H3 Topic 5 - RLC Circuits

Content

- Inductance
- Dielectrics and ferromagnetic materials
- Energy in an inductor
- Circuits with capacitors and inductors

Learning Outcomes

Candidates should be able to:

(a) define self-inductance as the ratio of the e.m.f. induced in an electrical circuit / component to the rate of change of current causing it and use $V = L(dI / dt)$ to solve problems

(b) show an understanding that mutual inductance is the tendency of an electrical circuit / component to oppose a change in the current in a nearby electrical circuit / component

(c) show a qualitative understanding that dielectric materials enhance capacitance, and that dielectric breakdown can occur when the electric field is sufficiently strong (knowledge of the quantitative modification of electric fields in matter through the permittivity is not required)

(d) show a qualitative understanding that ferromagnetic materials enhance inductance and that this enhancement is non-linear especially near saturation (knowledge of the quantitative modification of magnetic fields in matter through the permeability is not required)

(e) derive, by considering work done on charges, the expression for potential energy stored in an inductor, $U = LI^2/2$, and use this to solve problems

(f) solve problems using the formulae for the combined inductance of two or more inductors in series and in parallel

(g) solve problems involving circuits with resistors, inductors, and sources of constant e.m.f. (includes solving first-order differential equations) [RL series circuits with constant e.m.f. source]

(h) solve problems involving circuits with inductors and capacitors only (includes solving second-order differential equations) [LC series circuits without e.m.f. source]

(i) solve problems involving circuits with resistors, inductors and capacitors only (candidates are not expected to solve the general second-order differential equations, though they can be asked to verify and use particular solutions). [RLC series circuits without e.m.f. source]

5.0 Introduction

Capacitors and inductors are electrical components that store energy. Together with resistors, they are the key components in modern electrical devices. Depending on the intended purpose of the circuit, these components are used in a variety of combinations and configurations so as to influence how the electric current in the circuit (and potential difference across components) vary with time.

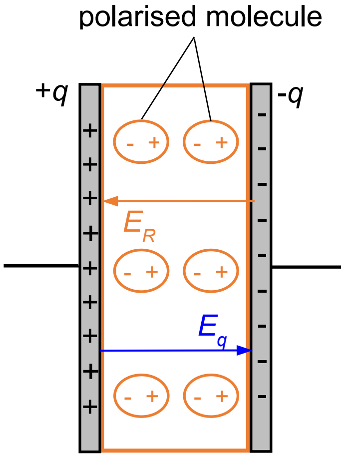
5.1 Effects of Dielectric Materials in a Capacitor

In H2 Physics, we learned about capacitor and capacitance. A capacitor is essentially two conductors (usually plates or sheets) separated by a gap (or an insulator). The insulator in the gap between the plates is called the dielectric. It prevents charges from flowing across the plates of the capacitor.

While the dielectric can simply be air, dielectric materials such as paper, paper soaked in mineral oil, plastic or mica are commonly used in practical capacitors which help increases capacitance:

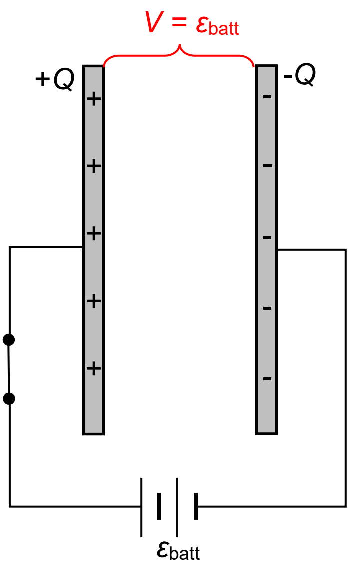
1. They physically separate the plates and prevent electrical contact. This allows the plates to be closer to each other.
2. They have a *higher dielectric constant* than air. This allows *more charge* to be stored on the plates for the same potential difference between them.
3. They have *higher dielectric strength* than air. This allows a *higher p.d.* to be applied across the capacitor all else being constant *(before dielectric breakdown, see next Section).*

When a p.d. is applied across the dielectric, the molecules of the dielectric become polarised by the electric field setup between the plates. The positive ends are induced towards the negative plate and vice versa. This implies that *within* the dielectric, the electric field is opposite in direction to the electric field originally set up between the plates, causing the resultant electric field to decrease.

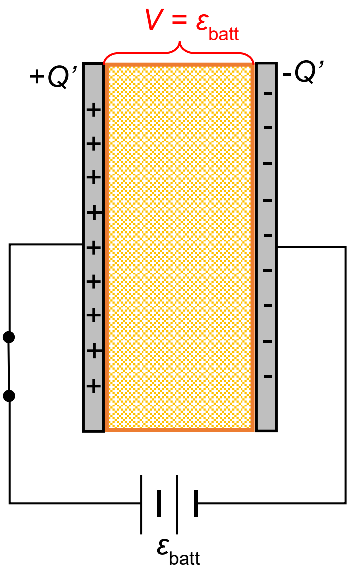


Polarised dielectric molecules setting up a reverse electric field ER

The reduction of the resultant electric field reduces the p.d. across the plates. If the capacitor is still connected to the same battery, more charges flow into the capacitor until the p.d. across the capacitor increases to match the e.m.f. of the battery.



A capacitor with a vacuum between the conductors



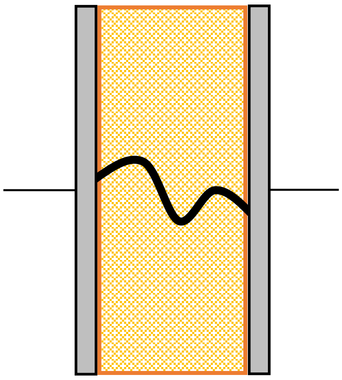
A capacitor with a dielectric between the conductors, where Q' > Q

The amount of additional charge that a capacitor with a dielectric can store depends on the latter's dielectric constant. More details in *Annex A*.

When the externally-applied electric field between the plates is greater than the dielectric strength of the dielectric material, dielectric breakdown occurs.

> *Dielectric breakdown* is the failure of a dielectric to withstand the electric field between the two plates.

When dielectric breakdown occurs, there will be permanent physical and/or chemical changes. The capacitor can no longer store charge as it will have a permanent conducting path etched within.



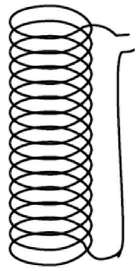
A permanent path (in black) etched into the solid dielectric after dielectric breakdown has occurred

Hence, each practical capacitor will have a *maximum p.d.* that it can be operated under, this limiting the maximum amount of charge that it can store.

5.2 Inductors

An inductor is a conductor that is wound up into a large coil. The most common form of inductors come in the form of solenoids and toroids.

An assortment of inductors



A Solenoid

A Toroid

Like a capacitor, inductors also store energy and are used in circuits to control the time-variation of current through a circuit or the potential difference across other components

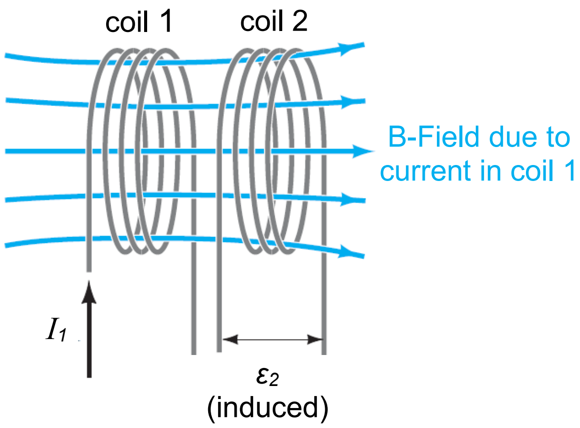
The circuit symbol of a inductor is .

5.2.1 Inductance of an Inductor

Mutual Inductance

We build on our knowledge from Electromagnetic Induction: *mutual induction* is the phenomenon in which an e.m.f. is induced in an inductor when the current in a neighbouring inductor changes.

Consider 2 coils placed next to each other with their axis aligned. When a current I_1 flows through coil 1, it produces a magnetic field that links with coil 2.



As I_1 changes, the magnetic flux linkage through coil 2 changes and in accordance to Faraday's Law, an e.m.f. ε_2 is induced in coil 2. This induced e.m.f. is proportional to the rate of change of current I_1.

$$\varepsilon_2 \propto \frac{dI_1}{dt}$$

$$\varepsilon_2 = M\frac{dI_1}{dt}$$

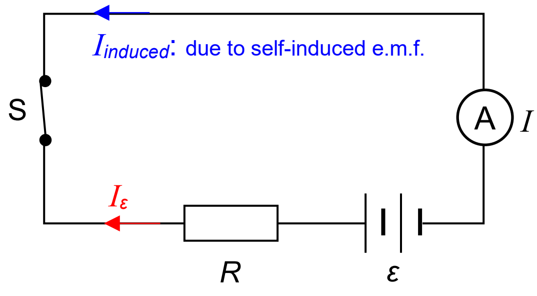
Mutual Inductance between two inductors is the tendency of one inductor to oppose a change in the current in the other circuit / component.

where *M* is the proportionality constant called the mutual inductance between the 2 inductors.

Self-Inductance

Self-induction is when an e.m.f. is induced in an inductor when the current flowing in it (i.e. the same inductor) changes.

Consider a circuit consisting of a switch S, a resistor R and an e.m.f. source ε. At the instant switch S is closed, the current in the circuit increases. This increase in current produces a changing magnetic flux that links the circuit, which in turn leads to a self-induced e.m.f. in the circuit.



At instant when switch S is closed

By Lenz's Law, the direction of this <u>*self-induced e.m.f.*</u> *will establish a magnetic field that opposes the original change in magnetic flux linkage through it*, i.e. opposite to the direction of the e.m.f. of the battery. This explains why the current in the circuit *does not instantaneously jump from zero* to I but, rather increases gradually (albeit in a short time interval) from zero to I.

This self-induced e.m.f. is proportional to the rate of change of current I.

$$\varepsilon \propto \frac{dI}{dt}$$

$$\varepsilon = -L\frac{dI}{dt}$$

where L is the proportionality constant called the inductance of the inductor, and the negative sign a reflection of the induced e.m.f. ε opposing the change in current I.

Unit for self-inductance is the henry (1 H).

Self-Inductance is the
ratio of self-induced e.m.f. across inductor to the rate of change of current flowing through inductor.

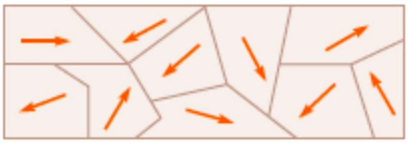
$$L = -\frac{\varepsilon}{\left(\frac{dI}{dt}\right)}$$

One *henry* is the
inductance when an e.m.f. of one volt is induced across an inductor
as the current through the inductor is changing at the rate of one ampere per second.

Inductors typically range from 1 μH to 20 H. Like capacitance, inductance depends on *geometric* (e.g. number of turns and the radius of the coil) and *dielectric* material. See **Annex B** for more info.

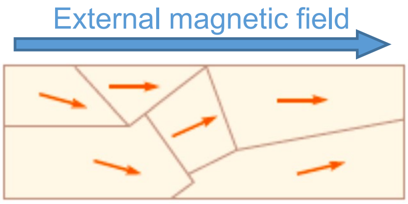
5.2.2 Effects of Ferromagnetic Materials

Ferromagnetic materials consist of regions called *domains*. Within each domain, the magnetic field due to the orbiting electrons of atoms in it are aligned. However, the domains in an unmagnetized ferromagnetic material are randomly oriented and it does not produce a resultant magnetic field.



Randomly oriented domains

When a ferromagnetic material is placed within a magnetic field, the domains will align with its direction and as such the material is able to produce a magnetic field in the same direction as the external magnetic field.



Aligned domains

Hence the magnetic field will now be stronger as it is now the sum of the original external magnetic field and that produced from the alignment of the domains.

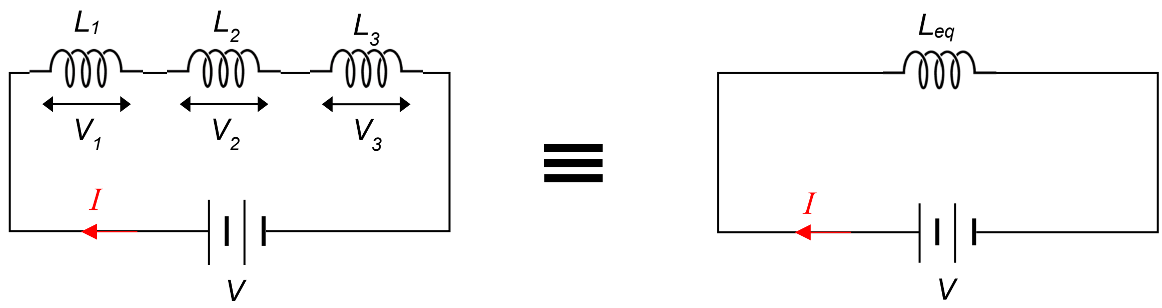
A stronger external magnetic field will lead to larger enhancement of the resultant magnetic field. However, this increase is non-linear; as the domains get more aligned with a stronger external magnetic field, *it starts to approach saturation* (a phenomenon where the domains are aligned perfectly with the field) and thus the limit to higher magnetization of the ferromagnetic material.

Hence inserting a ferromagnetic core into an inductor has the effect of increasing the magnetic field, which leads to an increase in flux linkage. This results a larger self-induced e.m.f. for the same rate of charge of current, thereby increasing the inductance of the inductor. See **Annex B** for more info.

5.2.3 Combination of Inductors

Inductors in Series

Three inductors of inductance L_1, L_2 and L_3 in series are connected with a battery.



Since the <u>same current flowing through the inductors</u>: $I = I_1 = I_2 = I_3 \Rightarrow \frac{dI}{dt} = \frac{dI_1}{dt} = \frac{dI_2}{dt} = \frac{dI_3}{dt}$

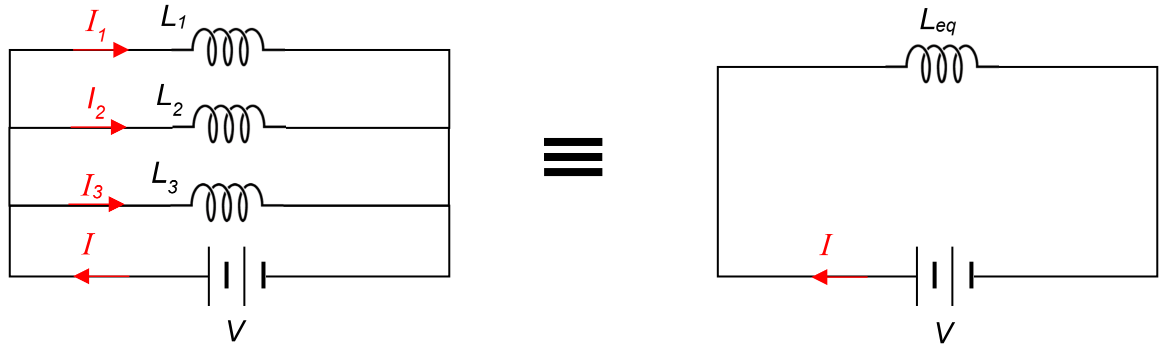
$$V = V_1 + V_2 + V_3$$
$$\left(L_{eq}\frac{dI}{dt}\right) = L_1\frac{dI_1}{dt} + L_2\frac{dI_2}{dt} + L_3\frac{dI_3}{dt}$$
$$\therefore \quad L_{eq} = L_1 + L_2 + L_3$$

Therefore, the equivalent inductance of N inductors in series: $\boxed{L = L_1 + L_2 + ... + L_N}$

Similar to resistors in series, the equivalent inductance will be larger than the inductance of each individual inductor in series.

Inductors in Parallel

Three inductors of inductance L_1, L_2 and L_3 in parallel are connected with a battery,.



All <u>inductors in parallel have the same potential difference *V*</u>.

$$I = I_1 + I_2 + I_3$$

$$\therefore \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt} + \frac{dI_3}{dt}$$

$$\frac{V}{L_{eq}} = \frac{V}{L_1} + \frac{V}{L_2} + \frac{V}{L_3}$$

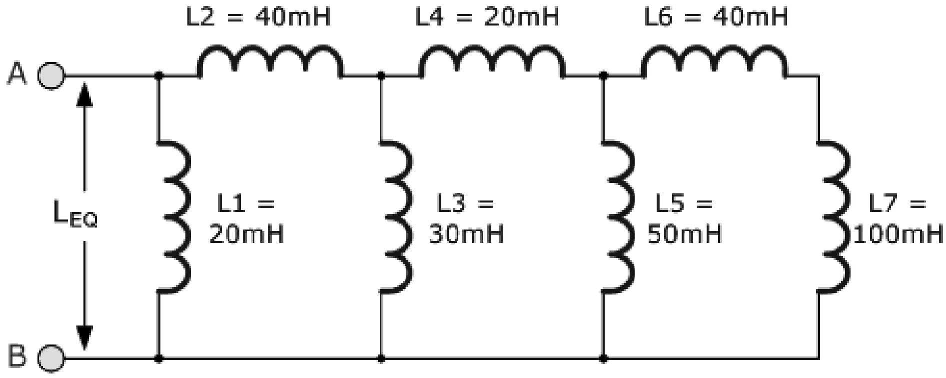
$$\therefore \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Therefore, the equivalent inductance of N inductors in parallel: $\boxed{\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots\ldots + \frac{1}{L_N}}$

Similar to resistors in series, the equivalent inductance will be smaller than the inductance of each individual inductor in parallel.

Example 8

Calculate the equivalent inductance of the following inductive circuit.



$$\frac{1}{L_{eq}} = \frac{1}{\dfrac{1}{\dfrac{1}{\dfrac{1}{\dfrac{1}{50}+\dfrac{1}{100+40}}+20}+\dfrac{1}{30}}+40} + \frac{1}{20}$$

$$\therefore \quad L_{eq} = 15 \text{ mH}$$

5.2.4 Energy stored in an Inductor

Like a capacitor, an inductor also stores energy.

Let's derive an expression for the energy stored in an inductor. For an inductor of inductance *L*, the potential difference induced in the inductor is

$$\varepsilon = L\frac{dI}{dt},$$

where $\frac{dI}{dt}$ is the rate of change of the current I in the inductor. The work done by the battery, *dW*, to move an small charge *dq* across the inductor is given by

$$dW = \varepsilon dq = L\frac{dI}{dt}dq \underbrace{=}_{I=\frac{dq}{dt}} LIdI\,.$$

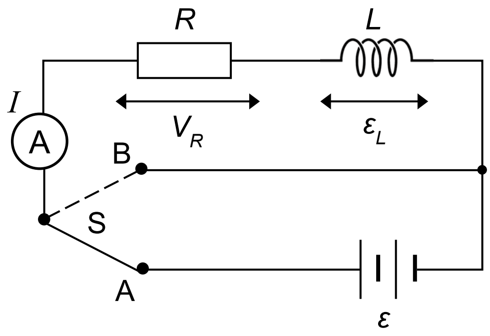
For the current to increase from 0 to its steady-state value, the total work done *W* is then

$$\begin{aligned} W &= \int_0^I LIdI \\ &= \frac{1}{2}LI^2 \end{aligned}$$

Hence the energy stored in an inductor $\boxed{U = \frac{1}{2}LI^2}$.

5.3 Circuits with Capacitors and Inductors

5.3.1 *RL* Circuits



RL circuits are often used as timers to turn on devices at particular intervals and can also be used to filter out noise in electrical signals. The inductor can act as a "surge protector" for sensitive electronic equipment which can be easily damaged by high currents.

The most basic circuit analysis involving inductors is the *RL* circuit. Apart from the resistance of other components in the circuit, *R* can also include the resistance of the inductor as well.

Connecting to a constant e.m.f. source

At the instant when switch S is connected to A, $t = 0$:

i. the current in the circuit $I = 0$

ii. the potential difference across the resistor $V_R = RI = 0$

After the switch S is closed for some time t:

i. the current in the circuit I

ii. the potential difference across the resistor $V_R = RI$

iii. the induced e.m.f. across the inductor $\varepsilon_L = -L\dfrac{dI}{dt}$

Since

$$V_R = \varepsilon + \varepsilon_L$$
$$RI = \varepsilon - L\frac{dI}{dt}$$
$$L\frac{dI}{dt} = \varepsilon - RI$$
$$\frac{1}{\varepsilon - RI}dI = \frac{1}{L}dt$$

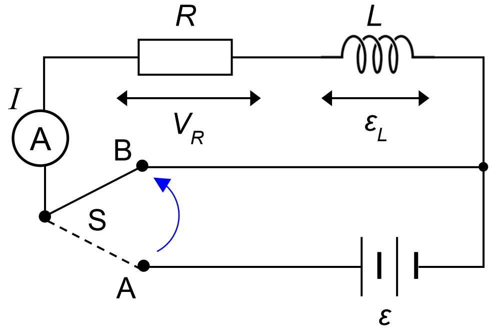
Integrating:

$$\int_0^I \frac{1}{\varepsilon - RI}dI = \frac{1}{L}\int_0^t dt$$
$$-\frac{1}{R}\left[\ln(\varepsilon - RI)\right]_0^I = \left[\frac{t}{L}\right]_0^t$$
$$\ln(\varepsilon - RI) - \ln\varepsilon = -\frac{R}{L}t$$
$$\ln\frac{\varepsilon - RI}{\varepsilon} = -\frac{R}{L}t$$
$$1 - \frac{R}{\varepsilon}I = e^{-\frac{R}{L}t}$$
$$I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$
$$\boxed{I = I_{\max}\left(1 - e^{-\frac{R}{L}t}\right)}$$

$$\boxed{\text{Hence} \quad V_R = RI = R\frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right) = \varepsilon\left(1 - e^{-\frac{R}{L}t}\right)}$$

When connecting a *RL* circuit to a constant e.m.f. source, the time constant of a *RL* circuit $\boxed{\tau = \frac{L}{R}}$ is the time for the current in the circuit (and the potential difference across the resistor) to increase to 63.2 % of its maximum value.

Disconnecting from an e.m.f. source



At the instant when switch S is moved from A to B, t = 0:

i. the current in the circuit $I = I_{max}$

ii. the potential difference across the capacitor $V_R = \varepsilon$

iii. the induced e.m.f. across the inductor $\varepsilon_L = 0$

After the switch S is at B for some time t:

i. the current in the circuit I

ii. the potential difference across the capacitor $V_R = RI$

iii. the induced e.m.f. across the inductor $\varepsilon_L = -L\dfrac{dI}{dt}$

Since
$$V_R = \varepsilon_L$$
$$RI = -L\frac{dI}{dt}$$
$$\frac{1}{RI}dI = -\frac{1}{L}dt$$

Integrating:
$$\int_{I_{max}}^{I} \frac{1}{RI}dI = -\frac{1}{L}\int_0^t dt$$
$$\frac{1}{R}\left[\ln(RI)\right]_{I_{max}}^{I} = -\frac{t}{L}$$
$$\ln(RI) - \ln(RI_{max}) = -\frac{R}{L}t$$
$$\ln\left(\frac{RI}{RI_{max}}\right) = -\frac{R}{L}t$$
$$\boxed{I = I_{max}e^{-\frac{R}{L}t}}$$

Hence $$V_R = RI = R\left(\frac{\varepsilon}{R}e^{-\frac{R}{L}t}\right) = \varepsilon e^{-\frac{R}{L}t}$$

When disconnecting a *RL* circuit from a constant e.m.f. source, the time constant of a *RL* circuit $\boxed{\tau = \frac{L}{R}}$ is the time for the current in the circuit (and the potential difference across the resistor) to decrease to 36.8 % of its maximum value

A *RL* circuit has similar properties to a *RC* circuit in terms of their variation of the circuit current with time. However, the time constant of a *RL* circuit is inversely proportional to *R* while the time constant of a *RC* circuit is directly proportional to *R*.

Example 9

A series *RL* circuit consisting of a 50 Ω resistor and a 10 H inductor is connected to a constant voltage supply of 100 V at t = 0 s.

(a) Derive the equation for the current I flowing in the circuit,

Since
$$V_R = \varepsilon + \varepsilon_L$$
$$RI = \varepsilon - L\frac{dI}{dt}$$
$$L\frac{dI}{dt} = \varepsilon - RI$$
$$\frac{1}{\varepsilon - RI}dI = \frac{1}{L}dt$$

Integrating:
$$\int_0^I \frac{1}{\varepsilon - RI}dI = \frac{1}{L}\int_0^t dt$$
$$-\frac{1}{R}\left[\ln(\varepsilon - RI)\right]_0^I = \left[\frac{t}{L}\right]_0^t$$
$$\ln(\varepsilon - RI) - \ln\varepsilon = -\frac{R}{L}t$$
$$\ln\frac{\varepsilon - RI}{\varepsilon} = -\frac{R}{L}t$$
$$1 - \frac{R}{\varepsilon}I = e^{-\frac{R}{L}t}$$
$$I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$
$$= \frac{100}{50}\left(1 - e^{-\frac{50}{10}t}\right)$$
$$= 2\left(1 - e^{-5t}\right)$$

(b) Determine

i. the current at t = 0.5 s,
$$I_{t=0.5\text{ s}} = 2\left(1 - e^{-5(0.5)}\right) = 1.84 \text{ A}$$

ii. the expression for the potential difference across the resistor V_R,
$$V_R = RI$$
$$= (50)\left[2\left(1 - e^{-5t}\right)\right]$$
$$= 100\left(1 - e^{-5t}\right)$$

iii. the expression for the potential difference across the inductor V_L
$$V_L = \varepsilon - V_R$$
$$= 100 - 100\left(1 - e^{-5t}\right)$$
$$= 100e^{-5t}$$

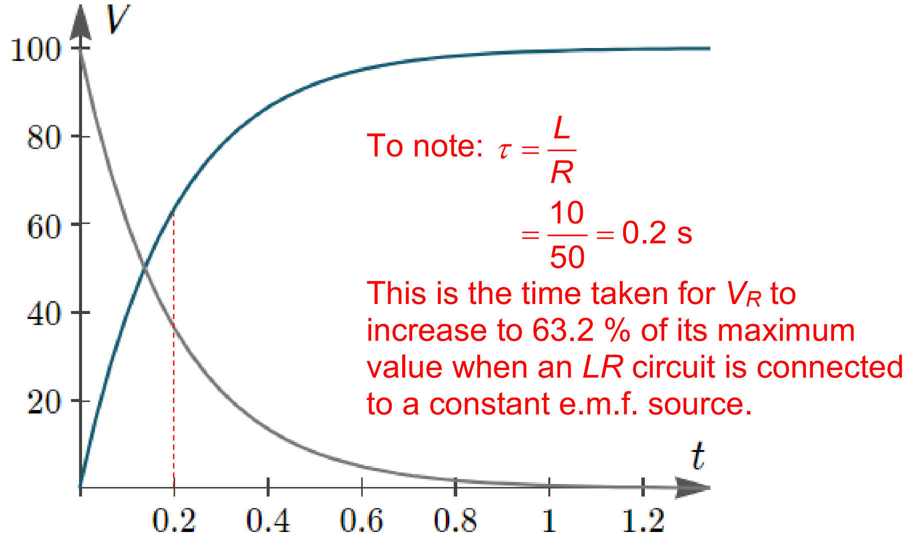
iv. the time which $V_R = V_L$
$$V_R = V_L$$
$$100\left(1 - e^{-5t}\right) = 100e^{-5t}$$
$$2e^{-5t} = 1$$
$$t = 0.138 \text{ s}$$

For a general case:
$$V_L = V_R$$
$$\varepsilon\left(1 - e^{-\frac{R}{L}t}\right) = \varepsilon e^{-\frac{R}{L}t}$$
$$2e^{-\frac{R}{L}t} = 1$$
$$t = \frac{R}{L}\ln 2$$

(c) On the axis below, sketch the graph showing the variation of

i. V_R,

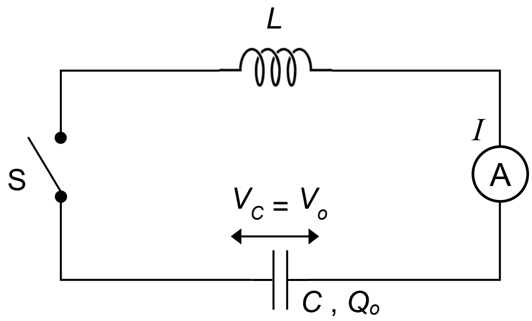
ii. V_L



5.3.2 *LC* Circuits

In the earlier sections, the two-element combinations of *RC* and *RL* have been analysed. In both types of circuits, it was determined that the current and potential difference either grows or decays exponentially and the time scale is given by a time constant τ .

The last two-element combination is that of *LC*. In such a circuit where both components store energy, if the circuit has energy to begin with (i.e. a charged capacitor), it would be expected that the energy be transferred back and forth between the two components. Hence the subsequent analysis will be taken from an energy perspective.



The capacitor has an initial charge Q_o.

When switch S is closed, t = 0.

i. the energy stored in the capacitor $U_C = \frac{1}{2}\frac{Q_o^2}{C}$

ii. the energy stored in the inductor $U_L = 0$

After the switch S is closed for some time t:

i. the energy stored in the capacitor $U_C = \frac{1}{2}\frac{Q^2}{C}$

ii. the energy stored in the inductor $U_L = \frac{1}{2}LI^2$

The total energy at any instant $U_T = U_C + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2$

Assuming there are no energy losses: $\frac{dU_T}{dt} = 0$

Hence,

$$\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2\right) = 0$$

$$\frac{Q}{C}\frac{dQ}{dt} + L\left(\frac{dQ}{dt}\right)\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$$

Solving: Comparing to the defining equation for a simple harmonic motion:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

The general solution is

$$Q = Q_o \sin(\omega t + \phi)$$

where $\omega = \sqrt{\frac{1}{LC}}$

and ϕ is a phase constant which is dependent on the initial conditions

Initial condition: At $t = 0$, Q must be equal to Q_o

Hence $Q_o = Q_o \sin(0 + \phi) \Rightarrow \phi = \frac{\pi}{2}$

Thus $Q = Q_o \cos \omega t$

The current in the circuit

$$I = \frac{dQ}{dt} = \frac{d}{dt}(Q_o \cos \omega t) = \omega Q_o \sin \omega t \qquad \Rightarrow I_o = \omega Q_o = \frac{1}{\sqrt{LC}}Q_o$$

As shown, the current *and* charge vary sinusoidally and the resulting oscillations of the capacitor's electric field and the inductor's magnetic field are said to be electromagnetic oscillations. Hence *LC* circuits are often termed electrical oscillators, akin to an electrical tuning fork (e.g. for tuning radio transmitters and receivers).

Energy in the Circuit

i. The electric energy stored in the capacitor $U_C = \frac{1}{2}\frac{Q_o^2}{C}\cos^2 \omega t$

ii. The magnetic energy stored in the inductor $U_L = \frac{1}{2}L\omega^2 Q_o^2 \sin^2 \omega t$

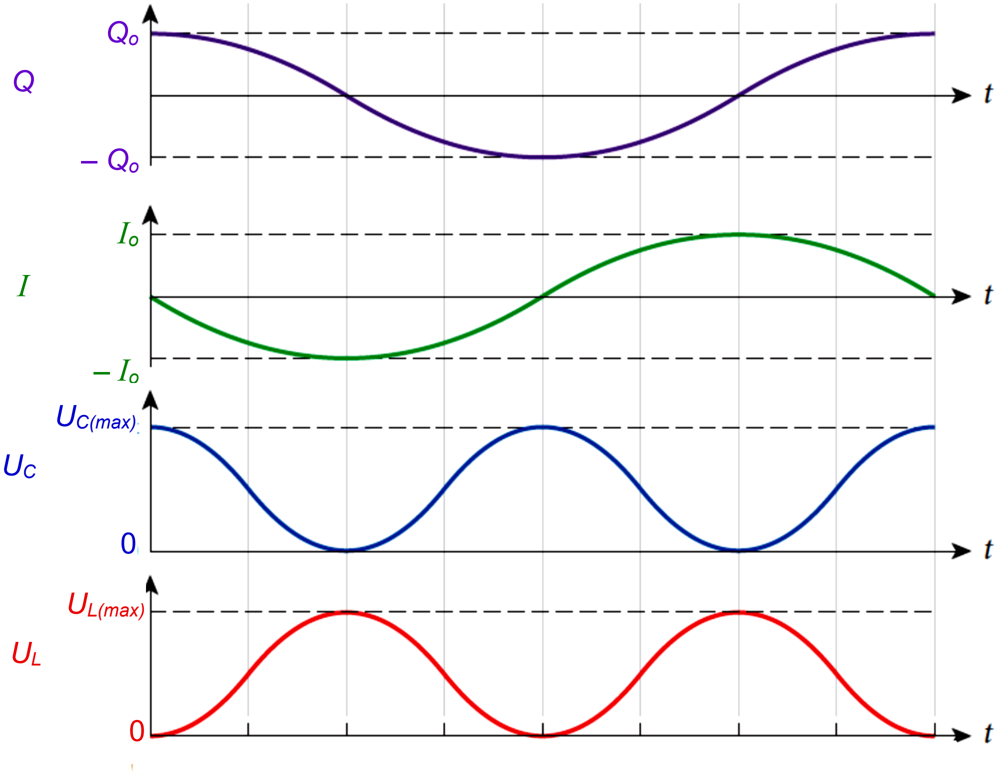
iii. The total energy in the circuit $U_T = U_C + U_L$

$$= \frac{1}{2}\frac{Q_o^2}{C}\cos^2 \omega t + \frac{1}{2}L\omega^2 Q_o^2 \sin^2 \omega t$$

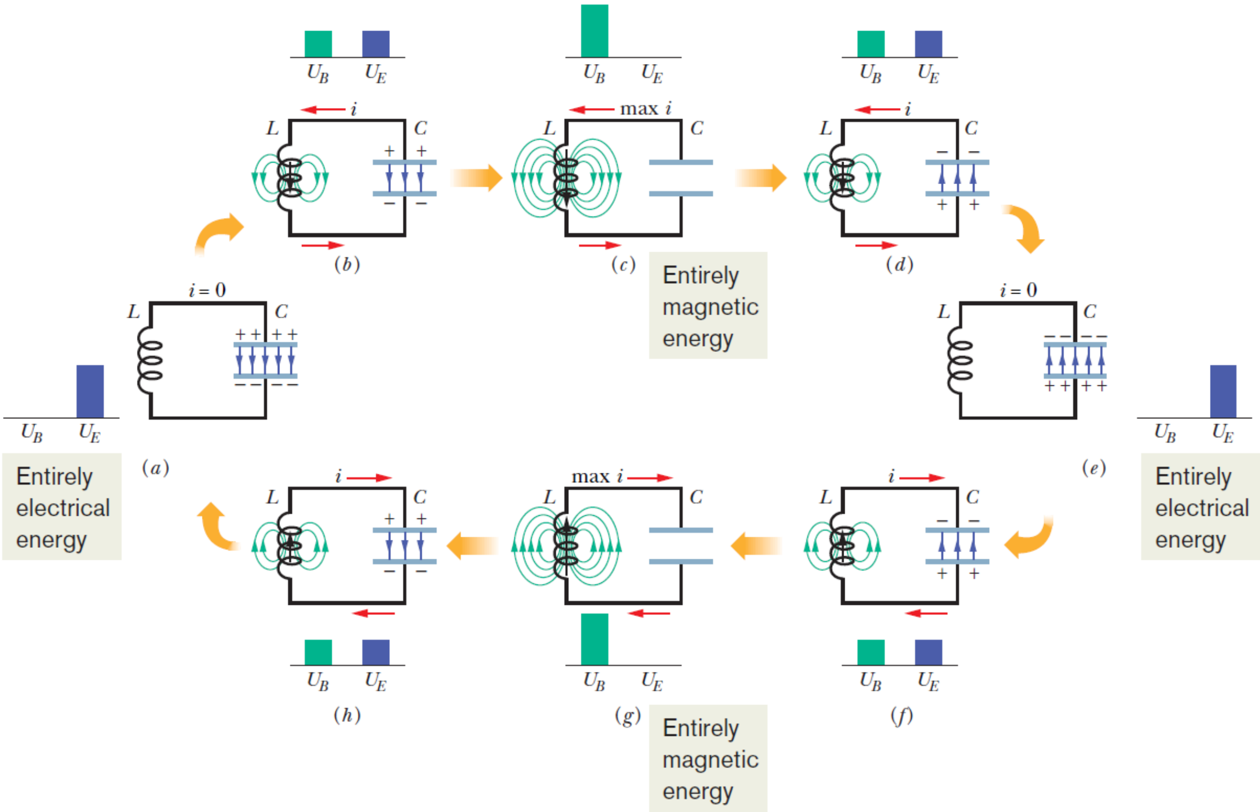
$$= \frac{1}{2}\frac{Q_o^2}{C}\cos^2 \omega t + \frac{1}{2}L\left(\frac{1}{\sqrt{LC}}\right)^2 Q_o^2 \sin^2 \omega t$$

$$= \frac{1}{2}\frac{Q_o^2}{C}\cos^2 \omega t + \frac{1}{2}\frac{Q_o^2}{C}\sin^2 \omega t$$

$$= \frac{1}{2}\frac{Q_o^2}{C} \quad = \text{ the initial energy stored in the capacitor}$$



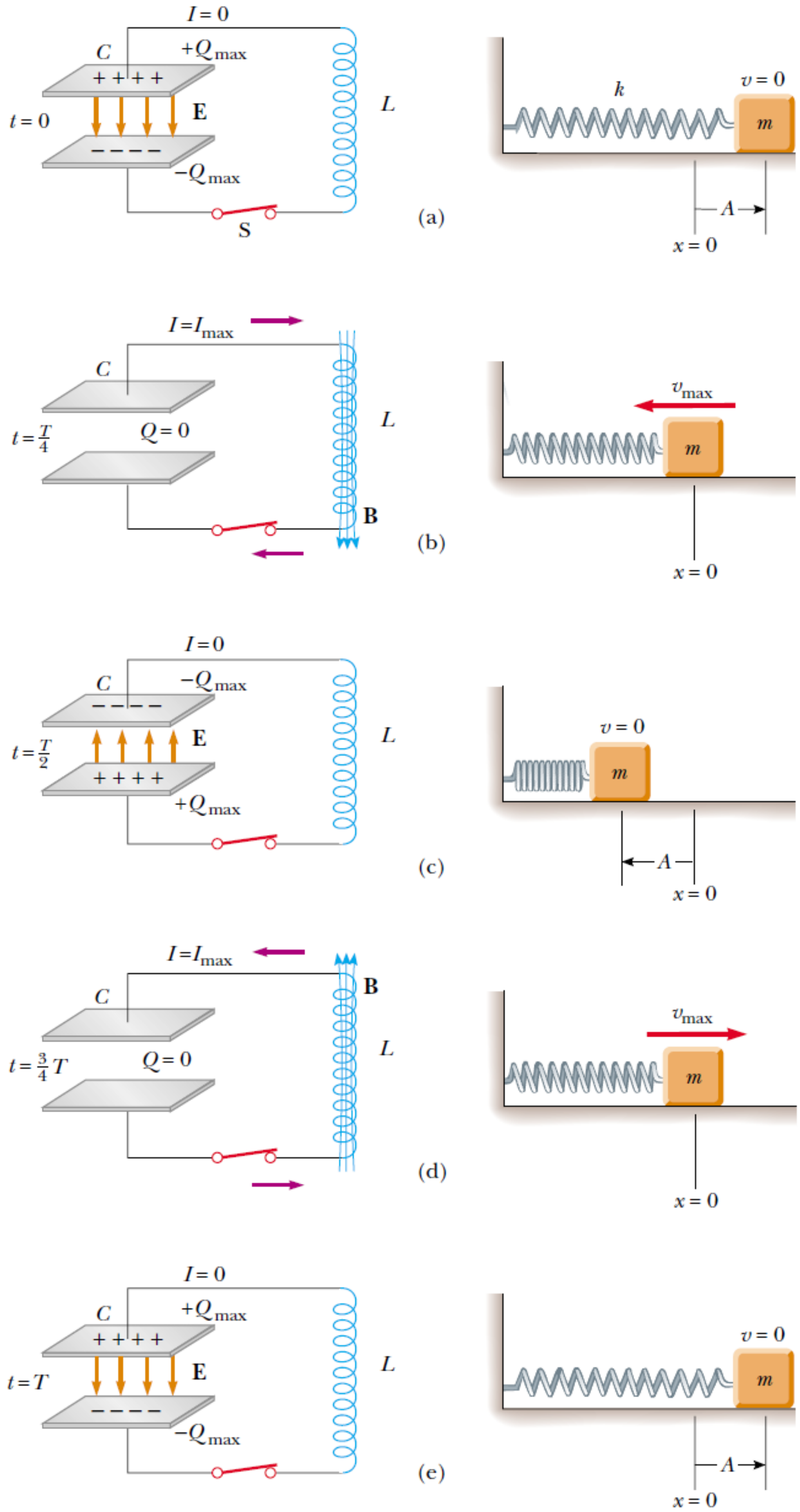
Graphs showing the variation of various quantities with time in a *LC* circuit



The energy conversion cycle of a *LC* circuit

Due to the simple harmonic nature of a *LC* circuit, it is often compared to a mechanical oscillator in S.H.M. such as a mass-spring system. As such

1. the energy stored in the capacitor is analogous to the potential energy of a mechanical oscillator
2. the energy stored in the inductor is analogous to the kinetic energy of a mechanical oscillator



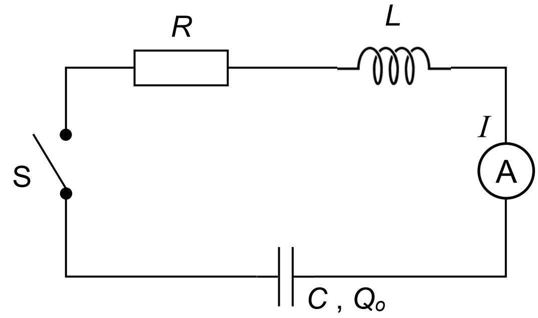
A mechanical oscillator analogy of an electrical oscillator

5.3.3 *RLC* Circuits

The analysis for *LC* circuits in the preceding section assumes the ideal situation where the circuit has zero resistance and energy is not radiated away from the circuit via EM radiation, both of which are rarely the case. When the resistance in the circuit is to be considered, it is analysed as an *RLC* circuit.

The key difference in the analysis of an *RLC* circuit is that the total energy decreases as dissipative losses in the resistor is factored in.

The capacitor has an initial charge Q_o.



When switch S is closed, $t = 0$.

i. the energy stored in the capacitor $U_C = \frac{1}{2}\frac{Q_o^2}{C}$

ii. the energy stored in the inductor $U_L = 0$

After the switch S is closed for some time *t*:

i. the energy stored in the capacitor $U_C = \frac{1}{2}\frac{Q^2}{C}$

ii. the energy stored in the inductor $U_L = \frac{1}{2}LI^2$

With the energy loss in the resistor: $\frac{dU_T}{dt} = -I^2R$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2\right) = -I^2R$$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2\right) = -\left(\frac{dQ}{dt}\right)^2 R$$

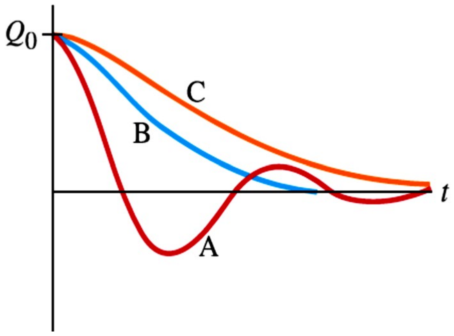
$$\frac{Q}{C}\left(\frac{dQ}{dt}\right) + L\left(\frac{dQ}{dt}\right)\frac{d^2Q}{dt^2} = -\left(\frac{dQ}{dt}\right)^2 R$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

This is a homogenous 2nd order differential equation.

The first step in solving this equation for the function of $Q(t)$ is to determine its solution of its auxiliary equation, which will indicate the form of the general solution. The exact solution can be then determined based on the initial conditions of the circuit. The method is given in **Annex C**. For the rest of this section, the focus will be on the results.

Depending on the values of R, L and C in the circuit used, the function of Q with time will take one of the 3 variations as shown.



Curve A: Light damping which occurs when $R^2 < \frac{4L}{C}$

Curve B: Critical damping which occurs when $R^2 = \frac{4L}{C}$

Curve C: Heavy damping which occurs when $R^2 > \frac{4L}{C}$

As expected for all 3 scenarios, Q must approach zero as t approach infinity as energy is constantly dissipated via the resistor.

In the case where $R^2 << \frac{4L}{C}$, curve A essentially approaches an undamped situation and the curve will approach that obtained for a LC circuit.

Electric Circuit		**One-Dimensional Mechanical System**
Charge	$Q \leftrightarrow x$	Position
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving object
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped object on a spring

Analogous quantities between electrical and mechanical systems

Annex A: Calculating Capacitance

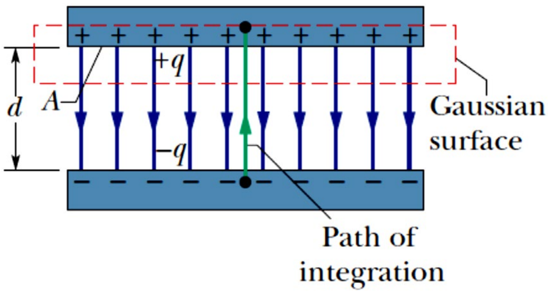
To determine the expression for the capacitance C of a capacitor based on its geometry, this general approach will be taken:

1. Assume a charge q on the conductors.
2. Determine an expression the electric field $\vec{E}$ between the conductors in terms of this charge using Gauss' law: $\varepsilon_o \oint \vec{E} \cdot d\vec{A} = q$.
3. Use the expression for $\vec{E}$ to determine the expression for the potential difference V between the plates using $V = -\int_i^f \vec{E} \cdot d\vec{s}$ (from the relationship $\vec{E} = -\dfrac{dV}{d\vec{s}}$).

   If evaluate the integral along a path that follows an electric field line from the negative plate to the positive plate, the vectors $\vec{E}$ and $d\vec{s}$ will always be in opposite direction. Then the dot product $\vec{E} \cdot d\vec{s}$ will always be negative and the expression can be rewritten as $V = \int_-^+ E \cdot ds$
4. Determine the expression for C using $C = \dfrac{Q}{V}$

Parallel Plate Capacitor

Assuming that the parallel plates are large and that the separation d between the plates are small that the fringing effects of the electric field at the edges of the plate may be ignored and $\vec{E}$ between the plates is constant.



Considering a Gaussian surface that just encloses the charge q on the positive plate:

$$q = \varepsilon_o \oint \vec{E} \cdot d\vec{A}$$
$$= \varepsilon_o EA$$, where A is the area of the plate.

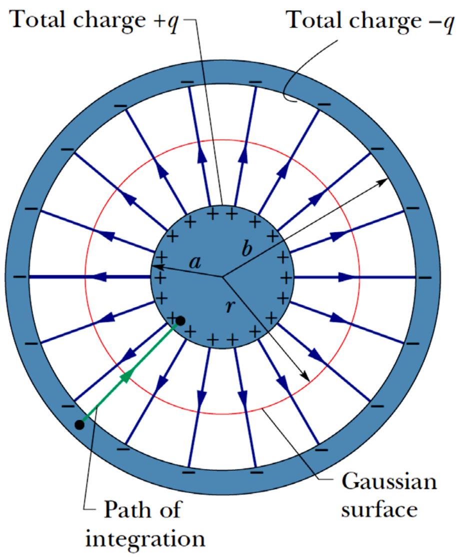
$$V = \int_-^+ E \cdot ds$$
$$= E\int_-^+ ds$$
$$= Ed$$, where d is the separation of the plates

Since $C = \dfrac{q}{V}$

$$= \frac{\varepsilon_o EA}{Ed}$$
$$= \frac{\varepsilon_o A}{d}$$

Cylindrical Capacitor

A cylindrical capacitor consists of 2 co-axial cylinders; an inner solid cylinder of radius a and an outer hollow cylinder of radius b.



Assuming that the length of the cylinder L is significantly greater than the radius b of the outer cylinder. This is so that the fringing effects of the electric field at the ends of the cylinder may be ignored.

Considering a Gaussian surface of a cylinder length L and radius r, closed by end caps. It is co-axial with the cylinders and encloses the charge q on the central cylinder:

$$\begin{aligned} q &= \varepsilon_o \oint \vec{E} \cdot d\vec{A} \\ &= \varepsilon_o EA \\ &= \varepsilon_o E(2\pi rL) \quad \text{(since there is no flux through the end caps)} \end{aligned}$$

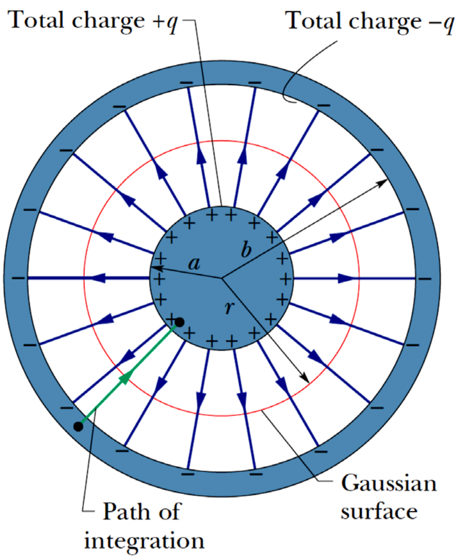
$$\begin{aligned} V &= \int_{-}^{+} E \cdot ds \\ &= -\int_b^a \frac{q}{\varepsilon_o (2\pi rL)} dr \qquad (ds = -dr \text{ as we integrated radially inwards}) \\ &= -\frac{q}{2\pi\varepsilon_o L} \int_b^a \frac{1}{r} dr \\ &= \frac{q}{2\pi\varepsilon_o L} \ln\left(\frac{b}{a}\right) \end{aligned}$$

Since $C = \dfrac{q}{V}$

$$\begin{aligned} &= \frac{q}{\frac{q}{2\pi\varepsilon_o L} \ln\left(\frac{b}{a}\right)} \\ &= 2\pi\varepsilon_o \frac{L}{\ln(b/a)} \end{aligned}$$

Spherical Capacitor

A spherical capacitor consists of 2 concentric spheres; an inner solid sphere of radius *a* and an outer hollow shell of radius *b*.



Considering a Gaussian surface of a sphere of radius *r*, concentric with the 2 shells and encloses the charge *q* on the inner shell:

$$
\begin{aligned}
q &= \varepsilon_o \oint \vec{E} \cdot d\vec{A} \\
&= \varepsilon_o E A \\
&= \varepsilon_o E \left(4\pi r^2\right)
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_{-}^{+} E \cdot ds \\
&= -\int_b^a \frac{q}{4\pi\varepsilon_o r^2} dr \\
&= -\frac{q}{4\pi\varepsilon_o} \int_b^a \frac{1}{r^2} dr \\
&= \frac{q}{4\pi\varepsilon_o} \left(\frac{1}{a} - \frac{1}{b}\right)
\end{aligned}
$$

Since $C = \dfrac{q}{V}$

$$
\begin{aligned}
&= \frac{q}{\dfrac{q}{4\pi\varepsilon_o}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)} \\
&= 4\pi\varepsilon_o \frac{ab}{b-a}
\end{aligned}
$$

Isolated Sphere

A single isolated sphere has capacitance as well; it can be considered as a case of a spherical conductor whose outer shell is of infinite radius and an inner sphere of radius *R*.

The expression above then becomes:

$$
\begin{aligned}
C &= \frac{q}{\dfrac{q}{4\pi\varepsilon_o}\left(\dfrac{1}{R} - \dfrac{1}{\infty}\right)} \\
&= 4\pi\varepsilon_o R
\end{aligned}
$$

Enhancement by a Dielectric Material

The dielectric constant *K* is the factor by which a dielectric material increases the capacitance of a capacitor as compared to the situation where the gap between the conductors is a vacuum. This is achieved via the modification of the permittivity of the space occupied by the electric field.

The permittivity of the dielectric material $\boxed{\varepsilon = K\varepsilon_o}$

Dielectric constants and dielectric strengths of typical materials

Material	**Dielectric Constant (at 20 °C)**	**Dielectric Strength (V / m)**
Vacuum	1.0000	
Air (1 atm)	1.0006	3×10^6
Paper	3.7	15×10^6
Vinyl (plastic)	2 to 4	50×10^6
Mica	7	150×10^6

Annex B: Calculating Inductance

Long Solenoid

Consider a long solenoid of length x, cross sectional area A and consisting of N turns.

The current I it carries creates a magnetic field inside the solenoid. The magnetic flux density within the solenoid is given by

$B = \mu_0 n I$, where n is the number of turns per unit length.

If the current in the solenoid varies with time, the flux linkage changes and induces an e.m.f. in the solenoid.

From Faraday's law, the self-induced e.m.f.

$$\begin{aligned}\varepsilon_{induced} &= -\frac{d\left(N\Phi_B\right)}{dt}\\ &= -nx\frac{d\Phi_B}{dt}\\ &= -nxA\frac{d}{dt}\left(\mu_0 n I\right) \quad \left(\because \Phi_B = BA\cos\theta \text{ and } \theta = 0°\right)\\ &= -\mu_0 A n^2 x\frac{dI}{dt}\end{aligned}$$

Hence the inductance of a long solenoid is

$$\begin{aligned}L &= \mu_0 A n^2 x\\ &= \frac{\mu_0 A N^2}{x}\end{aligned}$$

Enhancement by a Ferromagnetic Material

The permeability of the material μ, is related to the permeability of free space μ_0 by the equation

$$\mu = \mu_0\left(1 + \chi_m\right)$$

where χ_m is the magnetic susceptibility of the material. For ferromagnets, magnetic susceptibility is positive and large and thus μ is much greater than μ_0. For all other materials, the value is very close to μ_0.

Hence inserting a ferromagnetic core into an inductor has the effect of increasing its inductance through the enhancement of the permeability of the space occupied by the magnetic field.

Annex C: Solution for solving the homogenous 2nd ODE of a *RLC* Circuit

For a 2nd ODE in the of the form: $am''+bm'+c=0$

The associated auxiliary equation is $am^2+bm+c=0$

with solution given by $m=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

The form of the general solution will depend on the nature of the root of the auxiliary equation:

Condition	**Nature of roots**	**General Solution**
$b^2>4ac$	Real and distinct roots: m_1 and m_2	$y=Ae^{m_1x}+Be^{m_2x}$
$b^2=4ac$	Real and equal roots: m	$y=\left(A+Bx\right)e^{mx}$
$b^2<4ac$	Complex roots: $m_1=-\dfrac{b}{2a}+\left(\dfrac{\sqrt{4ac-b^2}}{2a}\right)j$ $m_2=-\dfrac{b}{2a}-\left(\dfrac{\sqrt{4ac-b^2}}{2a}\right)j$	$y=e^{\alpha x}(A\cos\omega x+B\sin\omega x)$ where $\alpha=-\dfrac{b}{2a}$ $\omega=\dfrac{\sqrt{4ac-b^2}}{2a}$

Hence for the differential equation: $L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=0$

Condition	**General Solution**
$R^2>\dfrac{4L}{C}$	$Q=Ae^{\frac{-R+\sqrt{R^2-4\frac{L}{C}}}{2L}t}+Be^{\frac{-R-\sqrt{R^2-4\frac{L}{C}}}{2L}t}$
$R^2=\dfrac{4L}{C}$	$Q=\left(A+Bt\right)e^{-\frac{R}{2L}t}$
$R^2<\dfrac{4L}{C}$	$Q=e^{-\frac{R}{2L}t}\left(A\cos\dfrac{\sqrt{\frac{4L}{C}-R^2}}{2L}t+B\sin\dfrac{\sqrt{\frac{4L}{C}-R^2}}{2L}t\right)$ $=e^{-\frac{R}{2L}t}\left(A\cos\sqrt{\dfrac{1}{LC}-\left(\dfrac{R}{2L}\right)^2}t+B\sin\sqrt{\dfrac{1}{LC}-\left(\dfrac{R}{2L}\right)^2}t\right)$

* Constants A and B are to be determined based on the initial conditions of the circuit.