup consisted of a pair of cogwheels set at opposite ends of a long axis. The wheels are arranged such that the cogs of one were opposite the intercog openings of the other, i.e. the light beam from the source on the right cannot be seen by the eye on the left no matter how the cogwheels are positioned. However, when the wheels are set in fast rotation such that during the time taken for the light to propagate from one wheel to the other, the cogwheels have turned half the distance between neighbouring cogs, light was expected to pass through the assembly unobtrusively. Fizeau concluded that the velocity of light was 3 x 10^{8} m s^{-1}, a value that coincided with that obtained three decades after Galileo's death by the Danish astronomer Olaus Roemer based on his observations on the apparent delay of lunar eclipses of Jupiter's moons when Jupiter was at different distances from Earth.

1.1.2 Maxwell's Equations

Seemingly unrelatedly, by 19th century, physicists have started to make progress in the field of electromagnetism. They began to recognize the relationship between electricity and magnetism. Through the works of Hans Christian Oersted, Andre Marie Ampere and Michael Faraday, it was already clear that moving charges (electric current) sets up a magnetic field and a changing magnetic field induces an emf which would drive a current. In 1865, Scottish mathematician James Clerk Maxwell formalized these observations with a set of four equations which we now call Maxwell's Equations. From these equations, Maxwell was able to show that oscillating charges set up a propagating transverse wave traveling at a speed of

$$\sqrt{\varepsilon_0 \mu_0}^{-1} = 3.0 \times 10^8 \text{ m s}^{-1}.$$

A young admirer of Faraday, Maxwell believed that the closeness of this constant to the speed of light was more than mere coincidence, especially since Faraday demonstrated in 1845 that a magnetic field produces a measurable effect on a beam of light passing through glass. Furthermore, experimental evidences such as Young's Double Slit interference and polarization of light proved beyond any doubt, the transverse wavelike nature of light. These prompted him to speculate that light involves transverse oscillations of electric and magnetic field lines.

This conclusion was a grand synthesis of the hitherto separate fields of optics and electromagnetism and is considered a triumph of mind equal to that of Newton in mechanics. Its experimental verification by H. Hertz in 1887 and its commercial exploitation by M. G. Marconi et al led to our present radio, TV, and satellite communications.

1.1.3 Luminiferous Ether

This new understanding of the nature of light immediately led to new problems. All the other wave phenomena known at the time, sound waves, water waves and waves in strings, are mechanical waves, whose propagation necessitates a propagating "medium". Furthermore, the speeds of the waves are constant relative to their respective medium. For example, the speed of sound is about 340 m s^{-1} relative to its medium, the air.

But it wasn't clear what medium electromagnetic waves travel in. To reconcile this, physicists of that time believed that there exists something that fills up all transparent matter and space. It serves as the substratum for the propagation of light and other electromagnetic waves. By the end of 19th century, the existence of such a medium, which was named the "luminiferous ether", was firmly established in physicists' mind.

The presumed properties of the luminiferous ether are such that it is perfectly transparent, and does not hinder the motion of, or interact with, planets, stars as well as all earthly objects that have been moving through it for billions of years. To detect it experimentally, physicists made use of the fact that the speed of light is presumably constant relative to it: the luminiferous ether reference frame will be an inertial frame in which the speed of light is a constant value c. However, if an observer were to be moving with respect to the luminiferous ether frame with velocity v, this observer would measure a different speed for light, ranging from $c + v$ to $c - v$, depending on the direction of relative motion (recall what you learned in the chapter on inertial frames). There is, thus, a 'simple' experiment which we can now carry out to establish the existence of the luminiferous ether – by measuring the speed of light in a variety of inertial systems, noting whether the measured speed is different in different systems. This forms the basis of the now-famous Michelson–Morley Experiment.

1.2 The Michelson-Morley Experiment

In 1881, American Physicist A. A. Michelson invented an optical interferometer which has a remarkable sensitivity that allows one to measure small differences in displacement (and hence through an indirect means, the velocity of light) to the precision of a fraction of a wavelength. Almost immediately, the scientific community became excited with the prospect of verifying the existence of the luminiferous ether, as Michelson embarked on his investigations.

The setup of the Michelson-Morley experiment is illustrated in Fig. 1.4. A light beam from lamp S fell on a glass plate P_1, which has a thin layer of silver such that half the incident beam is reflected (– – –) and half the incident beam is transmitted (·········). The two beams were further reflected by mirrors M_1 and M_2 located equal distances from the centre. Finally, partial reflections at P_1 recombine the two beams before they enter telescope T. The orientation of the apparatus with respect to Earth's revolution around the sun is such that the "ether wind" would be blowing against/along one of the arms (A) but perpendicular to the other arm (B). Note that glass plate P_2 is introduced to compensate for the additional path travelled in glass by light beam in arm (A).

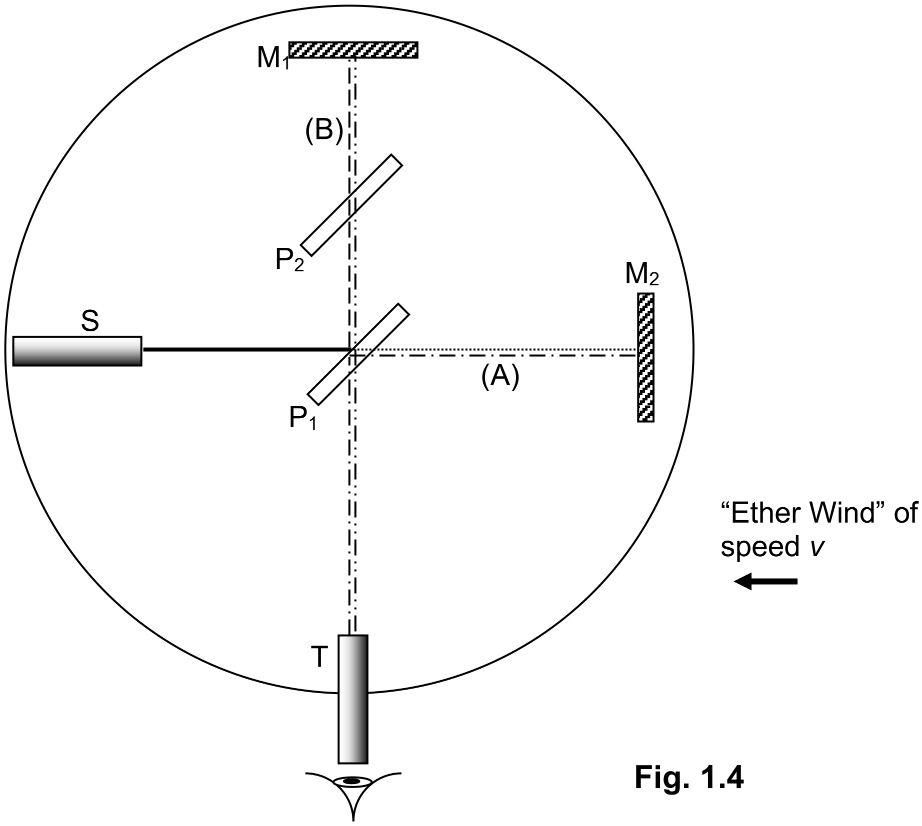


Fig. 1.4

If there were no "ether wind", both beams in arms (A) and (B) will reach T in phase (i.e. crest meeting crest, trough meeting trough) and produce a maximum illumination of the field of the telescope. However, if an 'ether wind' of speed v were to be present, the speed of light as measured in Earth's frame would be $c - v$ from P_1 to M_2, and $c + v$ from M_2 back to P_1. Due to this 'effect', the time delay for the round trip of arm (A) would be longer than that of arm (B), causing a small, but detectable, partial destructive interference – a decrease in illumination[1]. See **Appendix A** for more details.

When the experiment was performed in 1881 (and repeated with much improved precision by Michelson and his collaborator E. W. Morley in 1887), the results were null! No change in the interference pattern was detected. Repetition of the experiment on a balloon floating high above ground (to minimize any possible effect due to the Earth) made no difference. Improved versions of this experiment were repeated for more than half a century after Michelson's original investigations and every time, the results were null.

In a last bid to save the luminiferous ether, British Physicist G. F. Fitzgerald made a revolutionary proposal that all material moving with velocity v through the luminiferous ether shrink in the

[1] In the actual experiment, it was designed to observe a shift in the interference fringes rather than the decrease in illumination intensity, but the principle behind it remains unchanged.

direction of motion by a factor of $\sqrt{1-v^2/c^2}$. Such shrinking would reduce the distance travelled by beam (A) by a proper amount to make the arrival times equal and explain the empirical observations. However, this was only a "half-truth", since numerous attempts to explain the hypothetical shrinking were unsuccessful. As we shall see in the next section, this shrinking follows naturally from Einstein's Principle of Special Relativity.

2 Postulates of Einstein's Special Relativity

2.1 Galileo's Principle of Relativity

For thousands of years, the teachings of Aristotle dominated the scientific thought. It was believed that an object like the stone moves only when it is being pushed and will stop as soon as the force disappears. According to this point of view, a falling stone released from the top of the mast will fall vertically down, hence hit the deck closer to the stern (back of the ship), because the ship continues to move forward while the stone falls. It is characteristic of medieval scholasticism that problems like this kind were discussed pro and con for ages, but nobody cared to climb the mast and drop a stone.

Galileo, again, was among the first to recognize that this interpretation of Aristotle teachings was erroneous. In his book *Dialogue on the Great World Systems*, Galileo discussed this scenario of a stone released from a tower erected on the moving Earth, and argued for what we now accept to be true: at the moment of release, the stone will have the same horizontal velocity as the tower (due to Earth's rotation), and will thus continue to move with this horizontal velocity while accelerating downwards, remaining all the time just above the base of the mast.

Galileo further proposed a very interesting experiment. He suggested that the experimenter shut himself up with some insects, birds and fishes; and observe their movements while the ship was anchored at the port and while the ship was moving smoothly across the sea. He also suggested that the experimenter bring along a friend so that they can throw things to each other, again while the ship was anchored at the port and while the ship was moving smoothly across the sea. In both sets of experiment, we should not expect to notice any difference in our observations, whether the ship was moving or not. It is not clear if Galileo indeed performed these "experiments", but it was obvious to him that it is impossible to find out whether a ship is lying at anchor or moving across the sea by performing any mechanical experiment in a closed cabin. This is now know as "Galileo's principle of relativity", which states that

> The laws of mechanics have the same form in all inertial reference frames.

2.2 Einstein's Postulates of Special Relativity

Albert Einstein was born on Mar 14, 1879, just a few years before the Michelson–Morley experiment was performed. As a student, Einstein was not fantastic. In fact, he was rejected in his application to continue with postgraduate studies in his alma mater in Zurich, forcing him to take up a not too well-paid job as a patent clerk in the Swiss Patent Office in Berne, Switzerland. However, in 1905 at the age of 26, he published three articles (in the fields of heat, electricity and light, respectively) in a German magazine, *Annalen der Physick*, which shook the scientific world. Eventually, he was to win a Nobel Prize for one of these articles, though it was oft said that any of the three could have earned him a Nobel Prize.

As a boy, Einstein had always wondered what would happen if he were riding on a light wave and looking at the mirror. Would he see his own reflection? Would he see stationary sinusoidal variation in space of electric and magnetic fields that constitute the wave? These, and perhaps the paradoxes in the measurements of the velocity of light at that time, prompted him to develop a paper carrying a rather dull title, "On the Electrodynamics of Moving Bodies". This was the first paper on the theory of Special Relativity. As it turned out, it led us to question the Newtonian way of thinking about mechanics, and ultimately revolutionized our understanding of space and time.

2.2.1 First Postulate of Einstein's Principle of Special Relativity

Faced with the accumulated difficulties that come with the idea of the luminiferous ether, Einstein threw it out of the window. The elusive luminiferous ether was considered a universal reference for all motions. In the absence of such a universal reference, the motion of one object must be described as relative to another, and no object assumes the special status of being 'absolute stationary'. Therefore, together with the luminiferous ether, we must let go of the idea of absolute motion. Whatever experiment, be it mechanical or electromagnetic, we do in a 'closed cabin', we should not be able to tell if the cabin is stationary or moving with constant speed along a straight line. This is the first postulate of Einstein's principle of special relativity:

> The laws of physics have the same form in all inertial reference frames.

Notice the difference between Einstein's and Galileo's statements. Now, whether the experimenters shut themselves up with insects, birds and fishes, or, like Michelson and Morley, with light beams and some optical instruments, they will not be able to detect the motion of the "cabin" they are in without looking out of it.

2.2.2 Second Postulate of Einstein's Principle of Special Relativity

Recognizing that absolute motions do not exist, Einstein ran into another difficulty. Recall that Maxwell's theory of electrodynamics predicted a value for the speed of light, namely $c = 1/\sqrt{\mu_0\varepsilon_0} = 3.0 \times 10^8$ m s^{-1}, without naming any frame of reference. This was one of the reasons why physicists invented the luminiferous ether, the existence of which had now been disproved. To address this issue, Einstein put forth the second postulate of his principle of relativity:

> The velocity of light in free space is the same in all inertial reference frames.
> It does not depend on the motion of the source or the observer.

This idea was so counterintuitive that it took a genius like Einstein to conceive it. As we shall see, below, this new idea leads to even more counterintuitive phenomena.

We ought to notice that in coming up with the two postulates, Einstein simply asserted that this is the way things are and offered no "explanations". He would take them to be true and study their logical outcomes. The point to keep in mind is that all experimental consequences thus far have confirmed the correctness of these postulates.

3 Consequences of Einstein's Postulates of Special Relativity

As we shall see in this section, to accept Einstein's postulates of special relativity is to abandon the notion of absolute space and universal time. Two events simultaneous in one frame of reference are, in general, not simultaneous in a different frame of reference. The time duration between two events depends on the frame of reference in which the measurement is done, and so does the distance between two points in space. We will reach these conclusions by performing a few thought experiments.

3.1 The relativity of simultaneity

As shown in Fig 3.1, a train is moving with constant velocity v. There are two observers, O, who sits at the midpoint of the train and moves with it, and O', who is stationary on the ground. At time t = 0, O' is equidistant from the ends of the train, just as O. At the same moment, by some great coincidence, lightning strikes both ends of the train (A and B), which reflects the light towards O and O'.

Since O' was at the midpoint of the train when lightning struck, the two light signals reach him at the same instant ($t = t_2$). He concludes that lightning struck the two ends at the same instant, i.e., the two events were simultaneous. From the perspective of O, however, light signal from B reaches him first ($t = t_1$), and that from A later ($t = t_3$). Since he was moving towards B and away from A, the light signal from B has a shorter distance to travel before reaching O. Thus, the two events were not simultaneous according to O.

Notice that the conclusions of both O and O' are justified by two facts: the events were equidistant from them, and the speed of the light signal is constant. If we accept the constancy of the speed of light, we must accept that the concept of simultaneity is relative the frame of reference.

Concept of Simultaneity
Two events that are simultaneous in one reference frame are in general not simultaneous in a second frame moving relative to the first. Simultaneity is not an absolute concept but one that depends upon the state of the motion of the observer.

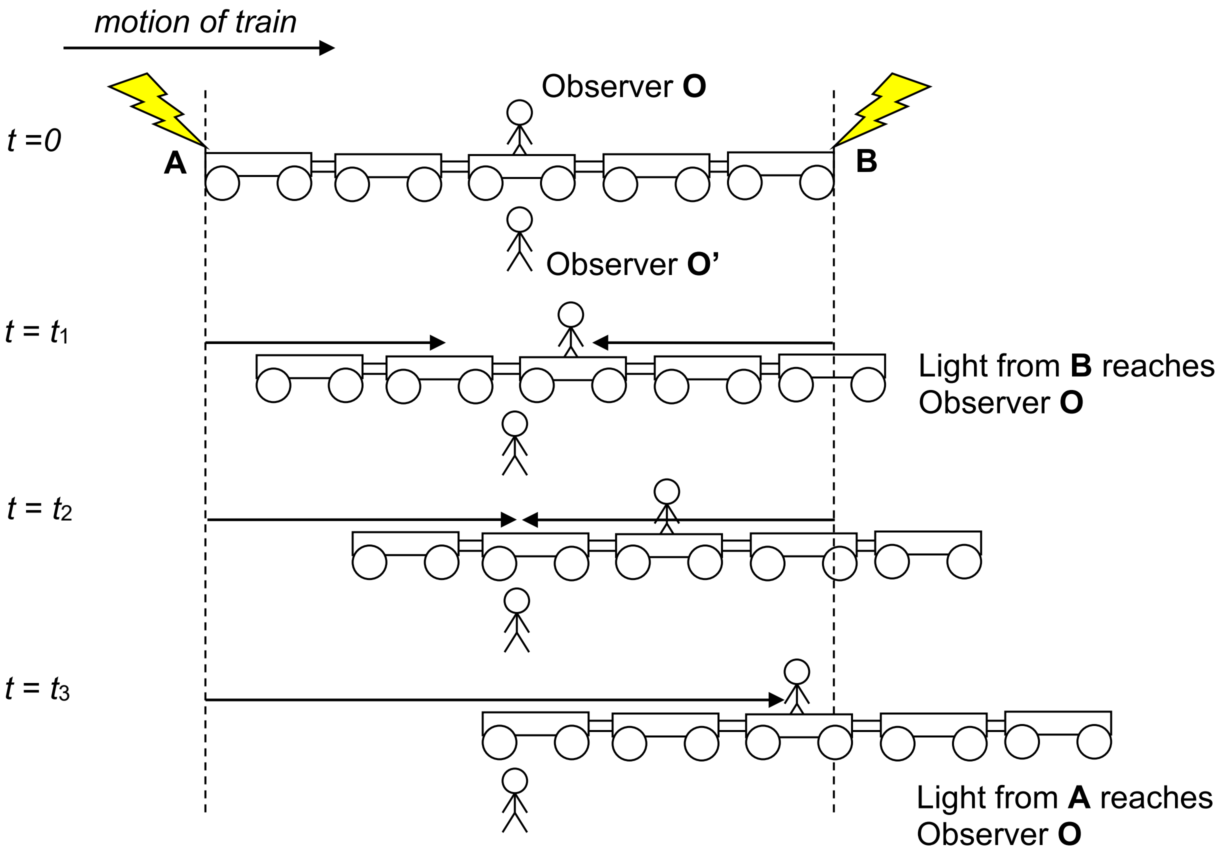
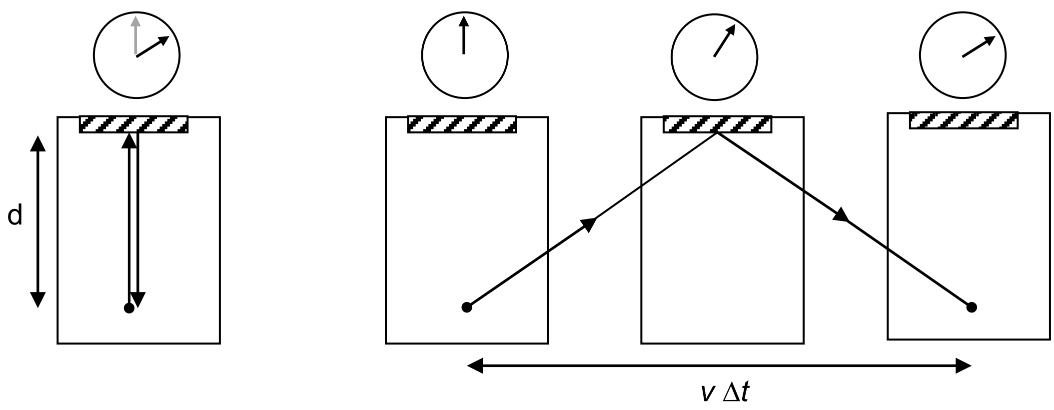


Fig. 3.1



(a)
Event viewed from an observer O who is sitting on the moving train

(b)
Event viewed from an observer O' who is standing on the stationary track

Fig 3.2

3.2 Time dilation

The observer O, sitting on the train moving with constant velocity v and carrying a clock, now measures the time duration Δt taken for a light pulse to leave from the source and returning to a detector just beside the source via a mirror on the ceiling (Fig. 3.2(a)). The source, the mirror and the detector are all attached to the train. Knowing the speed of light, c, and the distance between the source and the mirror, d, she verifies that

$$\Delta t = \text{distance traveled / speed of light} = \frac{2d}{c}$$

The observer O', standing by the track, measures the time taken by the same events (starting and receiving the pulse) to be time $\Delta t'$ (Fig. 3.2(b)).

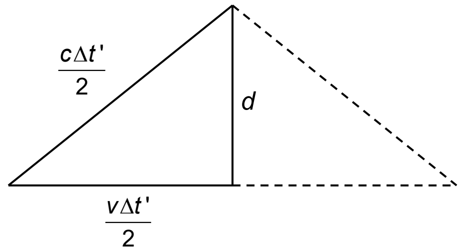


Fig 3.3

However, by the time the light reached the mirror, the mirror has moved horizontally a distance $v\Delta t'/2$. The total distance travelled by the beam of light can be calculated from Pythagoras' theorem, as shown in Fig. 3.3, to be

$$2\sqrt{\left[\left(\frac{v\Delta t'}{2}\right)^2 + d^2\right]}$$

Since the speed of light c is a constant, this distance must be equal to $c\Delta t'$. Therefore,

$$c\Delta t' = 2\sqrt{\left[\left(\frac{v\Delta t'}{2}\right)^2 + d^2\right]} \quad \Rightarrow \Delta t' = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

So, the time duration between the two events is different for the two observers:

Time measured by O: $\Delta t = \frac{2d}{c}$ (3.1)

Time measured by O': $\Delta t' = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$ (3.2)

Comparing Equations (3.1) and (3.2), we note that the time durations measured by the two observers, moving relative to each other by velocity v, are related by

$$\boxed{\Delta t' = \gamma \Delta t,} \quad (3.3)$$

where

$$\boxed{\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}} \quad (3.4)$$

is called the **Lorentz factor**, which appears frequently in relativistic expressions. Notice that, if $v > c$, the Lorentz factor γ becomes an imaginary number, so does the time duration $\Delta t'$, which is physically impossible. In other words, the form of the Lorentz factor γ implies that no object can move faster than the speed of light.

For $v < c$, γ is real and greater than 1, implying that $\Delta t' > \Delta t$, i.e. the time duration as measured by O', who is moving relative to the setup, is longer. Intuitively, this means that any process appears slower when observed by someone moving relative to the setup. This phenomenon, as described by Equation (3.3), is called **time dilation**. Notice that if we take the speed of light to be infinity, as was previously believed, the Lorentz factor would be equal to 1, the time duration would be the same in any frame of reference (no time dilation).

The frame of reference of the observer O is the *rest* frame of reference for this experiment, since in this frame, the emission of the beam and the receiving of it happen at the same point in space. The time as measured by a clock that is in the rest frame of reference, namely Δt above, is called the **proper time**.

Proper time between two events is the time interval measured in the frame of reference in which the two events occur at the same point in space.

Example 1

The period of a pendulum is measured to be 3.0s in the rest frame of a pendulum. What is the period when measured by an observer moving at a speed of 0.95*c* relative to the pendulum?

Example 2

The pion has an average lifetime of 26.0 ns when at rest. In order for it to travel 10.0 m, how fast must it move?

3.3 Length contraction

Now that we have shown the phenomenon of time dilation, we can make use of it to show the related phenomena of length contraction, through another thought experiment.

Imagine a train moving along a straight line between two locations P and Q, with constant velocity v. The observer O, who moves with the train, measures the time taken to get from P to Q to be Δt. Another observer, O', who stands on the ground, measures the time interval between the same two events (leaving P and getting to Q) to be $\Delta t'$. The time dilation formula says that,

$$\Delta t' = \gamma \Delta t\,.$$

The observers determine the distance between P and Q using

$$\text{distance} = \text{velocity} \times \text{time}.$$

The results are:

according to O: $L = v\Delta t$, (3.5)

according to O': $L_p = v\Delta t' = v\gamma\Delta t = \gamma L$. (3.6)

Although they agree on the relative velocity between the train and the ground (v), they don't agree on the distance between P and Q. Comparing Equations (3.5) and (3.6), we have

$$\boxed{L_p = \gamma L.} \qquad (3.7)$$

Since the Lorentz factor $\gamma > 1$, we conclude that $L < L_p$, i.e. the distance appears to be shorter according to the moving observer O. This phenomenon, as described by Equation (3.7), is called **length contraction**. Notice that if the speed of light c was infinity, there would be no length contraction.

The distance as measured by the observer O' who stands on the ground, L_p, is called the **proper length**, since P and Q are at rest relative to O'.

Proper length between two points in space is the length measured in the frame of reference in which the two points are at rest.

Note that **length contraction happens only for lengths along the direction of motion**, but not for lengths perpendicular to it. For example, for a vertical telegraph pole perpendicular to the direction of motion of the train (horizontal), O and O' would agree on its height.

Example 3

A spaceship is moving at one hundredth of the speed of light relative to the Earth. Inside the spaceship, a meter rule is oriented along the direction of motion of the spaceship.

Determine the length of the meter rule as measured by an observer on the Earth.

Example 4

When stationary on the Earth's surface, a spaceship has a length *d*. At what speed relative to the Earth will the spaceship be half as long?

3.4 Case Study: Decay of Muons

Both time dilation and length contraction, counter intuitive as they may appear, have been verified experimentally so often that they are to be taken as scientific facts. In the following case study, we look at the experimental evidence provided by the decay of muons.

Example 5

The muon (μ^+) is a type of elementary particles that are produced by the interaction between the high-energy cosmic rays and Earth's atmosphere at a height of about 10 km from Earth's surface, and they travel at a speed very close to the speed of light (0.999*c*) towards the Earth's surface. The muon is unstable. In its rest frame of reference, muons decay with a mean lifetime of 2.2 μs.

(a) Determine the distance a muon can travel during its mean lifetime of 2.2 μs.

Your answer to **(a)** should indicate that no (or, at least, very few) muons should be able to reach the ground from a height of 10 km. The reality, however, is that we are bombarded by them on the Earth's surface. To explain this, we need to take relativistic effects into consideration.

(b) The mean lifetime of 2.2 μs is measured in the rest frame of the muon, which moves at a speed of 0.999*c* relative to the surface of the Earth.

Determine

(i) the mean lifetime as measured by an observer on the Earth's surface;

(ii) the distance travelled by the muon in this mean lifetime.

Your answers to **(b)** offers one way to explain why muons can reach the ground, from the point of view of an observer on Earth's surface. How about in the rest frame of the muon? In its rest frame, the mean lifetime of a muon is still just 2.2 μs.

(c) In the Earth's frame of reference, a muon needs to travel through 10 km of atmosphere to reach the Earth's surface.
What is the corresponding distance a muon must travel as measured by the muon?

In the frame of reference of the muon, the distance it needs to travel is smaller than the distance it can travel during its mean lifetime (c.f. **(a)**). This result offers another perspective we can take to explain why muons can reach the Earth's surface.

4 Lorentz Transformation

In this section, we study how the spacetime coordinates in one frame of reference are related to those in another frame, in a way that satisfies Einstein's Principles of Special Relativity. The resulting equations are called the Lorentz Transformation. We will be able to rederive the equations of time dilation and length contraction using the Lorentz Transformation.

But before that, we need to see how the spacetime coordinates in different frames are related according to the Galilean principle of relativity, and where it fails.

4.1 Galilean Transformation

Let's consider two inertial frames of reference, S and S', attached respectively to two observers, O and O' that are moving relative to each other. In inertial frame S, the stationary observer O observes an 'event' (say, a tree is struck by a lightning bolt) and records the position where the event occurs as (x, y, z) and the time in which it occurs as t. Another observer O', stationary in the inertial frame S' that moves with velocity v in the positive x-direction relative to frame S, records the same event as occurring at position (x', y', z') and time t'. This is illustrated in Fig. 4.1 (z-axis is not shown):

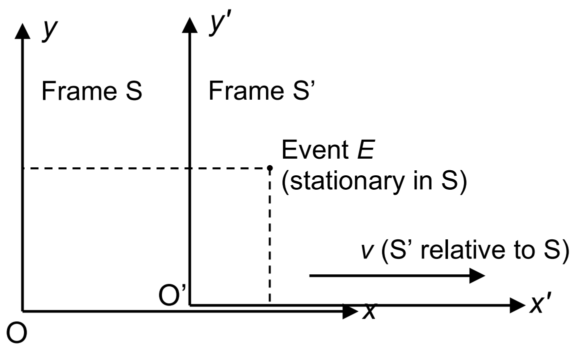


Fig. 4.1

For simplicity, we will also set the 'initial conditions' such that, at some instant in time, the positions of the two observers, O and O' (at the origins of the two frames) coincide. We will further require that the two observers synchronize their clocks such that $t = t' = 0$ when this happens.

It is almost "obvious" that both observers (O and O') will record the same time coordinate[2]

$$t' = t \tag{4.1}$$

[2] We say that it is "obvious" because of our conventional assumption of a *universal time*, i.e. we assume that once the clocks in the two inertial reference frames are synchronized, they remain that way indefinitely. We have disproved this when we looked time dilation in a previous section.

Due to the movement of frame *S'* in the *x* direction, we also see that

$$x' = x - vt \tag{4.2}$$
$$y' = y \tag{4.3}$$
$$z' = z \tag{4.4}$$

This set of four equations, which relates the coordinates in two inertial frames moving relative to each other, is known as the Galilean transformation of coordinates.

Taking the time derivative of equations (3.2) – (3.4), we have

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \tag{4.5}$$
$$\frac{dy'}{dt} = \frac{dy}{dt} \tag{4.6}$$
$$\frac{dz'}{dt} = \frac{dz}{dt} \tag{4.7}$$

This last set of three equations is hardly surprising, as they simply reiterate our concept of relative velocity. We may derive the corresponding relations between accelerations as measured by different inertial frames. For *x*-direction,

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt'^2} - v\frac{d^2}{dt'^2}t \overset{\text{since } t=t'}{=} \frac{d^2x}{dt^2} - v\frac{d^2}{dt^2}t = \frac{d^2x}{dt^2}$$

Hence,

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} \tag{4.5a}$$

Similarly, (4.6) and (4.7) lead to

$$\frac{d^2y'}{dt'^2} = \frac{d^2y}{dt^2} \tag{4.6a}$$
$$\frac{d^2z'}{dt'^2} = \frac{d^2z}{dt^2} \tag{4.7a}$$

Therefore, according to Galilean Transformations, the acceleration of a body as measured by different inertial observers is the same (given by $a = F/m$), i.e. Newton's 2nd Law will be obeyed in all frames of reference. This is the meaning of the Galilean principle of relativity.

4.1.1 The Failure of Galilean Transformation

The Galilean transformation equations are based on our intuitive notions of absolute space and universal time, which have been disproved by the phenomena of time dilation and length contraction. It clearly contradicts the second postulate of Einstein's principle of relativity, on the constancy of the speed of light. More explicitly, consider a source of light that emits a beam of light. The speed of light relative to the source is *c*. You are moving relative to the source, with speed *v*, in the direction opposite to that of the beam of light. According to the Galilean transformation, then, the speed at which the beam of light moves relative to you would be (*c* + *v*). In fact, you can make the speed of light relative to you anything between (*c* - *v*) and (*c* + *v*), by changing the direction of your motion.

The Galilean transformation works well enough for phenomena at low speeds, i.e. speeds << speed of light *c*. It can be considered an approximation, for speeds << *c*, to the more exact Lorentz transformation that will be introduced in the next section. Its failure, i.e. the discrepancy between its prediction and experiments, becomes evident only when the speed under consideration is close to the speed of light.

4.2 Lorentz Transformation

We state without proof the equations of the Lorentz Transformation here. There are a few ways to derive these equations, and one of these is presented in **Appendix B**.

Lorentz Transformations

(a) $x' = \gamma(x - vt)$

(b) $y' = y$

(c) $z' = z$

(d) $t' = \gamma(t - vx/c^2)$

where $\gamma = \left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}$, *v* being the relative velocity of *S'* with respect to *S*.

(4.8)

Here, *x*, *y*, *z*, and *t* denote the spacetime coordinates in the *S* frame and *x'*, *y'*, *z'*, and *t'* those in the *S'* frame, which is moving with velocity *v* relative to the *S* frame in the positive *x*–direction.

The Lorentz Transformations (4.8a-4.8d) transforms the spacetime coordinates in the S frame (*x*, *y*, *z*, *t*) into those in the S' frame (*x*', *y*', *z*', *t*'). To obtain the so-called *Inverse Lorentz Transformations*, which transforms (*x*', *y*', *z*', *t*') into (*x*, *y*, *z*, *t*), all we have to do is to replace *v* in (4.8a-4.8d) by −*v*. If the S' frame is moving with velocity *v* relative to S, S is, of course, moving with velocity −*v* relative to S'.

Notice that, if *c* is taken to be infinity, we recover the Galilean transformation equations, which is a valid approximation for *v* << *c*. It is only when we wish to study objects moving at unusually high speed that we need to invoke *the Theory of Relativity*.

Next, we are going to rederive the formulas for time dilation and length contraction using the Lorentz Transformations.

4.2.1 Rederiving time dilation using Lorentz Transformation

Consider a lamp that lights up and then dims. In the rest frame of the lamp, S, the lamp will be at the same spatial location. Thus, let the spacetime coordinates of the first event (lamp lights up) be (x, y, z, t_1), and those for the second event (lamp is dimmed) be (x, y, z, t_2). The spatial separation and time interval between the two events, in the rest frame, are given by

$$\Delta x = x - x = 0,$$
$$\Delta t = t_2 - t_1.$$

In another frame, S', which is moving along the +x-direction with velocity *v*, the time coordinates of the two events are, using Lorentz Transformations,

$$t'_1 = \gamma(t_1 - \frac{v}{c^2}x),\ t'_2 = \gamma(t_2 - \frac{v}{c^2}x).$$

Consequently, the time interval between the two events in frame S' is given by

$$\Delta t' = t'_2 - t'_1 = \gamma(t_2 - t_1) - \frac{v}{c^2}(x - x) = \gamma\Delta t$$
$$\text{or} \qquad \Delta t' = \gamma\Delta t,$$

which is the time dilation formula that we have derived previously.

4.2.2 Rederiving length contraction using Lorentz Transformation

Imagine you are measuring the length of a toy car moving relative to you (A real car also works. But that would be too dangerous an experiment even to imagine). You place a meter rule parallel to the direction of motion of the toy car, and take the readings at the head and the tail of the toy car on the meter rule *at the same instant in time*. You obtain the length of the toy car by calculating the difference between the two readings. The point to take home is that the length of an object, when measured in a frame of reference in which the object is moving, must be determined by events that are simultaneous in that frame. In the above example, for instance, it would be a silly mistake to take the readings at different instances in time at the head and the tail, and call the difference the length of the toy car. However, in the rest of frame of the toy car, one can take readings at different instances in time and still obtain the correct length.

Let the toy car move in the $-x$-direction relative to the floor. You, who is standing on the floor (hence moving relative to the toy car in the $+x$-direction), obtain (x'_1, y', z', t') and (x'_2, y', z', t') at the head and tail of the toy car when measuring its length. Notice that the two readings are taken simultaneously (same t'). Using Lorentz Transformation, the spatial coordinates of the two events in the rest frame of the toy car are

$$x_1 = \gamma(x'_1 - vt'),$$
$$x_2 = \gamma(x'_2 - vt').$$

Taking the difference,

$$\Delta x = (x_1 - x_2) = \gamma(x'_1 - x'_2) - \gamma v(t' - t') = \gamma(x'_1 - x'_2) = \gamma\Delta x',$$
$$\text{or } \Delta x = \gamma\Delta x'.$$

It should be clear that $\Delta x'$ is the length of the toy car as measured in the frame in which the toy car is moving, while Δx is the length of the toy car in its rest frame, i.e. the proper length. Hence, we have recovered the formula for length contraction Equation (3.7).

If you calculate the temporal coordinates t_1 and t_2 of the two events in the rest frame of the toy car, you'll notice that they are not equal, i.e. x_1 and x_2 are not simultaneous. This is fine, because the two readings need not be simultaneous in the *rest frame* to obtain the correct length.

Example 6

Use the Lorentz transformation equations to show that simultaneity is not an absolute concept.

Example 7

Two planets, *A* and *B*, are at rest with respect to each other, a distance *L* apart, with synchronized clocks. A spaceship flies at speed *v* past planet *A* toward planet *B* and synchronizes its clock with *A*'s right when it passes *A* (both their clocks read zero). The spaceship eventually flies past planet *B* and compares its clock with *B*'s. We know, from working in the planets' frame, that when the spaceship reaches *B*, *B*'s clock reads L/v. And the spaceship's clock reads $L/\gamma v$, because it runs slow by a factor of γ when viewed in the planets' frame.

How would someone on the spaceship quantitatively explain to you why *B*'s clock reads L/v (which is *more* than its own $L/\gamma v$), considering that the spaceship sees *B*'s clock running slow?

4.3 Relativistic Velocity Addition Formula

One of the glaring failures of the Galilean transformation was its inability to satisfy the constancy of the speed of light in all inertial frames. Let's see how Lorentz Transformation handles the speed of light.

Consider two frames of reference, S and S', as depicted in Fig 4.2. S' moves at velocity v relative to S in the $+x$-direction. An object, P, moves with velocity u', also in the $+x$-direction, when observed in the S' frame. The question now is, what is the velocity of the object when observed from the S frame?

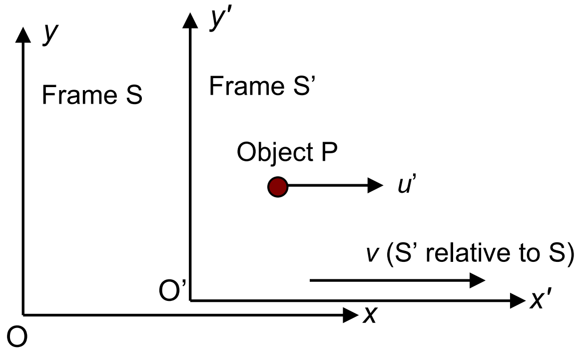


Fig. 4.2

According to Galilean transformation, the answer is obviously (v+u'). But, as we have discussed, this contradicts the constancy of speed of light, among other things. Lorentz Transformation tells us that

$$\left.\begin{aligned} x &= \gamma(x'+vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t'+\frac{vx'}{c^2}\right) \end{aligned}\right\} \Rightarrow \begin{cases} dx = \gamma(dx'+vdt') \\ dy = dy' \\ dz = dz' \\ dt = \gamma\left(dt'+\dfrac{vdx'}{c^2}\right) \end{cases} \tag{4.9}$$

Notice that, as compared to Equations (4.8), velocity v has been replaced by $-v$, since we are now transforming coordinates in S' into those in S. Let the velocity of P as observed in S be u. Note that

$$u = \frac{dx}{dt},\ u' = \frac{dx'}{dt'}. \tag{4.10}$$

Using Equations (4.9) and (4.10), we have

$$u = \frac{dx}{dt} = \frac{\gamma(dx'+vdt')}{\gamma\left(dt'+\dfrac{vdx'}{c^2}\right)} = \frac{(dx'/dt'+v)}{\left(1+\dfrac{vdx'/dt'}{c^2}\right)} = \frac{(u'+v)}{\left(1+\dfrac{vu'}{c^2}\right)},$$

$$\text{or} \quad u = \frac{(u'+v)}{\left(1+\dfrac{vu'}{c^2}\right)}. \tag{4.11}$$

Equation (4.11) is the *relativistic formula for the addition of velocities*. Now, what if the object is a photon, and the speed *u*' is the speed of light, *c*? Let's see:

$$u = \frac{(c+v)}{\left(1+\frac{cv}{c^2}\right)} = \frac{(c+v)}{\left(1+\frac{v}{c}\right)} = c.$$

Indeed, Equation (4.11) preserves the speed of light, whatever the relative velocity *v* between the frames of reference, consistent with the Principles of Special Relativity.

So far, we have focused on the *x*-components of the velocities. What happens to the *y*- and *z*-components of the velocities (if any)? Denoting them as u_y, u_z, u'_y and u'_z, where

$$u_y = \frac{dy}{dt},$$
$$u_z = \frac{dz}{dt},$$
$$u'_y = \frac{dy'}{dt'},$$
$$u'_z = \frac{dz'}{dt'},$$

and using Equations (4.9),

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt'+\frac{vdx'}{c^2}\right)} = \frac{dy'/dt'}{\gamma\left(1+\frac{vdx'/dt'}{c^2}\right)} = \frac{u'_y}{\gamma\left(1+\frac{u'v}{c^2}\right)},$$
$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt'+\frac{vdx'}{c^2}\right)} = \frac{dz'/dt'}{\gamma\left(1+\frac{vdx'/dt'}{c^2}\right)} = \frac{u'_z}{\gamma\left(1+\frac{u'v}{c^2}\right)},$$

or, putting all three formulas together,

$$\boxed{\begin{aligned} u_x &= \frac{(u'+v)}{\left(1+\frac{vu'}{c^2}\right)}, \\ u_y &= \frac{u'_y}{\gamma\left(1+\frac{u'v}{c^2}\right)}, \\ u_z &= \frac{u'_z}{\gamma\left(1+\frac{u'v}{c^2}\right)}. \end{aligned}} \qquad (4.12)$$

Notice that, even though the relative motions between the frames S and S' are in the *x*-direction, the *y*- and *z*-components of the velocities of P are also different when observed from in the two frames. Also, note that, if S' was moving in the −*x*-direction, we just need to replace *v* by −*v* in the above equations.

Example 8

Two spaceships A and B are moving in opposite directions. An observer on Earth measures the speed of A to be $0.75c$, and the speed of B to be $0.85c$. Find the velocity of B with respect to A using both Galilean and Lorentz Transformations.

5 Relativistic Mass, Momentum and Energy

In this section, we are going to state a few results without proof. These results can be obtained by various thought experiments, as we have been doing in the previous sections. For more details, students are encouraged to refer to, for example, Chapters 15 and 16, Volume I, *the Feynman Lectures* (freely available online here), or *Special Relativity* by A.P. French.

5.1 Relativistic Mass and Energy

Like spacetime coordinates, the mass of an object depends on the frame of reference in which it is observed. The mass as measured in the object's rest frame is called the rest mass, conventionally denoted m_0:

$$\text{rest mass} = m_0.$$

In a frame of reference in which the object is moving with speed v, its mass m is given by

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad (4.13)$$

For a low speed ($v << c$), we can expand the above expression using series expansion:

$$m = \gamma m_0 = m_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}+...\right).$$

Upon multiplying c^2 on both sides,

$$mc^2 = \gamma m_0 c^2 = m_0 c^2 + \frac{1}{2}m_0 v^2 + \dots \qquad (4.14)$$

The second term on the right side is the classical kinetic energy that we are familiar with. This suggests that each term in Equation (4.14) is some form of energy. The first term on the right side is called the rest energy:

$$\text{rest energy} = m_0 c^2, \qquad (4.15)$$

which is the energy contained in a stationary object. The sum of other terms on the right side is the relativistic kinetic energy of the object, since all these terms contain the speed of the object (v):

$$\text{relativistic kinetic energy} = \gamma m_0 c^2 - m_0 c^2 = \frac{1}{2}m_0 v^2 + \dots, \qquad (4.16)$$

where (…) denotes higher order corrections to the kinetic energy. The left side of Equation (4.14), namely

$$E = mc^2 = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad (4.17)$$

is the relativistic total energy, i.e. the sum of the rest energy and the relativistic kinetic energy:

$$\text{relativistic total energy} = mc^2 = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \overbrace{m_0 c^2}^{\text{rest energy}} + \underbrace{\frac{1}{2}m_0 v^2 + \dots}_{\text{relativistic kinetic energy}} \qquad (4.18)$$

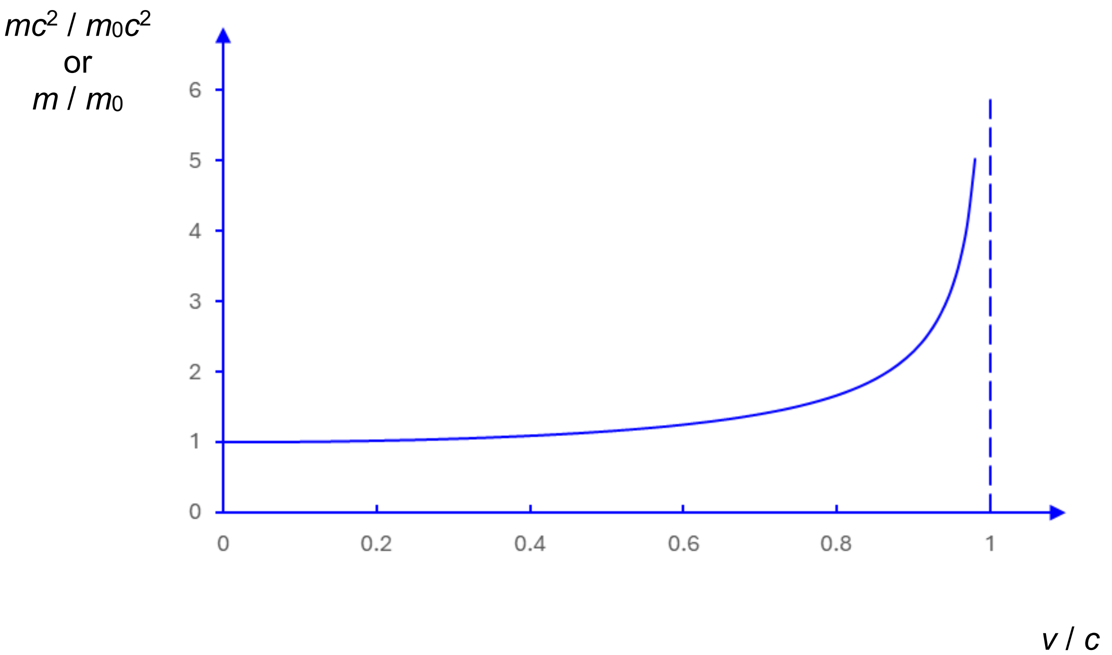


Fig. 4.3

In Fig. 4.3, we plot the variation of the total energy of an object (or its mass, since they differ by a constant factor, c^2) with the speed of the object. From the graph, we see that, as the speed of the object approaches the speed of light, its mass (or total energy) approaches infinity. In other words, infinite amount of energy is needed to accelerate a massive object to the speed of light. This is the reason why no massive object can achieve the speed of light.

5.2 Relativistic Energy-Momentum Relation

As a consequence of the relativistic mass, the relativistic momentum is given by

$$p = mv = \gamma m_0 v = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad (4.19)$$

Next, we are going to derive a relationship between the relativistic momentum and energy. Squaring both sides of Equation (4.19) and multiplying by c^2, we have

$$\begin{aligned}(pc)^2 &= \frac{m_0^2 v^2 c^2}{1-\frac{v^2}{c^2}} = \frac{m_0^2 c^4 \left(\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}} \\ &= \frac{m_0^2 c^4 \left(\frac{v^2}{c^2}-1\right)}{1-\frac{v^2}{c^2}} + \frac{m_0^2 c^4}{1-\frac{v^2}{c^2}} \\ &= -m_0^2 c^4 + \frac{m_0^2 c^4}{1-\frac{v^2}{c^2}}\end{aligned}$$

From Equation (4.17), for the total energy,

$$E = \gamma m_0 c^2$$

$$E^2 = \gamma^2 \left(m_0 c^2\right)^2 = \frac{\left(m_0 c^2\right)^2}{1-\frac{v^2}{c^2}}$$

Finally, comparing the above equations, we have the relativistic energy-momentum relation

$$\boxed{E^2 = (pc)^2 + \left(m_0 c^2\right)^2} \qquad (4.20)$$

5.2.1 Energy-Momentum Relation for Low Speed ($v \ll c$)

For speed $v \ll c$, or $v/c \ll 1$, we can perform series expansions, and keeping only the highest order in v/c. From Equation (4.20), we have

$$E = \sqrt{(pc)^2 + \left(m_0c^2\right)^2} = m_0c^2\sqrt{1+\left(\frac{pc}{m_0c^2}\right)^2} = m_0c^2\sqrt{1+\left(\frac{\gamma m_0 vc}{m_0c^2}\right)^2}$$

$$= m_0c^2\sqrt{1+\left(\frac{\gamma v}{c}\right)^2} = m_0c^2\sqrt{1+\frac{1}{1-\left(\frac{v}{c}\right)^2}\left(\frac{v}{c}\right)^2} \approx m_0c^2\sqrt{1+\left(1+\left(\frac{v}{c}\right)^2\right)\left(\frac{v}{c}\right)^2}$$

$$\approx m_0c^2\sqrt{1+\left(\frac{v}{c}\right)^2} \approx m_0c^2\left(1+\frac{1}{2}\left(\frac{v}{c}\right)^2\right) = m_0c^2 + \frac{1}{2}m_0v^2 .$$

or $$\boxed{E = m_0c^2 + \frac{1}{2}m_0v^2 \qquad (\text{for } v \ll c).} \qquad (4.21)$$

We recover Equation (4.18).

5.2.2 Energy-Momentum Relation for Massless Particles

Particles such as photons have no rest mass, i.e. $m_0 = 0$. For these particles, the energy-momentum relation becomes

$$\boxed{E = pc \qquad (\text{for massless particles}).} \qquad (4.22)$$

In the Chapter of Quantum Physics in H2 Physics, we learned that a photon's energy E and momentum p are given by

$$E = hf,\ p = \frac{h}{\lambda}, \qquad (4.23)$$

where f and λ are the frequency and wavelength of the photon. Hence,

$$\frac{E}{p} = \frac{hf}{h/\lambda} = f\lambda = c .$$

The relationships (4.23) agree with Equation (4.22).

Example 9

Determine the rest mass of a particle whose rest energy is equal to the energy of photon with frequency 3.0×10^{16} Hz.

Example 10

In the following calculations, all energies are to be calculated in MeV.

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of $0.75c$.

Determine its

(a) rest energy;
(b) total energy;
(c) classical kinetic energy;
(d) relativistic kinetic energy;
(e) classical momentum;
(f) relativistic momentum;

Example 11

Two protons (each with M = 1.67 x 10^{-27} kg) are initially moving with equal speeds in opposite directions. They continue to exist after a head-on collision that also produces a neutral pion (m = 2.40 x 10^{-28} kg). If the protons and the pion are all at rest after the collision, find the initial speed of the protons. Energy is conserved in the collision.