RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 2B: Vectors II – Equations of Straight Lines

Section A (Basic Questions)

1 For each of the following, write down a vector equation of the line l and convert it to Cartesian form.

(a) l passes through the point with position vector $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and is parallel to the vector $\mathbf{i}+\mathbf{j}$.

(b) l passes through the points $P(1,-1,3)$ and $Q(2,1,-2)$.

(c) l passes through the origin O and is parallel to the line $m:\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix},\quad \lambda\in\mathbb{R}$.

(d) l passes through the point $C(1,0,1)$ and is parallel to the $y-$axis.

Solution

(a) $l:\mathbf{r}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix},\quad \lambda\in\mathbb{R}$

Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

Then $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$

$x=-1+\lambda \quad\Rightarrow\quad \lambda=x+1$

$y=2+\lambda \quad\Rightarrow\quad \lambda=y-2$

$z=1$

Cartesian equation is $x+1=y-2,\quad z=1$

(b) $\overrightarrow{OP}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\overrightarrow{OQ}=\begin{pmatrix}2\\1\\-2\end{pmatrix}$. $\qquad \overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}1\\2\\-5\end{pmatrix}$

$l:\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-5\end{pmatrix},\quad \lambda\in\mathbb{R}$

Then $x=1+\lambda \quad\Rightarrow\quad \lambda=x-1$

$y=-1+2\lambda \quad\Rightarrow\quad \lambda=\dfrac{y+1}{2}$

$z=3-5\lambda \quad\Rightarrow\quad \lambda=\dfrac{z-3}{-5}$

Cartesian equation is $x-1=\dfrac{y+1}{2}=\dfrac{z-3}{-5}$

(c)	$\mathbf{r} = \mu \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mu \in \mathbb{R}$ Cartesian equation is $x = \dfrac{y}{2} = \dfrac{z}{3}$
(d)	A vector parallel to the y-axis is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. $l : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ $x = 1$, $z = 1$, $y = \lambda$ Cartesian equation is $x = 1$, $z = 1$.

2 For each of the following, find the acute angle between the lines l_1 and l_2.
Determine if l_1 and l_2 are parallel, intersecting or skew.
In the case of intersecting lines, find the position vector of the point of intersection.

(a) $l_1 : x-1=-y=z-2$ and $l_2 : \dfrac{x-2}{2}=-\dfrac{y+1}{2}=\dfrac{z-3}{2}$

(b) $l_1 : \mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix}$, $\alpha\in\mathbb{R}$ and $l_2 : \mathbf{r}=\begin{pmatrix}0\\10\\1\end{pmatrix}+\beta\begin{pmatrix}3\\8\\1\end{pmatrix}$, $\beta\in\mathbb{R}$

(c) $l_1 : \mathbf{r}=(\mathbf{i}-5\mathbf{k})+\lambda(\mathbf{i}-\mathbf{j}+\mathbf{k})$, $\lambda\in\mathbb{R}$ and $l_2 : \mathbf{r}=(\mathbf{i}-\mathbf{j}+\mathbf{k})+\mu(5\mathbf{i}-4\mathbf{j}-\mathbf{k})$, $\mu\in\mathbb{R}$

[**(a)** $0°$, parallel **(b)** $81.3°$, skew **(c)** $44.5°$, intersecting, $\begin{pmatrix}6\\-5\\0\end{pmatrix}$]

Solution

(a) $l_1 : x-1=-y=z-2$ and $l_2 : \dfrac{x-2}{2}=-\dfrac{y+1}{2}=\dfrac{z-3}{2}$

$l_1 : \mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+\alpha\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\alpha\in\mathbb{R}$ $l_2 : \mathbf{r}=\begin{pmatrix}2\\-1\\3\end{pmatrix}+\beta\begin{pmatrix}2\\-2\\2\end{pmatrix}$, $\beta\in\mathbb{R}$

Since $\begin{pmatrix}1\\-1\\1\end{pmatrix}=\dfrac{1}{2}\begin{pmatrix}2\\-2\\2\end{pmatrix}$, $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ is parallel to $\begin{pmatrix}2\\-2\\2\end{pmatrix}$.

Hence l_1 and l_2 are parallel and the angle between l_1 and l_2 is 0°.
[Note that l_1 and l_2 are actually the same line]

(b) $l_1 : \mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix}$, $\alpha\in\mathbb{R}$ and $l_2 : \mathbf{r}=\begin{pmatrix}0\\10\\1\end{pmatrix}+\beta\begin{pmatrix}3\\8\\1\end{pmatrix}$, $\beta\in\mathbb{R}$

Let θ be the angle between l_1 and l_2.

$$\cos\theta=\frac{\begin{pmatrix}4\\-2\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\8\\1\end{pmatrix}}{\sqrt{16+4+9}\sqrt{9+64+1}}=\frac{12-16-3}{\sqrt{29}\sqrt{74}}=\frac{-7}{\sqrt{2146}}$$
$\Rightarrow\ \theta=98.691°$

Hence required acute angle between l_1 and l_2 is $180^\circ-98.691^\circ=81.3^\circ$ (1 d.p.)

Since $\begin{pmatrix}4\\-2\\-3\end{pmatrix}\neq m\begin{pmatrix}3\\8\\1\end{pmatrix}$ for any $m\in\mathbb{R}$, l_1 and l_2 are not parallel.

[Note that the above step of checking whether the direction vectors of l_1 and l_2 may no longer be necessary since the angle between these 2 lines is non-zero.]

Let $\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix}=\begin{pmatrix}0\\10\\1\end{pmatrix}+\beta\begin{pmatrix}3\\8\\1\end{pmatrix} \Rightarrow \begin{cases}1+4\alpha=3\beta\\-2\alpha=10+8\beta\\-3\alpha=1+\beta\end{cases} \Rightarrow \begin{cases}-4\alpha+3\beta=1\\-2\alpha-8\beta=10\\-3\alpha-\beta=1\end{cases}$

Solving using GC, there is no solution for α and β.
Hence l_1 and l_2 do not intersect.

Read Appendix A on how to solve simultaneous equations with/without a GC

Since l_1 and l_2 are non-parallel and non-intersecting, they are skew lines.

(c) $l_1 : \mathbf{r}=(\mathbf{i}-5\mathbf{k})+\lambda(\mathbf{i}-\mathbf{j}+\mathbf{k}),\ \lambda\in\mathbb{R}$ and $l_2 : \mathbf{r}=(\mathbf{i}-\mathbf{j}+\mathbf{k})+\mu(5\mathbf{i}-4\mathbf{j}-\mathbf{k}),\ \mu\in\mathbb{R}$

$$l_1 : \mathbf{r}=\begin{pmatrix}1\\0\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \lambda\in\mathbb{R} \quad\text{and}\quad l_2 : \mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}5\\-4\\-1\end{pmatrix},\ \mu\in\mathbb{R}$$

Let θ be the angle between l_1 and l_2.

$$\cos\theta=\frac{\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}5\\-4\\-1\end{pmatrix}}{\sqrt{3}\sqrt{25+16+1}}=\frac{5+4-1}{\sqrt{3}\sqrt{42}}=\frac{8}{\sqrt{126}}$$

$\Rightarrow\ \theta=44.5^\circ$ (to 1d.p)

Since $\begin{pmatrix}1\\-1\\1\end{pmatrix}\neq k\begin{pmatrix}5\\-4\\-1\end{pmatrix}$ for any $k\in\mathbb{R}$, l_1 and l_2 are not parallel.

[Note that the above step of checking whether the direction vectors of l_1 and l_2 may no longer be necessary since the angle between these 2 lines is non-zero. (similar to Qn 2(b)]

To check if the lines intersect:

Let $\begin{pmatrix}1\\0\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\mu\begin{pmatrix}5\\-4\\-1\end{pmatrix} \Rightarrow \begin{cases}\lambda-5\mu=0\\-\lambda+4\mu=-1\\\lambda+\mu=6\end{cases}$

Solving using GC, we obtain a unique solution $\lambda=5$ and $\mu=1$.
Hence l_1 and l_2 are intersecting lines.

Read Appendix A on how to solve simultaneous equations with/without a GC

Position vector of point of intersection $=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\begin{pmatrix}5\\-4\\-1\end{pmatrix}=\begin{pmatrix}6\\-5\\0\end{pmatrix}$

(We can of course also use $\lambda=5$ to obtain the same answer)