ELECTRIC AND MAGNETIC FIELDS

Content

- Electric fields in a conductor
- Gauss's law for electric and magnetic fields
- Ampère's law for magnetic fields
- Electric and magnetic dipoles

Learning Outcomes

Candidates should be able to:

(a) show an understanding that ideal conductors form an equipotential volume and that the electric field within an ideal conductor is zero.

(b) show an understanding that electric charge accumulates on the surfaces of a conductor and that the electric field at the surface of a conductor is normal to the surface.

(c) recall and apply Gauss's law for electric and magnetic fields (knowledge of the differential form of Gauss's law is not required) and

(i) solve problems involving symmetric charge distributions by relating the electric flux (in a vacuum) through a closed surface with the charge enclosed by that surface.

(ii) show an understanding of the non-existence of "magnetic charge" expressed by Gauss's law for magnetism.

(d) recall and apply Ampère's law relating the line integral of the magnetic field (in a vacuum) around a closed loop with the electric current enclosed by the loop to solve problems involving symmetric field configurations (knowledge of the differential form of Ampère's law is not required). [Note further that candidates are not required to know Maxwell's generalisation of Ampère's law including the term related to the rate of change of electric flux, nor the Biot-Savart law.]

(e) define the magnitude of the electric dipole moment as the product of the charge and the separation.

(f) show an understanding of and use the torque on an electric dipole and the potential energy of an electric dipole to solve related problems.

(g) define the magnitude of the magnetic dipole moment for a current loop as the product of the current and the area of the loop.

(h) show an understanding of and use the torque on a magnetic dipole and the potential energy of a magnetic dipole to solve related problems.

(i) appreciate that while electric and magnetic dipoles behave analogously, the theoretical framework at this level of study does not admit the possibility of magnetic monopoles.

show an understanding that ideal conductors form an equipotential volume and that the electric field within an ideal conductor is zero.

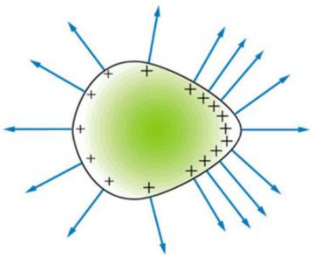
Electric fields in a conductor

When an electrical conductor is charged, all the charges (are able to move freely) will immediately move about inside the conductor due to mutual repulsion. The charges will very quickly redistribute themselves and stop moving reaching electrostatic equilibrium.

This happens when the deposited charges distribute themselves on the surface of the conductor in such a way that:

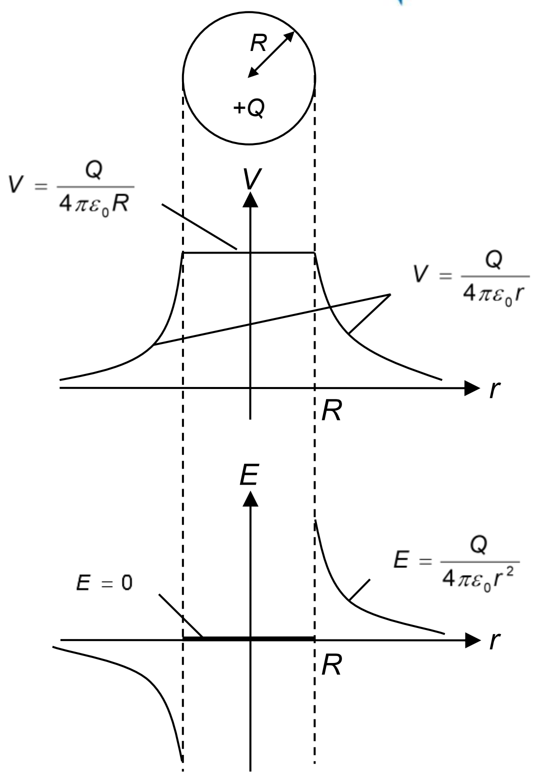
(a) the electric field inside the conductor is zero

(b) the electric field at every point of the surface of the conductor is perpendicular to the surface.



i.e. The electric potential **inside** the conductor is **constant** and is **equal** to the electric potential at the **surface** $\Rightarrow$ the entire conductor is an **equipotential volume**.

Conducting Sphere



Since charges on a spherical conductor will be uniformly distributed on its surface, the charge distribution will also be spherically symmetric. Hence E and V outside the sphere are given by the same expressions as that of a point charge.

Outside of the sphere: $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$, $V = \dfrac{Q}{4\pi\varepsilon_0 r}$

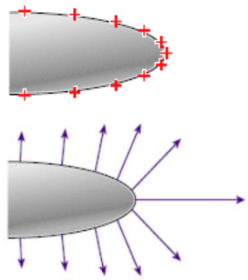
Inside of the sphere:

Since $E = -\dfrac{dV}{dr}$, the electric field **within** the conductor must be **zero** as there is no potential difference within (or on the surface of) the conductor $\Rightarrow$ potential V within and on the surface of the conductor must be uniform.

show an understanding that electric charge accumulates on the surfaces of a conductor and that the electric field at the surface of a conductor is normal to the surface.

Note:

1. For a conductor in electrostatic equilibrium, all charges reside on the surface and E field is zero everywhere inside the conductor **regardless** of the shape and size of the conductor.
2. The E field is zero everywhere inside the conductor (regardless whether **solid** or **hollow)**.
3. The E field outside a charged **spherical** conductor is the same as if all the charge is located at the **centre** of the sphere (i.e. point charge).
4. Since there is no E field within a conductor, and the entire conductor is at the same electric potential, no work is done to move a charge through it.
5. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest i.e. charges are closer together at sharp points. The E field looks as if due to a point charge, with the field lines spreading out radially. $\Rightarrow$ the **field** is **stronger** near the **sharp tip** than on the flat part of the conductor.
6. The E field at a point just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude $\frac{\sigma}{\varepsilon_0}$ (σ = surface charge density at that point) .



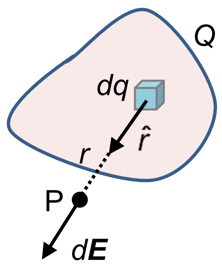
<u>Continuous Charge Distribution</u>

When the distances between charges in a group of charges is much smaller than the distance between the group to a point where the electric field strength or electric potential is to be calculated, the system of charges can be modelled as a <u>continuous charge distribution</u>.

The electric field strength or electric potential due to a continuous charge distribution must be evaluated for each infinitesimally small charge element *dq*. The electric field strength (vector) and electric potential (scalar) at a distance *r* due to a charge element d*q* is

$$d\boldsymbol{E} = \frac{dq}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$$

and
$$dV = \frac{dq}{4\pi\varepsilon_0 r}$$



respectively, where $\hat{\boldsymbol{r}}$ is a unit vector directed towards that point.

The total electric field strength and electric potential due to the continuous charge distribution is therefore

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2}\hat{\boldsymbol{r}}$$

and
$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}$$

When we determine the electric field strength and electric potential due to various charge distributions, it is convenient to use the concept of charge density defined in the following ways:

If a charge *Q* is uniformly distributed

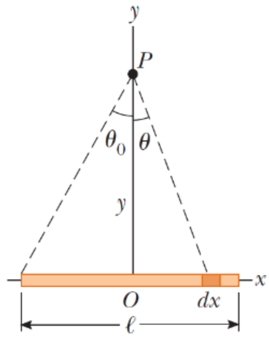
- along a line of length L, the ***linear charge density*** λ is

  $\lambda = Q/L$
- on a surface of area A, the ***surface charge density*** σ is

  $\sigma = Q/A$
- within a volume V, the ***volume charge density*** ρ is

  $\rho = Q/V$

Example 1



A thin rod of length l and uniform charge per unit length λ lies along the x axis.

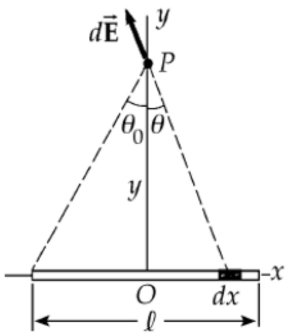
(a) Show that the electric field at P, a distance y from the rod along its perpendicular bisector and is given by $E = \dfrac{\lambda \sin\theta_0}{2\pi\varepsilon_0 y}$.

(b) Using your part above, show that the field of a rod of infinite length is $E = \dfrac{\lambda}{2\pi\varepsilon_0 y}$. Suggestion: First calculate the field at P due to an element of length dx, which has a charge of $\lambda\, dx$. Then change variables from x to θ, using the relationships $x = y\tan\theta$ and $dx = y\sec^2\theta\, d\theta$, and integrate over θ.

Solution:

(a) The electric field at point P due to each element of length dx is

$dE = \dfrac{dq}{4\pi\varepsilon_0 (x^2 + y^2)}$ and is directed along the line joining the element to point P.



By symmetry

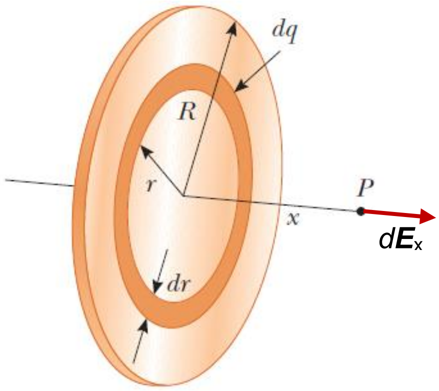
$E_x = \int dE_x = 0$ and since $dq = \lambda dx$

$E = E_y = \int dE_y = \int dE\cos\theta$ where $\cos\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

Therefore, $E = \dfrac{\lambda y}{2\pi\varepsilon_0}\displaystyle\int_0^{\ell/2} \dfrac{dx}{(x^2+y^2)^{3/2}} = \dfrac{\lambda \sin\theta_0}{2\pi\varepsilon_0 y}$

(b) For a bar of infinite length, $\theta_0 = 90^\circ$ and $E_y = \dfrac{\lambda}{2\pi\varepsilon_0 y}$.

Example 2 (Electric field of a uniformly charged disk)



A disk of radius R has a uniform surface charge density σ. Calculate the electric field at a point P that lies along the central perpendicular axis of the disk and a distance x from the centre of the disk. Hence or otherwise, derive the electric field strength of an infinite, uniformly charged plate.

Solution:

A disk can be considered to be a set of concentric rings. The charge of a ring of diameter r and thickness dr is

$$dq = \boxed{}$$

and the parallel electric field strength at P due to this ring is

$$dE_x = \frac{x\,dq}{4\pi\varepsilon_0\left(r^2+x^2\right)^{3/2}} = \frac{\sigma\,r\,x\,dr}{2\varepsilon_0\left(r^2+x^2\right)^{3/2}}$$

The total electric field strength at P is

$$E_x = \frac{\sigma x}{2\varepsilon_0}\int_0^R \frac{r\,dr}{\left(r^2+x^2\right)^{3/2}} = \frac{\sigma x}{2\varepsilon_0}\left[-\frac{1}{\left(r^2+x^2\right)^{1/2}}\right]_0^R = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{\left(R^2+x^2\right)^{1/2}}\right]$$

Similarly, the electric potential at P due to the ring is

$$dV = \boxed{}$$

and the total electric potential at P is

$$V = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r\,dr}{\left(r^2+x^2\right)^{1/2}} = \frac{\sigma}{2\varepsilon_0}\left[\left(r^2+x^2\right)^{1/2}\right]_0^R = \frac{\sigma}{2\varepsilon_0}\left[\left(r^2+x^2\right)^{1/2}-x\right]$$

For an infinite, uniformly charged plate, $R \to \infty$. So, $E = \dfrac{\sigma}{2\varepsilon_0}$

recall and apply Gauss's law for electric and magnetic fields

Gauss's law for electric and magnetic fields

In the section above, we have shown how to calculate the electric field due to a given charge distribution. We will now describe Gauss's law and an alternative method for calculating electric fields.

Gauss's law states that the net flux through *any* closed surface, of area dA is

$$\Phi_E = \oint E \cdot dA = \frac{q_{in}}{\varepsilon_o}$$

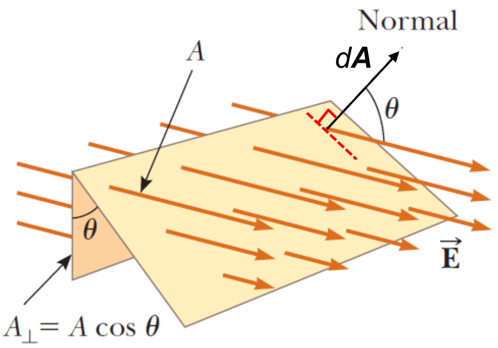
where q_{in} in represents the net charge inside the surface and E represents the electric field at any point on the surface.

Another way to quote **Gauss Law** is:

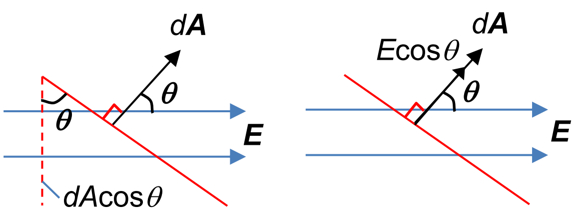
The total electric flux Φ_E coming out of any closed surface (that is, a surface enclosing a definite volume) is proportional to the total (net) electric charge q_{in} inside the surface.

Electric flux

The electric flux is the product of the electric flux density E (also known as the electric field strength) and the component of the area perpendicular to the field:



$$\Phi_E = EA_\perp = EA\cos\theta$$



side view

where θ is the angle between the electric field and the perpendicular (normal) to the area. In general, the electric field strength varies over a large surface. Hence we consider a surface divided into a large number of small elements each of area dA and corresponding area vector d$\boldsymbol{A}$ perpendicular to the infinitesimally small surface element. The angle between $\boldsymbol{E}$ and d$\boldsymbol{A}$ is θ.
The flux through this small element is:

$$d\Phi_E = \boldsymbol{E} \cdot d\boldsymbol{A} = E\ dA\cos\theta = E\cos\theta\ dA$$

The total electric flux over any surface area S is then

$$\Phi_E = \int_S \boldsymbol{E} \cdot d\boldsymbol{A}$$

Typically, the total electric flux is evaluated over a *closed surface* in order to apply Gauss's Law. In such cases, we will replace the integral by

$$\Phi_E = \oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \oint_S E_n dA$$

where E_n is the component of the electric field strength normal to the surface.

The total electric flus is positive when there is a source (positive charges) within the volume while it is negative when there is a sink (negative charges) within the volume.

Note:

1. Electric flux is a **scalar** quantity with unit N m^2 C^{-1}.
2. Electric flux through the surface:
   - maximum value *EA* when the surface is $\perp$ to the field. i.e. when the area vector d***A*** is parallel to the field, $\theta = 0°$.
   - Value of zero when the surface is // to the field. i.e. when the area vector d***A*** is $\perp$ to the field, $\theta = 90°$.
3. Electric flux may be positive or negative depending on the (arbitrary) choice of normal to area.
4. A closed surface encloses a volume of space, so that there is an inside and an outside.
5. Gauss's law is applicable for *any* closed surface and arbitrary shape.

Example 3

Consider a positive point charge *q* placed at the centre of a spherical surface of radius *r*. Calculate the total electric flux through this spherical surface.

Solution:

The electric field strength at a distance *r* from the charge *q* is

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}}$$

at every point on the surface.

So the total electric flux is

$$\begin{aligned}\Phi_E &= \oint_S \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}} \cdot d\boldsymbol{A} = \oint_S \frac{q}{4\pi\varepsilon_0 r^2} \cdot dA \quad (\because E \text{ is always} \perp \text{ to } dA) \\ &= \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)(4\pi r^2) \\ &= \frac{q}{\varepsilon_0}\end{aligned}$$

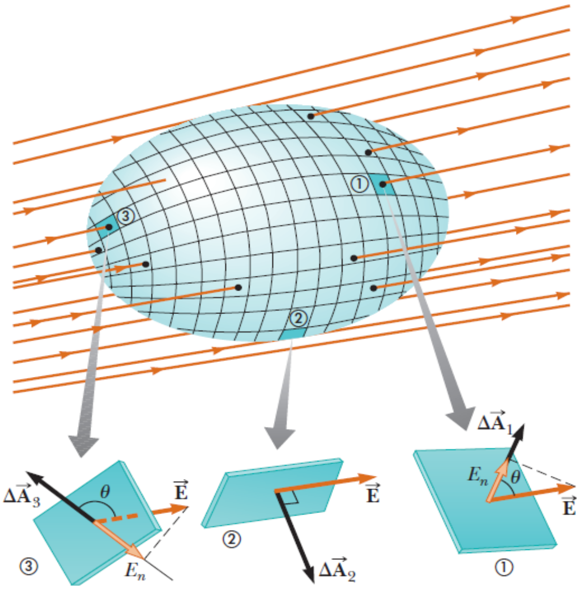
This relation is **independent of** the **radius** of the sphere and the equation holds for irregular surfaces of any shape and size. The general statement of **Gauss' law** is therefore

$$\boxed{\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0}}, \quad \text{where } q \text{ is the enclosed charge.}$$

For multiple charges inside the surface, Gauss's law asserts that the **electric flux** through any closed surface is **proportional** to the **total charge** *q* inside the surface.

Note:

1. Any surface can be chosen as long as it **encloses** the charge; an imaginary construct known as a **Gaussian surface.**
2. The area elements $d\boldsymbol{A}$ and the corresponding unit vectors $\hat{n}$ always point **out** of the volume enclosed by the surface.
3. The electric flux is **positive** in areas where the electric field points out of the surface and **negative** where it points inward. This depends on the sign of the enclosed charge.
4. The relation $\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \dfrac{q}{\varepsilon_0}$ is independent of the shape of the Gaussian surface and of the location of the charge inside it.
5. Charges outside the surface do not give a net electric flux through the surface.
6. The electric flux through a closed surface that encloses no net charge is zero.

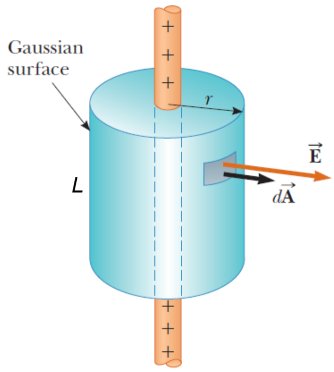


A closed surface in an electric field. The area vectors are, by convention, normal to the surface and point outward. The flux through an area element can be positive (element ①), zero (element ②), or negative (element ③).

Gauss's Law states that $\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \dfrac{q}{\varepsilon_0}$, and it can simplify the calculation of electric field strength for charge distributions is highly symmetric. These will be illustrated in the following examples.

Example 4 (cylindrical symmetry)

Calculate the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ

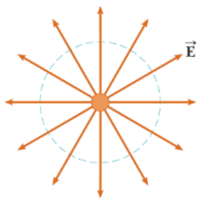


Solution:

Since the line of charge is infinitely long, ***E*** points radially outward from the wire. ***E*** is the same at all points equidistant from the line.

Because the charge is distributed uniformly along the line, the charge distribution has cylindrical symmetry and we can apply Gauss's law by selecting a cylindrical Gaussian surface of radius r and length L.

Since ***E*** is // to the flat ends and ⊥ to the curved surface of the cylinder, the area integral of this Gaussian surface is



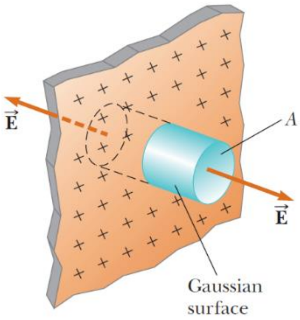
$$\int_{\text{curved}} \boldsymbol{E} \cdot d\boldsymbol{A} + \int_{\text{ends}} \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0}$$

$$E \cdot 2\pi r L + 0 = \lambda L / \varepsilon_0$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Example 5 (planar symmetry)

(a) A thin flat infinite plate has a uniform positive charge distribution over its entire surface, with uniform surface charge density σ. Determine the electric field strength.

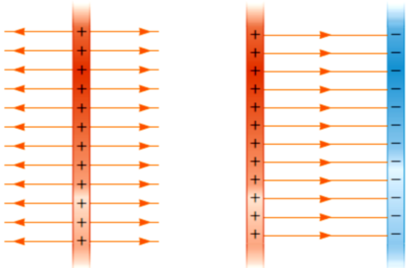


Solution:
Since the plate is infinitely large and positively charged, E is perpendicular to and pointing away from the plate. Consider the Gaussian surface as shown. Since E is // to the curved surface and $\perp$ to the flat ends of the cylinder, the area integral of this Gaussian surface is

$$\int_{\text{curved}} \boldsymbol{E}\cdot d\boldsymbol{A} + \int_{\text{ends}} \boldsymbol{E}\cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0}$$
$$0 + 2EA = \sigma A/\varepsilon_0$$
$$E = \frac{\sigma}{2\varepsilon_0}$$

This shows that the electric field due to an infinite charged plate is ***uniform*** everywhere.

(b) Suppose two infinite plates of charge are parallel to each other, one positively and the other negatively charged. Both planes have the same surface charge density. Determine the electric field strength around the plates.



The electric field is confined to between the plates and the electric field strengths add up.

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

The electric field strength everywhere else is zero.
(charged sheet approach)

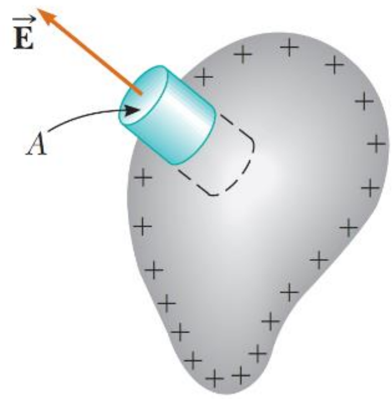
Example 6

Show that the electric field just outside the surface of any good conductor of arbitrary shape is given by $E = \frac{\sigma}{\varepsilon_0}$ where σ is the surface charge density on the conductor's surface at that point.

Solution:

The electric field everywhere inside the conductor is when it is in electrostatic equilibrium. Any net charge resides on the surface of the conductor and the field vector must be perpendicular to the surface.

We consider a Gaussian surface in the shape of a small cylinder with ends parallel to the conductor's surface. One end of the cylinder is just outside the conductor and the other just inside the surface. There is no flux at the sides of the Gaussian surface as $\boldsymbol{E}$ is parallel to the sides. There is no flux on the inner flat face where $\boldsymbol{E} = 0$. Hence, the net flux through the Gaussian surface is equal to that through only the flat face outside the conductor, where the field is perpendicular to it.

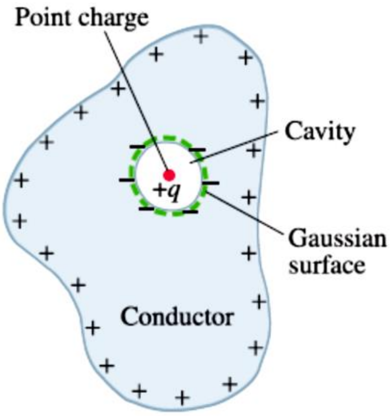


Applying Gauss's law:

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q}{\varepsilon_0}$$
$$EA = \frac{\sigma A}{\varepsilon_0}$$
$$E = \frac{\sigma}{\varepsilon_0}$$

Example 7

Given that a conductor carries a net charge $+Q$ and contains a cavity, inside of which resides a point charge $+q$. Determine the charges on the inner and outer surfaces of the conductor.



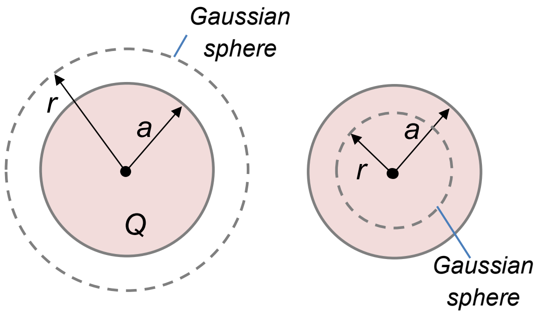
A Gaussian surface just inside the conductor surrounding the cavity must contain zero net charge as $E = 0$ in a conductor. Thus a net charge of $-q$ must exist on the cavity surface. The conductor itself carries a net charge $+Q$, so its outer surface must carry a charge equal to $Q + q$. These results apply to a cavity of any shape.

Example 8 (spherical symmetry)

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q. Determine the electric field strength at a point inside and outside the sphere.

Solution:

Since the charge is distributed uniformly throughout the sphere, the charge distribution has spherical symmetry.



Consider a spherical Gaussian surface of radius $r < a$, concentric with the sphere so that the magnitude of E is constant. The charge q_{in} within this Gaussian surface is

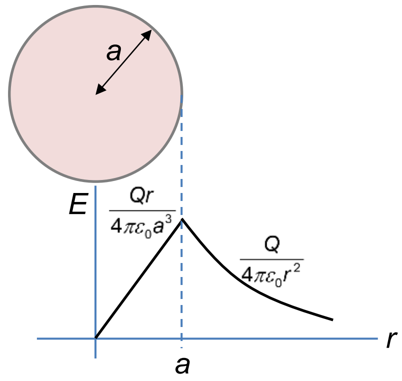
$q_{in} =$ $\boxed{}$

Since E is radial (by symmetry) and hence perpendicular to the Gaussian surface, the area integral of this Gaussian surface is

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q_{in}}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{r^3}{a^3} \cdot \frac{Q}{\varepsilon_0}$$

$$E = \boxed{}$$



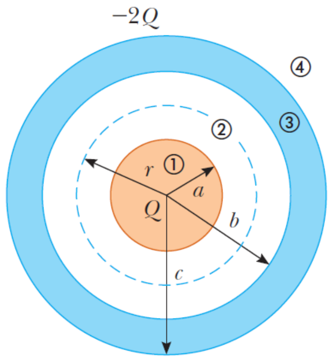
For the gaussian sphere of radius $r > a$, $q_{in} = Q$. So

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E = \boxed{}$$

This result shows that the electric field due to a uniformly charged sphere in the region external to the sphere is equivalent to that of a point charge located at the centre of the sphere.

Example 9 (Sphere inside a spherical shell)



A solid insulating sphere of radius a carries a net positive charge Q uniformly distributed throughout its volume. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and caries a net charge $-2Q$. Determine the electric field strength in the regions labelled ①, ②, ③ and ④ and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

Solution:

The charge is distributed uniformly throughout the sphere and the charge on the conducting shell distributes itself uniformly on the surfaces. Therefore, the system has spherical symmetry and we can apply Gauss's law to find the electric field.

Regions ① and ② is within the conducting shell. Hence the charge on the shell has no effect on the electric field inside the shell.

Therefore, the electric field strength in region ① and ② is identical to that found earlier $E_1 = \frac{Qr}{4\pi\varepsilon_0 a^3}$,

$E_2 = \frac{Q}{4\pi\varepsilon_0 r^2}$.

The electric field in region ③ is zero since it is within the conducting shell that is in electrostatic equilibrium ($b < r < c$, because $\boldsymbol{E}$ inside a conductor is zero.)

The electric field in region ④ is equivalent to a point charge of $Q+(-2Q)=-Q$ placed at the centre of the system.

$$\oint_s \boldsymbol{E}\cdot d\boldsymbol{A}=\frac{q}{\varepsilon_0}$$

$$-E_4\left(4\pi r^2\right)=\frac{Q+(-2Q)}{\varepsilon_0}$$

$$E_4=\frac{Q}{4\pi\varepsilon_0 r^2}\quad (r>c)$$ acting radially inwards

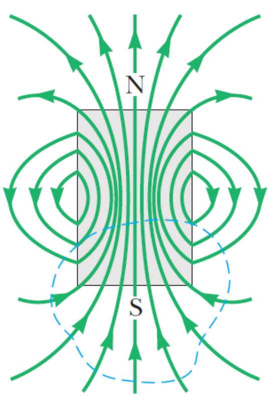
Therefore

$$E=\begin{cases} Qr/4\pi\varepsilon_0 a^3 & (r<a)\\ Q/4\pi\varepsilon_0 r^2 & (a<r<b)\\ 0 & (b<r<c)\\ -Q/4\pi\varepsilon_0 r^2 & (r>c)\end{cases}$$

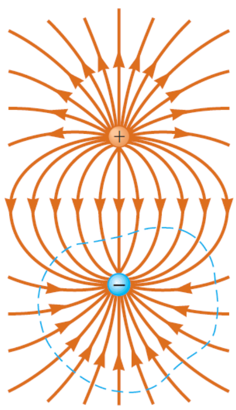
The charge on the inner surface of the spherical shell must be −Q to cancel the charge +Q on the solid sphere to give zero electric field in the shell. This implies that the charge on the outer shell is – Q.

Gauss's Law for Magnetic Field

Magnetic monopoles do not exist.
We have previously seen that the electric flux through a closed surface surrounding a net charge is proportional to that charge (Gauss's law). I.e. the number of electric field lines leaving the surface depends only on the net charge within it. This behaviour exists because electric field lines originate and terminate on electric charges. The situation is quite different for magnetic fields, which are continuous and form closed loops. This means that the magnetic field lines do not begin and end at any point, and that for any closed surface the number of lines entering and leaving the surface is the same. This results in the **net magnetic flux to be zero**. In contrast, for the closed surface surrounding one charge of an electric dipole, the net electric flux is not zero.



The magnetic field lines of the bar magnet form closed loops. Note that the net magnetic flux through a closed surface surrounding one of the poles (or any other closed surface) is zero.



Electric field line surrounding an electric dipole begin on the +ve charge and terminate on the –ve charge. The electric flux through a closed surface is not zero.

Gauss's law in magnetism states that:

the net magnetic flux through any closed surface is always zero:

$$\Phi_B = \oint_S \boldsymbol{B} \cdot d\boldsymbol{A}$$

This implies that isolated magnetic pole (monopoles) have never been detected and perhaps do not exist.

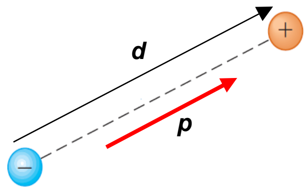
(j) define the magnitude of the electric dipole moment as the product of the charge and the separation.

(k) show an understanding of and use the torque on an electric dipole and the potential energy of an electric dipole to solve related problems.

Electric Dipole in an Electric Field

Electric Dipole moment

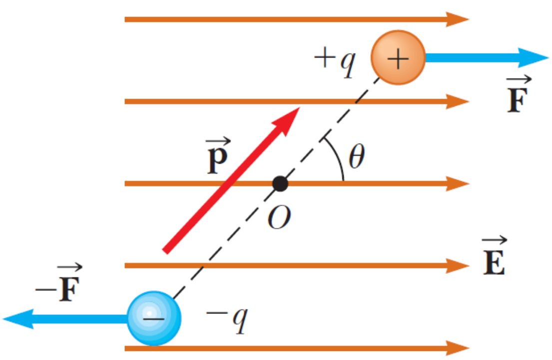
An electric dipole consists of two charges of equal magnitude and opposite sign separated by a distance d. The **electric dipole moment** is defined as the vector quantity



$$\boldsymbol{p} = q\boldsymbol{d}$$

where $\boldsymbol{d}$ is the displacement vector directed from $-q$ toward $+q$ along the line joining the charges. The units of $\boldsymbol{p}$ is coulomb metre (Cm). The magnitude of the electric dipole moment is given by $p = dq$.

Torque on an Electric Dipole



If the electric dipole is placed in an external uniform electric field $\boldsymbol{E}$ and makes and angle θ as shown, the two charges will experience electric forces equal in magnitude ($F = qE$) and opposite in direction. The resultant force acting on the dipole is hence zero.

The two forces produce a net torque on the dipole, and the dipole always tends to rotate in the direction that brings the dipole moment vector $\boldsymbol{p}$ into greater alignment with the field. The torque is greatest when $\boldsymbol{p}$ and $\boldsymbol{E}$ are perpendicular, and is zero when they are parallel or antiparallel.

The value of the net torque about O is:

$$\begin{aligned}\tau &= Fd\sin\theta \\ &= qEd\sin\theta \\ &= pE\sin\theta \quad (\because p = qd)\end{aligned}$$

We can express the torque in vector form as the cross product of vectors ***p*** and ***E***.

$$\tau = \boldsymbol{p} \times \boldsymbol{E}$$

Potential Energy of an Electric Dipole

The potential energy of the system of an electric dipole in an external electric field can be expressed as a function of the dipole's orientation with respect to the field. Work must be done by an external agent to rotate through an angle to cause the dipole moment vector to become *less aligned* with the field which is stored as potential energy in the system (c.f. GPE).

The work dW required to rotate the dipole through an angle $d\theta$ is $dW = \tau\, d\theta$. Since the work results in an increase in the potential energy U, for a rotation from θ_i to θ_f the change in potential energy of the system is

$$U_f - U_i = \int_{\theta_i}^{\theta_f} \tau d\theta = \int_{\theta_i}^{\theta_f} pE\sin\theta\ d\theta = pE\int_{\theta_i}^{\theta_f} \sin\theta\ d\theta$$

$$= pE\left[-\cos\theta\right]_{\theta_i}^{\theta_f} = pE\left(\cos\theta_i - \cos\theta_f\right)$$

The term $pE\cos\theta_i$ is a constant that depends on the initial orientation of the dipole. We choose a reference angle $\theta_i = 90°$ so that $\cos\theta_i = \cos 90° = 0$. Similarly choosing $U_i = 0$ at $\theta_i = 90°$ as our reference of potential energy. Hence we can express $U = U_f$ as

$$U = -pE\cos\theta$$

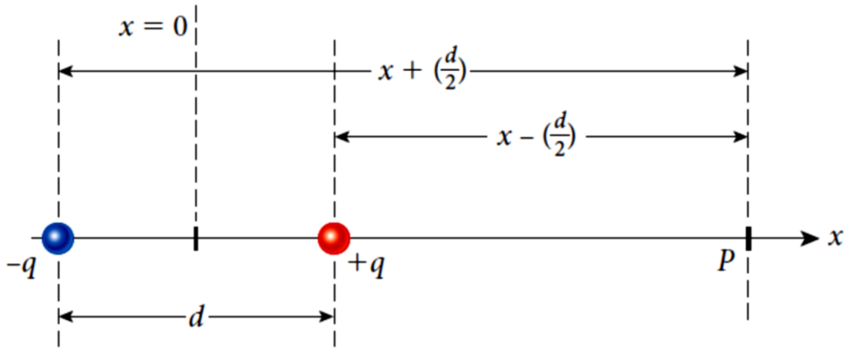
and also as the dot product of vectors ***p*** and ***E***:

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}$$

The potential energy is has its minimum (most negative) value $U = -pE$ at the stable equilibrium position where $\theta = 0$ and ***p*** is parallel to ***E***. The potential energy is maximum, $U = pE$ when $\theta = \pi$ where ***p*** is antiparallel to ***E***.

Example 10

Determine the electric field (self-field) at a point a distance *x* along the dipole axis from the centre of an electric dipole, with electric dipole moment *p*, where *x* is much larger than the separation between the charges.



Solution:

Considering the negative charge, taking left as positive,

$E_- =$

Considering the positive charge, taking right as positive,

$E_+ =$

Taking right as positive,

$E = E_+ - E_-$

$$= \frac{q}{4\pi\varepsilon_0\left(x - \frac{d}{2}\right)^2} - \frac{q}{4\pi\varepsilon_0\left(x + \frac{d}{2}\right)^2} = \frac{q}{4\pi\varepsilon_0}\left[\left(x - \frac{d}{2}\right)^{-2} - \left(x + \frac{d}{2}\right)^{-2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 x^2}\left[\left(1 - \frac{d}{2x}\right)^{-2} - \left(1 + \frac{d}{2x}\right)^{-2}\right] = \frac{q}{4\pi\varepsilon_0 x^2}\left[\left(1 + \frac{d}{x}\right) - \left(1 - \frac{d}{x}\right) + \ldots\right]$$

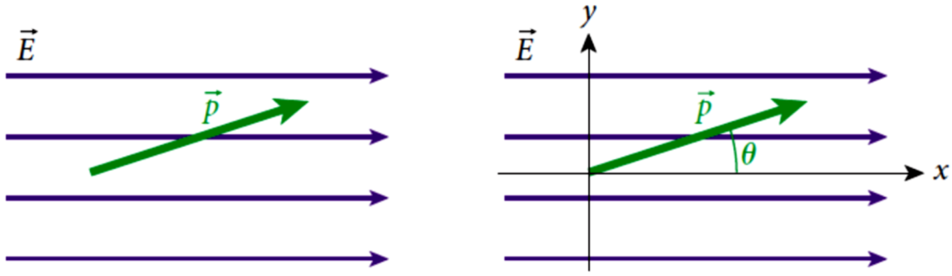
$$\approx \frac{qd}{2\pi\varepsilon_0 x^3}$$

$$\approx \frac{p}{2\pi\varepsilon_0 x^3}$$

On the dipole axis, the dipole produces an electric field $\boldsymbol{E} = \frac{\boldsymbol{p}}{2\pi\varepsilon_0 x^3}$ and this equation holds only for distant points along the dipole axis. The direction of electric field for distant points on the dipole axis is always in the same direction as the dipole moment vector. This is true whether point *P* is on the right or the left of the electric dipole.

Example 11

An electric dipole with dipole moment of magnitude p = 1.40 × 10^{-12} C m is placed in a uniform electric field of magnitude E = 498 N C^{-1}.



Solution:

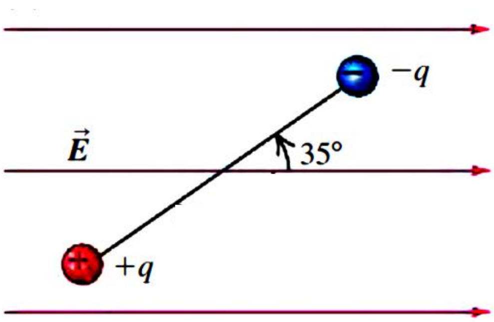
Given that θ = 14.5°, calculate the torque on the electric dipole.

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{p}\times\boldsymbol{E}\\ &= pE\sin\theta\\ &= \left(1.40\times10^{-12}\right)(498)\sin14.5^\circ\\ &= 1.75\times10^{-10}\ \text{N m}\end{aligned}$$

Torque is 1.75 × 10^{-10} N m in the negative z-direction.

Food for thought: If water molecules are exposed to an external electric field, they experience a torque and thus begin to rotate. If the direction of the external electric field changes very rapidly, the water molecules perform rotational oscillations, which create heat. This is the principle of operation of a microwave oven. Microwave ovens use a frequency of 2.45 GHz for the oscillating electric field.

Example 12



The diagram shows an electric dipole in a uniform electric field of magnitude 5.0 × 10^5 N C^{-1}.

The charges are 1.6 × 10^{-19} C; both lie in the plane and are separated by 0.125 nm. Calculate the

(a) net force exerted by the field on the dipole,
(b) magnitude and direction of the electric dipole moment,
(c) magnitude and direction of the torque,
(d) potential energy of the system in the position shown.

Solution:

(a) Since the forces acting on both charges are equal in magnitude but opposite in direction, the net force on the dipole is zero.

(b) $$\begin{aligned}p &= qL\\ &= \left(1.6\times10^{-19}\right)\left(0.125\times10^{-9}\right)\\ &= 2.0\times10^{-29}\ \text{C m}\end{aligned}$$

Electric dipole moment is 2.0 × 10^{-29} C m from the negative to the positive charge, 145° clockwise from the electric-field direction.

(c) $\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$

$= pE \sin\theta$

$= \left(2.0 \times 10^{-29}\right)\left(5.0 \times 10^{5}\right) \sin 145°$

$= 5.7 \times 10^{-24}$ N m

Torque is out of page.

(d) $U = -\boldsymbol{p} \cdot \boldsymbol{E}$

$= -\boldsymbol{pE} \cos\theta$

$= -\left(2.0 \times 10^{-29}\right)\left(5.0 \times 10^{5}\right) \cos 145°$

$= 8.2 \times 10^{-24}$ J

The charge magnitude, the distance between the charges, the dipole moment, and the potential energy are all very small, but are all typical of molecules.

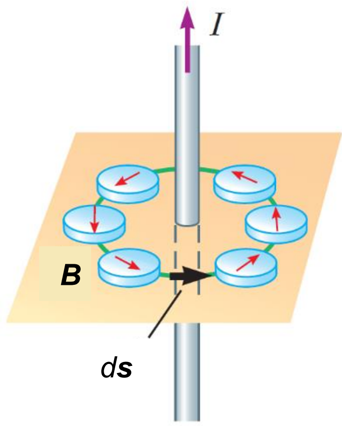
Magnetic Field Due To An Electric Current

In H2, we have discussed the magnetic force exerted on a charged particle moving in a magnetic field. To complete the description of the magnetic interaction, we explore the origin of the magnetic field, moving charges.
The Ampère's law will also be introduced, useful in calculating the magnetic field of a highly symmetric configuration carrying a steady current.

> (d) recall and apply Ampère's law relating the line integral of the magnetic field (in a vacuum) around a closed loop with the electric current enclosed by the loop to solve problems involving symmetric field configurations

Ampère's Law

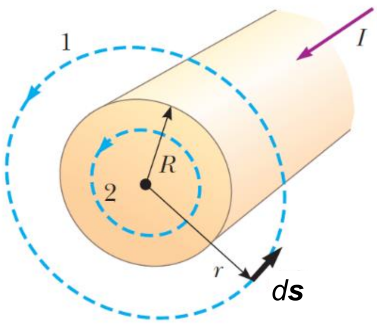
Ampère's law describes the creation of magnetic fields by all continuous current configurations, and it is useful for **calculating the magnetic field** of current configurations having a high degree of symmetry. Its use is similar to that of Gauss's law in calculating electric fields for highly symmetric charge distributions.



> Ampère's law states that the line integral of ***B***.***ds*** around any closed path (Ampèrian loop) is equal to $\mu_0 I$, where I is the **total current** enclosed by the closed path:
>
> $$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 I \ .$$

Example 14: Magnetic Field Inside/Outside of a Long, Straight Conductor

A long straight wire of radius *R* carries a steady current I that is uniformly distributed through the cross-section of the wire. Calculate the magnetic field at a distance *r* from the centre of the wire in the regions $r \geq R$ and $r < R$.



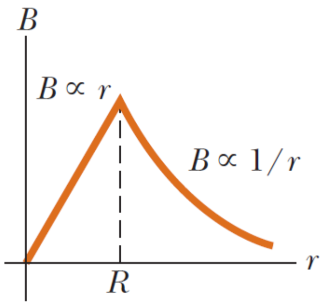
Solution:

For $r \geq R$, $\oint \boldsymbol{B} \cdot d\boldsymbol{s} = B\oint ds = B(2\pi r) = \mu_o I$ (choosing symmetric path enclosing current)

$\Rightarrow$ B =

For $r < R$, current within the closed loop 2 is I' such that

$$\frac{I'}{I} = \frac{J \times 2\pi r^2}{J \times 2\pi R^2} = \frac{r^2}{R^2}$$ where J = current density = $\frac{I}{A}$.

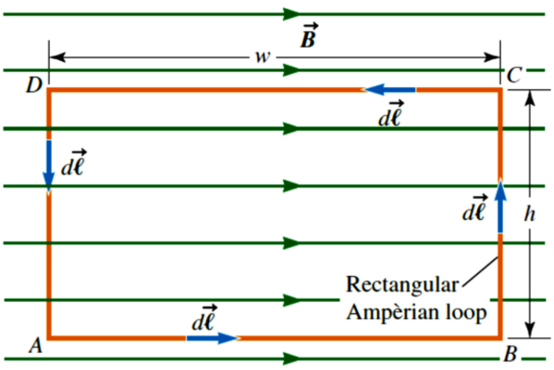


Therefore, $\oint \boldsymbol{B} \cdot d\boldsymbol{s} = B\oint ds = B(2\pi r) = \mu_o I' = \mu_o \left(\frac{r^2}{R^2} I\right)$

$\Rightarrow$ B =

<u>Example 15</u>

Consider a uniform magnetic field $\boldsymbol{B}$ and complete the circulation integral $\oint_s \boldsymbol{B} \cdot d\boldsymbol{s}$ around the rectangular Ampèrian loop as shown.



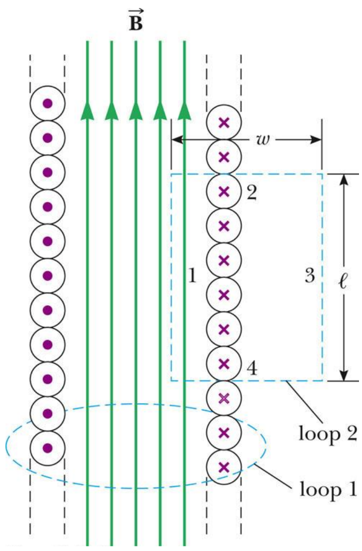
<u>Solution:</u>

$$\oint_s \boldsymbol{B} \cdot d\boldsymbol{s} = \int_A^B \boldsymbol{B} \cdot d\boldsymbol{s} + \int_B^C \boldsymbol{B} \cdot d\boldsymbol{s} + \int_C^D \boldsymbol{B} \cdot d\boldsymbol{s} + \int_D^A \boldsymbol{B} \cdot d\boldsymbol{s}$$

$$= Bw\cos 0 + Bh\cos\frac{\pi}{2} + Bw\cos\pi + Bh\cos\frac{\pi}{2}$$

$$= 0$$

<u>Example 16:</u> <u>Magnetic field of a solenoid</u>



Use Ampere's Law to obtain a quantitative expression for the interior magnetic field in a long solenoid.

<u>Solution:</u>

A solenoid is a long wire wound in the form of a helix. An ideal solenoid is approached when its turns are closely spaced and its length is much greater than the radius of the turns.

We can use Ampere's law to obtain a quantitative expression for the interior magnetic field in an ideal solenoid. Because the solenoid is ideal, $\boldsymbol{B}$ in the interior space is uniform and parallel to the axis.

Consider the rectangular loop 2 of length ℓ and width w as shown. For sides 2, 3 and 4, $\boldsymbol{B}.d\boldsymbol{s}$ is zero because the magnetic field is either perpendicular to the paths or it is near-zero.

Applying ampere's law, we have

$$\oint_s \boldsymbol{B}\cdot d\boldsymbol{s} = \mu_0 I_{\text{enclosed}}$$

$$\oint_s \boldsymbol{B}\cdot d\boldsymbol{s} = \int_A^B \boldsymbol{B}\cdot d\boldsymbol{s} + \int_B^C \boldsymbol{B}\cdot d\boldsymbol{s} + \int_C^D \boldsymbol{B}\cdot d\boldsymbol{s} + \int_D^A \boldsymbol{B}\cdot d\boldsymbol{s} = \mu_0 I$$

$$B\ell = \mu_0 NI \quad \text{(where } i \text{ is the current through each turn)}$$

$$B = \mu_0 nI \qquad \text{(where } n = \frac{N}{\ell} \text{ is the number of turns per unit length)}$$

The above expression is valid only for points near the middle of the solenoid, far from the ends. At the ends of the solenoid, the magnitude of the field is half that at the centre.

Example 17 : Magnetic Field Inside A Toroid

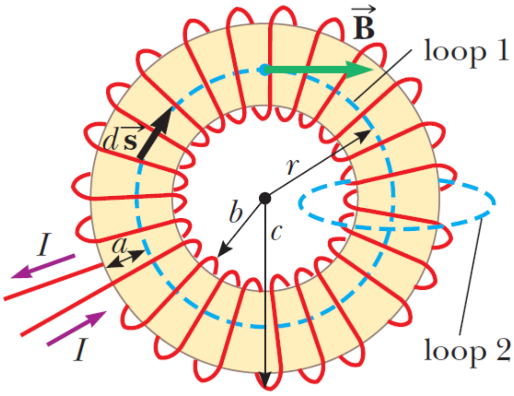
A device called a toroid is often used to create an almost uniform magnetic field in some enclosed area. The device consists of a conducting wire wrapped around a ring made of non-conducting material. For a toroid having *N* closely-spaced turns of wire, determine the magnetic field in the region occupied by the torus, a distance *r* from the centre.

Solution:

$$\oint \boldsymbol{B}\cdot d\boldsymbol{s} = B\oint ds = B(2\pi r) = \mu_o NI$$

$$B = \frac{\mu_o NI}{2\pi r}$$

The magnetic field will be approximately uniform within the toroid if *r* is very large compared with the cross sectional radius *a* of the torus. i.e. when *b* and *c* are approximately equal.

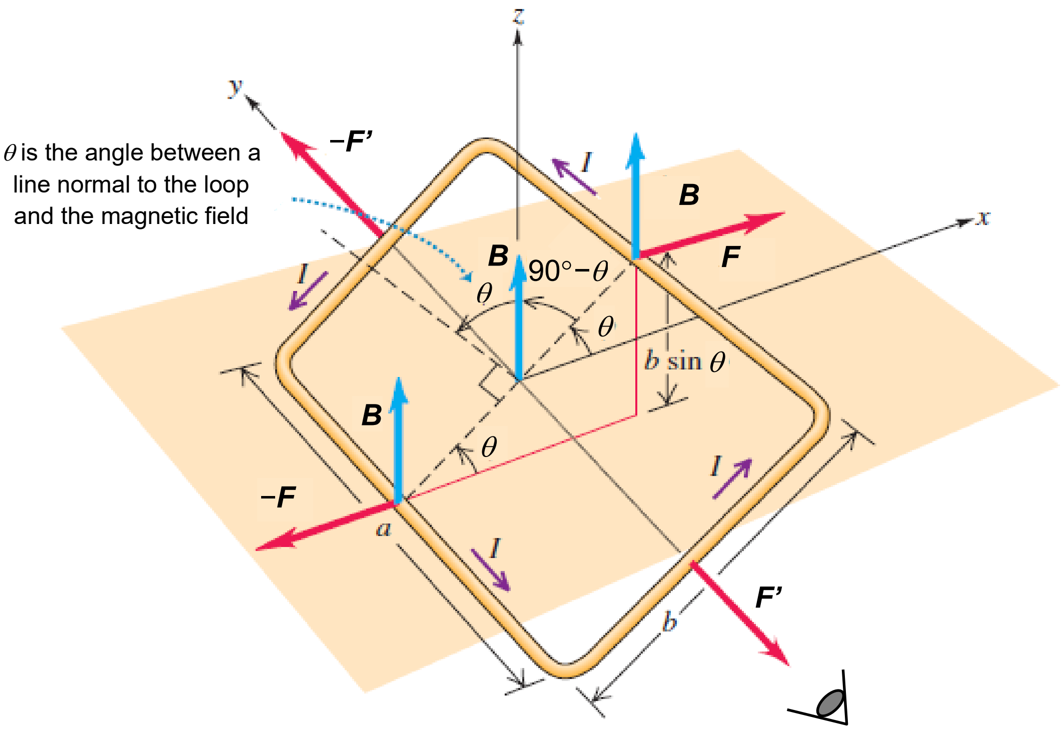


Magnetic Dipoles

Magnetic dipoles are an idealised system that is used for the approximate description of a field created by a more complex system of charges. Examples of dipoles are a closed circuit or current loop. The magnetic dipole moment (often called magnetic moment) $\boldsymbol{\mu}$ is a vector quantity used to characterise magnetic properties of some matter. One way to look at it would be to think of a maximum amount of torque that can be caused by magnetic force on a dipole arising per unit value of surrounding magnetic field in vacuum.

Torque on a Magnetic Dipole

The diagram below shows a rectangular loop of wire with side lengths *a* and *b*. A line perpendicular to the plane of the loop (i.e., a *normal* to the plane) makes an angle θ with the direction of the magnetic field ***B***, and the loop carries a current I.



The force ***F*** on the right side of the loop (length *a*) is to the right, in the positive *x*-direction as shown. On this side, ***B*** is perpendicular to the current direction, and the force on this side has magnitude

$$F = BIa$$

A force $-$***F*** acts on the opposite side of the loop, as shown in the diagram. This results in a torque τ in the positive *y*-direction (use right-hand rule), and the magnitude is

$$\tau = 2F\left(\frac{b}{2}\right)\sin\theta = BI(ab)\sin\theta$$

The torque is zero at $\theta = 0°$ or $180°$, with the former being the position of stable equilibrium.

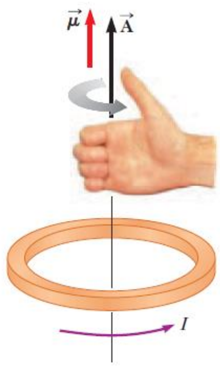
Since *ab* is equal to the area *A* of the loop, we have

$$\tau = BIA\sin\theta$$

where A = ab is the area of the loop. This result shows that the torque has its maximum value *BIA* when the field is perpendicular to the normal to the plane of the loop ($\theta = 90°$) and is zero when the field is parallel to the normal to the plane of the loop ($\theta = 0$).

(g) define the magnitude of the magnetic dipole moment for a current loop as the product of the current and the area of the loop.
(h) show an understanding of and use the torque on a magnetic dipole and the potential energy of a magnetic dipole to solve related problems.

Magnetic Dipole Moment



A convenient expression for the torque exerted on a loop placed in a uniform magnetic field $\boldsymbol{B}$ is

$$\boldsymbol{\tau} = I\boldsymbol{A} \times \boldsymbol{B}$$

where $\boldsymbol{A}$ the vector perpendicular to the plane of the loop and has a magnitude equal to the area of the loop. The direction of $\boldsymbol{A}$ can be determined by using the right hand grip rule.

The product $I\boldsymbol{A}$ is defined to be the **magnetic dipole moment** $\boldsymbol{\mu}$ (often simply called the "magnetic moment") of the loop:

$$\boldsymbol{\mu} \equiv I\boldsymbol{A}$$

The SI unit of magnetic dipole moment is A.m^2. If a coil of wire contains N loops of the same area, the magnetic moment of the coil is

$$\boldsymbol{\mu}_{\text{coil}} \equiv NI\boldsymbol{A}$$

The torque exerted on a current-carrying loop in a magnetic field can be expressed as

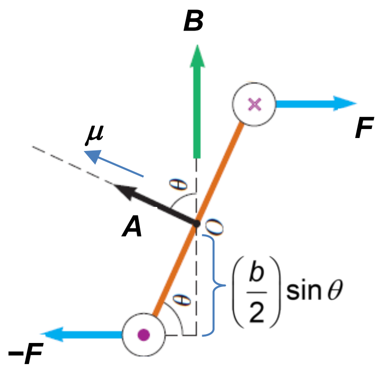
$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \boldsymbol{B}$$

Which is analogous to the torque exerted on an electric dipole $\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$ in the presence of an electric field $\boldsymbol{E}$.

Energy of a magnetic dipole in an external magnetic field

Similar to the case of an electric dipole in an electric field, the potential energy of a system of a magnetic dipole in a magnetic field depends on the orientation of the dipole in the magnetic field and is given by

$$U = -\mu B \cos\theta = -\boldsymbol{\mu} \cdot \boldsymbol{B}$$



which shows that the system has its lowest energy $U_{\text{min}} = -\mu B$ when $\boldsymbol{\mu}$ points in the same direction as $\boldsymbol{B}$, and the highest energy $U_{\text{max}} = +\mu B$ when $\boldsymbol{\mu}$ points in the direction opposite $\boldsymbol{B}$. (refer to energy of an electric dipole for mathematical treatment).

Example 18

A coil of wire has an area of 2.0×10^{-4} m^2, consists of 100 turns, and carries a current of 0.045 A. The coil is placed in a uniform magnetic field of magnitude 0.15 T.

(a) Determine the magnetic moment of the coil.

(b) Find the maximum torque that the magnetic field can exert on the coil.

Solution:

(a)
$$\begin{aligned}\mu &= NIA \\ &= 100(0.045)(2.0\times10^{-4}) \\ &= 9.0\times10^{-4}\ \text{A m}^2\end{aligned}$$

(b)
$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{\mu}\times\boldsymbol{B} \\ &= \mu B\sin\theta \\ &= \mu B\ \ (\text{maximum when }\sin\theta = 1) \\ &= (9.0\times10^{-4})(0.15) \\ &= 1.4\times10^{-4}\ \text{N m}\end{aligned}$$

Example 19

A circular loop of mass m, radius r and current I in it lies on a horizontal surface placed in a region with horizontal magnetic field B. Calculate the minimum current required for one edge of the loop to lift off the surface.

Solution:

$$I = \frac{mg}{\pi r B}$$

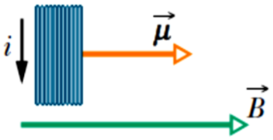
The current is proportional to the mass for constant B. The larger the mass, the greater the current needed to start to rotate the loop.

Example 20

A circular coil with 250 turns, area A of 2.52×10^{-4} m^2, and carries a current of 100 μA as shown below. The coil is at rest in a uniform magnetic field of magnitude 0.85 T, with its magnetic dipole moment initially $\vec{\mu}$ aligned with $\vec{B}$.

(a) Determine the direction of the current in the coil.

(b) Calculate the work required of the torque applied by an external agent to rotate the coil 90° from its initial orientation, so that $\vec{\mu}$ is perpendicular to $\vec{B}$.



Solution:

(a) The current is clockwise as viewed from the left of the coil.

(b) $U = -\vec{\mu} \cdot \vec{B}$

$$\begin{aligned} W &= \Delta U \\ &= -\mu B \cos\theta_f - (-\mu B \cos\theta_i) \\ &= \mu B(\cos\theta_i - \cos\theta_f) \\ &= NIAB(\cos\theta_i - \cos\theta_f) \\ &= 250(100 \times 10^{-6})(2.52 \times 10^{-4})(0.85)(\cos 0 - \cos 90°) \\ &= 5.36 \times 10^{-6} \text{ J} \end{aligned}$$

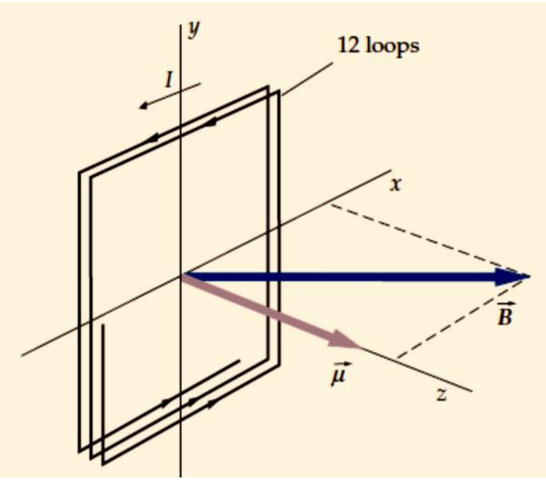
Example 21

A square coil with 12 turns has sides 40.0 cm and carries a current of 3.00 A. It lies in the $z = 0$ plane in a uniform magnetic field $\vec{B} = (0.300\hat{\underset{\sim}{i}} + 0.400\hat{\underset{\sim}{k}})$ T. The current is counterclockwise as viewed from a point on the positive z- axis. Calculate the

(a) magnetic moment of the coil,

(b) torque exerted on the coil,

(c) potential energy of the coil.



Solution:

(a) $\vec{\mu} = NIA\hat{\underset{\sim}{k}}$

$$\begin{aligned} &= 12(3.00)(40.0 \times 10^{-2})^2 \hat{\underset{\sim}{k}} \\ &= 5.76 \text{ A m}^2 \ \hat{\underset{\sim}{k}} \end{aligned}$$

(b)

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$
$$= 5.76\begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}0.300\\0\\0.400\end{pmatrix}$$
$$= 5.76\begin{pmatrix}0\\0.300\\0\end{pmatrix}$$
$$= 1.73 \text{ N m } \underset{\sim}{\hat{j}}$$

(c)

$$U = -\vec{\mu} \cdot \vec{B}$$
$$= -5.76\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}0.300\\0\\0.400\end{pmatrix}$$
$$= -2.30 \text{ J}$$