RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 2B: Vectors II – Equations of Straight Lines

SYLLABUS INCLUDE

- Vector and cartesian equations of lines
- Finding the foot of the perpendicular and distance from a point to a line
- Finding the angle between two lines
- Relationships between two lines (coplanar or skew)

CONTENT

INTRODUCTION

In this chapter, we shall see how the equation of a straight line can be expressed in three forms : vector, parametric and cartesian. We will then use the equation to solve problems involving distances, intersections and angles.

1 Equations of Straight Lines

1.1 Vector Equation of a Straight Line

Recall that to find the equation of a straight line in the xy plane, we need a point which lies on the line and the gradient of the line.

Similarly, to obtain a vector equation of a line, we need
(i) the position vector of a point on the line, and
(ii) a vector parallel to the line (known as the direction vector).

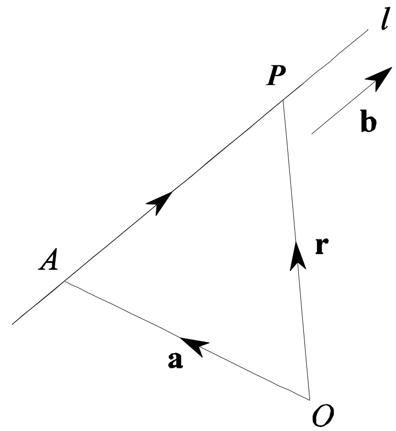
A straight line l which passes through a fixed point A with position vector $\mathbf{a}$ and is parallel to a vector $\mathbf{b}$ has vector equation given by

$$l: \quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \quad \lambda \in \mathbb{R},$$

where $\mathbf{r}$ represents the **position vector of any point** on the line l.

Each real value of λ gives the position vector of a point on the line l.

To see how this formula arises,
consider any point P on the line that is different from A.



Since l is parallel to $\mathbf{b}$,
the vector $\overrightarrow{AP}$ is also parallel to $\mathbf{b}$

$\Rightarrow \overrightarrow{AP} = \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

Thus $\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

When point P is A, clearly $\mathbf{r} = \overrightarrow{OP} = \mathbf{a}$ (i.e. $\lambda = 0$)

Example 1

Relative to the origin O, the points A, B and C have position vectors given by

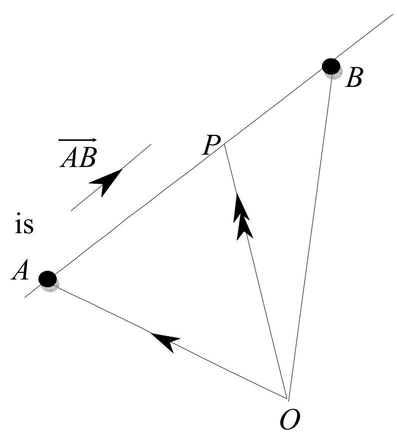
$$\mathbf{a} = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \ \mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{c} = 10\mathbf{i} + s\mathbf{j} + t\mathbf{k}$$

respectively, where s and t are constants.

(a) Find a vector equation of the line passing through A and B.
(b) Find a vector equation of the line passing through O and parallel to $\mathbf{a}$.
(c) The line passing through A and C has equation given by $\mathbf{r} = (7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j})$, $\lambda \in \mathbb{R}$. Find s and t.

Solution

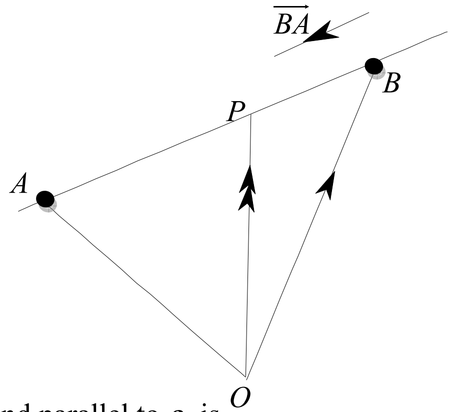
(a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$



A vector equation of the line passing through A and B is

$$\mathbf{r} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{R},$$

or $\mathbf{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+\beta\begin{pmatrix}5\\2\\-1\end{pmatrix},\ \beta\in\mathbb{R}$, or …



Remark:
Vector equation of a line is not unique!

(b) A vector equation of the line passing through O and parallel to $\mathbf{a}$ is

$$\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\mu\begin{pmatrix}7\\3\\2\end{pmatrix},\ \mu\in\mathbb{R} \qquad \text{or} \qquad \mathbf{r}=\mu\begin{pmatrix}7\\3\\2\end{pmatrix},\ \mu\in\mathbb{R}$$

(c) $\overrightarrow{AC}=\mathbf{c}-\mathbf{a}=\begin{pmatrix}3\\s-3\\t-2\end{pmatrix}$

Since $\overrightarrow{AC}$ is parallel to $\begin{pmatrix}1\\3\\0\end{pmatrix}$, we have $\begin{pmatrix}3\\s-3\\t-2\end{pmatrix}=m\begin{pmatrix}1\\3\\0\end{pmatrix}$ for some $m\neq 0$

$$\Rightarrow \begin{cases} 3=m \\ s-3=3m \quad \Rightarrow \quad s=3(3)+3=12 \\ t-2=0 \quad \Rightarrow \quad t=2 \end{cases}$$

1.2 Parametric Form of Equation of a Straight Line

Let the vector equation of a straight line be $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{b},\ \ \lambda\in\mathbb{R}$.

By writing $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$,

we have $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

$$\boxed{\Rightarrow \begin{cases} x=a_1+\lambda b_1 \\ y=a_2+\lambda b_2, \quad \lambda\in\mathbb{R} \\ z=a_3+\lambda b_3 \end{cases}}$$

This set of three equations is known as the parametric form of the equation of the line l.

1.3 Cartesian Equation of a Straight Line

From the parametric form of the equation of line l, by making λ the subject,

we have $\begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \\ z = a_3 + \lambda b_3 \end{cases} \Rightarrow \begin{cases} \lambda = \dfrac{x - a_1}{b_1} \\ \lambda = \dfrac{y - a_2}{b_2}, \\ \lambda = \dfrac{z - a_3}{b_3} \end{cases}$ provided $b_1,\ b_2,\ b_3 \neq 0$.

The equation $\boxed{\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}}$ is known as the cartesian equation of the line l.

Remarks

1. In 2D-plane, the cartesian equation of a line $y = mx + c$ can be written as $\dfrac{y - c}{m} = \dfrac{x}{1} = \lambda \Rightarrow x = \lambda, y = m\lambda + c$. The vector equation of the line is thus $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m \end{pmatrix} = \mathbf{a} + \lambda \mathbf{b},\ \lambda \in \mathbb{R}$. We can see that $\mathbf{a}$ is the position vector of a point on the line and $\mathbf{b}$, the direction vector of the line, is essentially the gradient (for every increase in 1 unit in the x-direction, there is an increase by m units in the y-direction).

2. In 3D, when $b_1 = 0$, $\begin{cases} x = a_1 + \lambda b_1 & \Rightarrow & x = a_1 \\ y = a_2 + \lambda b_2 & \Rightarrow & \lambda = \dfrac{y - a_2}{b_2} \\ z = a_3 + \lambda b_3 & \Rightarrow & \lambda = \dfrac{z - a_3}{b_3} \end{cases}$

The cartesian equation of l becomes

$$x = a_1,\ \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3},$$

which is a line parallel to the $yz-$plane. Similar results for $b_2 = 0$ or $b_3 = 0$.

3. When $b_1 = b_2 = 0$, $\begin{cases} x = a_1 + \lambda b_1 & \Rightarrow & x = a_1 \\ y = a_2 + \lambda b_2 & \Rightarrow & y = a_2 \\ z = a_3 + \lambda b_3 & \Rightarrow & z = a_3 + \lambda b_3 \end{cases}$

Since $\lambda \in \mathbb{R}$, $z \in \mathbb{R}$. The cartesian equation of l becomes

$$x = a_1,\ y = a_2,\ z \in \mathbb{R},$$

which is a line perpendicular to the $xy-$plane (or parallel to the $z-$axis).
Similar results for the cases where $b_1 = b_3 = 0$ or $b_2 = b_3 = 0$.

Example 2

(a) Find the cartesian equation of the line $l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \ \lambda \in \mathbb{R}$.

(b) Find the vector equation of the line $l_2 : \dfrac{x-3}{4} = \dfrac{y+1}{2} = 2z+1$.

Solution

(a) From $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+3\lambda \\ 5-\lambda \\ 4\lambda \end{pmatrix}$,

we get $\begin{cases} x = 2+3\lambda \\ y = 5-\lambda \\ z = 4\lambda \end{cases} \Rightarrow \ \lambda = \dfrac{x-2}{3} = \dfrac{y-5}{-1} = \dfrac{z}{4}$

Cartesian equation of l_1 is $\dfrac{x-2}{3} = 5 - y = \dfrac{z}{4}$

(b) Let $\dfrac{x-3}{4} = \dfrac{y+1}{2} = 2z+1 = \lambda$ for $\lambda \in \mathbb{R}$ and solve for x, y and z,

we get $\begin{cases} x = 3+4\lambda \\ y = -1+2\lambda \\ z = \dfrac{1}{2}(-1+\lambda) \end{cases} \Rightarrow \ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -\frac{1}{2} \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ \frac{1}{2} \end{pmatrix}, \ \lambda \in \mathbb{R}$

Vector equation of l_2 is $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ -\frac{1}{2} \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R}$

2 Calculations Involving a Point and a Line

2.1 Determining whether a Point lies on a Line or a Line passes through a Point

To determine if a point C lies on a line $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}, \ \lambda \in \mathbb{R}$, we let $\overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b}$ and see if we can solve for a unique value of λ.
If it is possible, it means that C lies on l (or l passes through C).
Otherwise, C does not lie on l (or l does not pass through C).

Example 3

The points A and B have position vectors $\mathbf{a} = 2\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = \mathbf{i}+3\mathbf{k}$.
The line l_1 passes through A and B while the line l_2 passes through A and is parallel to the vector $\mathbf{c} = \mathbf{i}+3\mathbf{j}+5\mathbf{k}$.

Determine if the lines l_1 or l_2 passes through the point $P\left(\frac{1}{2}, -\frac{3}{2},\ 3\right)$.

Solution

$$\overrightarrow{AB} = \mathbf{b}-\mathbf{a} = \begin{pmatrix}1\\0\\3\end{pmatrix} - \begin{pmatrix}2\\3\\3\end{pmatrix} = \begin{pmatrix}-1\\-3\\0\end{pmatrix}$$

$$\Rightarrow\quad l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\0\end{pmatrix},\ \lambda\in\mathbb{R} \quad \text{and} \quad l_2 : \mathbf{r} = \begin{pmatrix}2\\3\\3\end{pmatrix} + \mu\begin{pmatrix}1\\3\\5\end{pmatrix},\ \mu\in\mathbb{R}$$

If P lies on l_1, then $\overrightarrow{OP} = \begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\\3\end{pmatrix} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\0\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

Consider $\begin{cases} \frac{1}{2} = 1+\lambda \quad\Rightarrow\quad \lambda = -\frac{1}{2} \\ -\frac{3}{2} = 0+3\lambda \quad\Rightarrow\quad \lambda = -\frac{1}{2} \\ 3 = 3 \end{cases}$

Since there is a unique solution for λ, l_1 pass through P.

Alternatively, by observation, since $\overrightarrow{OP} = \begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\\3\end{pmatrix} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\0\end{pmatrix}$ for $\lambda = -\frac{1}{2}$, l_1 passes through P.

If P lies on l_2, then $\overrightarrow{OP} = \begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\\3\end{pmatrix} = \begin{pmatrix}2\\3\\3\end{pmatrix} + \mu\begin{pmatrix}1\\3\\5\end{pmatrix}$ for some $\mu\in\mathbb{R}$.

However $\begin{cases} \dfrac{1}{2} = 2 + \mu \quad \Rightarrow \quad \mu = -\dfrac{3}{2} \\ -\dfrac{3}{2} = 3 + 3\mu \quad \Rightarrow \quad \mu = -\dfrac{3}{2} \\ 3 = 3 + 5\mu \quad \Rightarrow \quad \mu = 0 \end{cases}$

Since there is no unique solution for μ, l_2 does not pass through P.

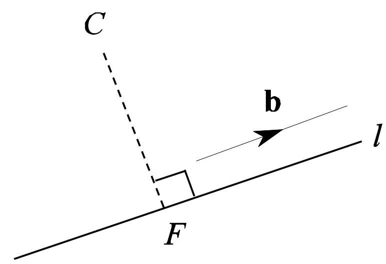
2.2 Finding the Foot of Perpendicular from a Point to a Line and the Corresponding Perpendicular Distance

Let C be a point not on the line $l : \mathbf{r} = \mathbf{a} + \lambda \mathbf{b}, \quad \lambda \in \mathbb{R}$.
Let F be the foot of perpendicular from point C to line l.

To find $\overrightarrow{OF}$, we use the fact that

- F lies on l, i.e. $\overrightarrow{OF}$ satisfies equation of l, and
- $\overrightarrow{CF}$ is perpendicular to l, i.e. $\overrightarrow{CF} \cdot \mathbf{b} = 0$

The corresponding perpendicular distance (or shortest distance) between point C and line l is $\left|\overrightarrow{CF}\right|$.



Example 4

The equation of the line l is given by $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mu \in \mathbb{R}$.

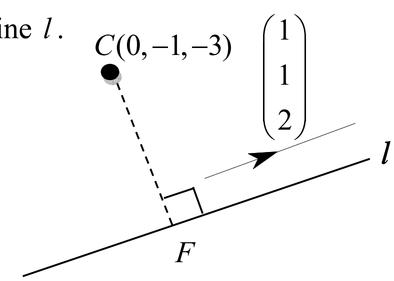
(i) Find the position vector of the foot of the perpendicular from the point $C(0, -1, -3)$ to l.
(ii) Deduce the shortest distance between the point C and the line l.

Solution

(i) Let F be the foot of perpendicular from point C to line l.

F lies on l, $\overrightarrow{OF} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ for some $\mu \in \mathbb{R}$

$$\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$



Since $\overrightarrow{CF} \perp l$, $\overrightarrow{CF} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0 \quad \Rightarrow \quad \left[\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$

$$(2 + 4 + 10) + \mu(1 + 1 + 4) = 0 \quad \Rightarrow \quad \mu = -\frac{8}{3}$$

Hence $\overrightarrow{OF}=\begin{pmatrix}2\\3\\2\end{pmatrix}-\frac{8}{3}\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}-\frac{2}{3}\\\frac{1}{3}\\-\frac{10}{3}\end{pmatrix}$

(ii) Shortest distance between C and l

= Perpendicular distance between C and l $=\left|\overrightarrow{CF}\right|$

$$\overrightarrow{CF}=\begin{pmatrix}2\\4\\5\end{pmatrix}-\frac{8}{3}\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}-\frac{2}{3}\\\frac{4}{3}\\-\frac{1}{3}\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-2\\4\\-1\end{pmatrix}$$

Hence $\left|\overrightarrow{CF}\right|=\left|\frac{1}{3}\begin{pmatrix}-2\\4\\-1\end{pmatrix}\right|=\frac{1}{3}\sqrt{(-2)^2+4^2+(-1)^2}=\frac{\sqrt{21}}{3}$

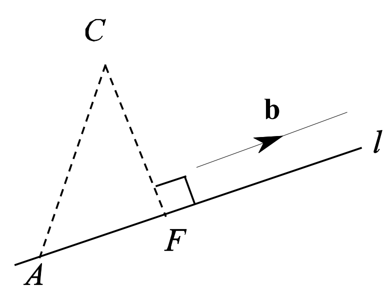
Remark:
We can also solve part **(ii)**, i.e. to find the shortest distance between C and l, in Example 4 using the following two methods:

1. We can use the length of projection and Pythagoras Theorem.
2. We can use vector product.
   (Refer to Example 5 below.)

Example 5

The equation of the line l is given by $\mathbf{r}=\begin{pmatrix}2\\3\\2\end{pmatrix}+\mu\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\mu\in\mathbb{R}$. Find the shortest distance between the point $C(0,-1,-3)$ and the line l.

Solution 1



Let $\overrightarrow{OA}=\begin{pmatrix}2\\3\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\1\\2\end{pmatrix}$.

$\overrightarrow{AC}=\begin{pmatrix}0\\-1\\-3\end{pmatrix}-\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}-2\\-4\\-5\end{pmatrix}$ and $\left|\overrightarrow{AC}\right|=\left|\begin{pmatrix}-2\\-4\\-5\end{pmatrix}\right|=\sqrt{45}$

Let F be the foot of perpendicular from point C to line l.

$\left|\overrightarrow{AF}\right|=$ Length of the projection of $\overrightarrow{AC}$ onto l $=\frac{\left|\overrightarrow{AC}\cdot\mathbf{b}\right|}{\sqrt{1^2+1^2+2^2}}=\frac{\left|\begin{pmatrix}-2\\-4\\-5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}\right|}{\sqrt{6}}=\frac{16}{\sqrt{6}}$

Shortest distance between l and C $=\left|\overrightarrow{CF}\right|=\sqrt{(AC)^2-(AF)^2}=\sqrt{45-\dfrac{16^2}{6}}=\sqrt{\dfrac{7}{3}}=\dfrac{\sqrt{21}}{3}$

Solution 2

Let $\overrightarrow{OA}=\begin{pmatrix}2\\3\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\1\\2\end{pmatrix}$.

Then ,

$\overrightarrow{AC}=\begin{pmatrix}0\\-1\\-3\end{pmatrix}-\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}-2\\-4\\-5\end{pmatrix}$ and $|\mathbf{b}|=\left|\begin{pmatrix}1\\1\\2\end{pmatrix}\right|=\sqrt{6}$.

Shortest distance between l and C is

$$CF=\left|\overrightarrow{AC}\right|\sin\angle CAF$$

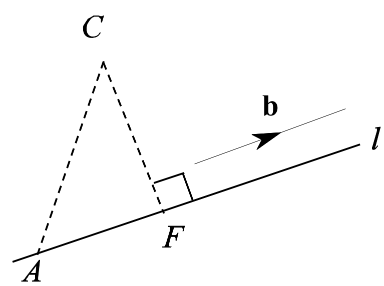
$$=\frac{\left|\overrightarrow{AC}\times\mathbf{b}\right|}{|\mathbf{b}|}$$

Since $\left|\overrightarrow{AC}\times\mathbf{b}\right|=\left|\overrightarrow{AC}\right||\mathbf{b}|\sin\angle CAF$,

$\left|\overrightarrow{AC}\right|\sin\angle CAF=\dfrac{\left|\overrightarrow{AC}\times\mathbf{b}\right|}{|\mathbf{b}|}$.

$$=\frac{\left|\begin{pmatrix}-2\\-4\\-5\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}\right|}{\sqrt{6}}$$

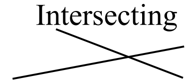
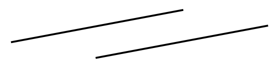
$$=\frac{\left|\begin{pmatrix}-3\\-1\\2\end{pmatrix}\right|}{\sqrt{6}}=\frac{\sqrt{21}}{3}.$$



3 Calculations Involving a Pair of Lines

3.1 Parallel Lines, Intersecting Lines and Skew Lines

Given a pair of lines, there are four possible scenarios:

	Parallel	**Not Parallel**
Intersecting	Same line 	Intersecting 
Not Intersecting		skew lines

Remark:
If 2 lines are parallel or intersecting, then they are coplanar.

Example 6

Determine if the following pairs of lines are parallel, intersecting or skew.
In the case of intersecting lines, find the point of intersection.

(a) $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \mu \begin{pmatrix} -9 \\ 12 \\ -3 \end{pmatrix}, \ \mu \in \mathbb{R}$

(b) $l_3 : x - 4 = \dfrac{y-8}{2} = z - 3$ and $l_4 : \dfrac{x-7}{6} = \dfrac{y-6}{4} = \dfrac{z-5}{5}$

(c) $l_5 : \mathbf{r} = (\mathbf{i} + 3\mathbf{k}) + s(2\mathbf{i} + \mathbf{j} + \mathbf{k}), \ s \in \mathbb{R}$ and $l_6 : \mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(\mathbf{i} - 2\mathbf{j}), \ t \in \mathbb{R}$

Solution

(a) When $\lambda = -1$, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -9 \\ 12 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$,

l_1 and l_2 are parallel.

(b) The vector equations of l_3 and l_4 are

$$l_3 : \ \underset{\sim}{r} = \begin{pmatrix} 4 \\ 8 \\ 3 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}, \qquad l_4 : \ \underset{\sim}{r} = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}, \quad \beta \in \mathbb{R}$$

Since $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \neq k \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$ for any $k \in \mathbb{R}$, l_3 and l_4 are not parallel.

(To check if the lines intersect)

Let $\begin{pmatrix}4\\8\\3\end{pmatrix}+\alpha\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}7\\6\\5\end{pmatrix}+\beta\begin{pmatrix}6\\4\\5\end{pmatrix} \Rightarrow \begin{cases}\alpha-6\beta=3\\2\alpha-4\beta=-2\\\alpha-5\beta=2\end{cases}$

From GC, we obtain a unique solution $\alpha=-3$ and $\beta=-1$.

Hence l_3 and l_4 are intersecting lines.

Position vector of point of intersection $=\begin{pmatrix}4\\8\\3\end{pmatrix}-3\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$,

i.e. point of intersection is $(1,2,0)$.

(We can of course also use $\beta=-1$ to obtain the same answer)

*Refer to **Appendix A** for instructions on using the **GC** to solve simultaneous equations, as well as methods for solving them **without a GC**.*

(c) $l_5:\mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+s\begin{pmatrix}2\\1\\1\end{pmatrix}$, $s\in\mathbb{R}$ and $l_6:\mathbf{r}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\0\end{pmatrix}$, $t\in\mathbb{R}$

Since $\begin{pmatrix}2\\1\\1\end{pmatrix}\neq m\begin{pmatrix}1\\-2\\0\end{pmatrix}$ for any $m\in\mathbb{R}$, l_5 and l_6 are not parallel.

Let $\begin{pmatrix}1\\0\\3\end{pmatrix}+s\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+t\begin{pmatrix}1\\-2\\0\end{pmatrix} \Rightarrow \begin{cases}2s-t=1\\s+2t=-1\\s=-2\end{cases}$

From GC, there is no solution for s and t.

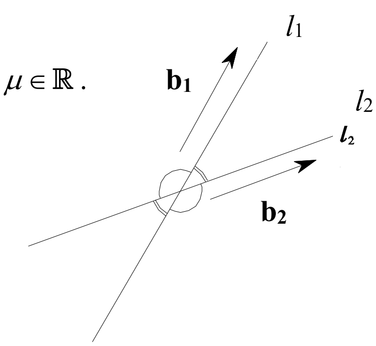
Hence l_5 and l_6 do not intersect.

Since l_5 and l_6 are non-parallel and non-intersecting, they are skew lines.

3.2 Acute Angle between Two Lines

Let the vector equations of two lines l_1 and l_2 be given by

$$l_1:\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1 \text{ and } l_2:\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}_2,\quad \lambda,\ \mu\in\mathbb{R}.$$



If θ is the angle between the two direction vectors $\mathbf{b}_1$ and $\mathbf{b}_2$, then

$$\boxed{\cos\theta=\frac{\mathbf{b}_1\cdot\mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}}.$$

Note that θ may be acute or obtuse, depending on the sign of $\mathbf{b}_1\cdot\mathbf{b}_2$.

If $\mathbf{b}_1\cdot\mathbf{b}_2>0$, then θ is acute and the acute angle between l_1 and l_2 is θ.

If $\mathbf{b}_1\cdot\mathbf{b}_2<0$, then θ is obtuse and the acute angle between l_1 and l_2 is $180^\circ-\theta$.

Remark:

The angle between the two lines l_1 and l_2 depends on $\mathbf{b}_1$ and $\mathbf{b}_2$, not $\mathbf{a}_1$ and $\mathbf{a}_2$.

Example 7

Find the acute angle between each of the following pairs of lines, giving your answer to 1 decimal place.

(a) $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \ s \in \mathbb{R}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 8 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}$

(b) (2-D vectors)

$l_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R}$ and $l_4 : \mathbf{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R}$

Solution

(a) Let θ_1 be the angle between $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

$$\cos\theta_1 = \frac{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{1^2 + 2^2 + 1^2}\sqrt{2^2 + 1^2 + 1^2}} = \frac{2+2+1}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

$$\theta_1 = 33.6^\circ \ \text{(1dp)}$$

Hence required acute angle between l_1 and l_2 is 33.6° (1dp)

(b) Let θ_2 be the angle between $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

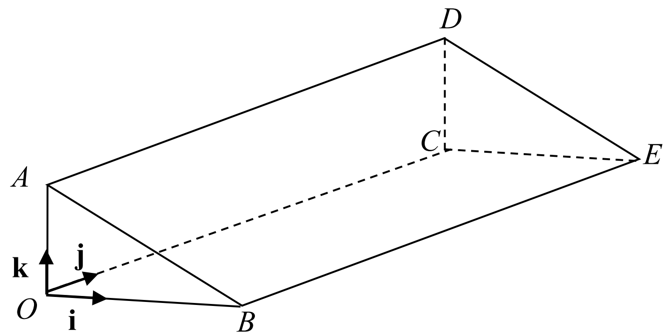
$$\cos\theta_2 = \frac{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\sqrt{1^2 + (-1)^2}\sqrt{(-3)^2 + 1^2}} = \frac{-3-1}{\sqrt{2}\sqrt{10}} = -\frac{2}{\sqrt{5}}$$

$$\theta_2 = 153.435^\circ \ \text{(3dp)}$$

Hence required acute angle between l_3 and l_4 is $180^\circ - 153.435^\circ = 26.6^\circ$ (nearest 0.1°)

Example 8 [NYJCPrelim9233/2006/01/Q15 Modified]

The diagram below shows a prism with O as the origin of position vectors and the unit vectors **i**, **j** and **k** are parallel to OB, OC and OA respectively. The point M is the mid-point of CE and the point N divides AD in the ratio $2:1$. It is given that $OA = CD = 3$ units, $OB = CE = 4$ units and $OC = BE = AD = 10$ units.



The line l_1 passes through A and B and line l_2 passes through M and N.

(i) Without the use of GC, show that l_1 and l_2 are skew lines.

(ii) Find the acute angle between l_1 and l_2, giving your answer to the nearest $0.1°$.

(iii) Find the coordinates of the foot of perpendicular from point E to the line BC.

(iv) Find the coordinates of the mirror image of point E about the line BC.

Solution

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \overrightarrow{OC} = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}, \overrightarrow{OD} = \begin{pmatrix} 0 \\ 10 \\ 3 \end{pmatrix}, \overrightarrow{OE} = \begin{pmatrix} 4 \\ 10 \\ 0 \end{pmatrix}$$

M is the mid-point of CE $\Rightarrow \overrightarrow{OM} = \begin{pmatrix} 2 \\ 10 \\ 0 \end{pmatrix}$

N divides AD in the ratio $2:1$ $\Rightarrow \overrightarrow{ON} = \dfrac{2\overrightarrow{OD} + \overrightarrow{OA}}{3} = \dfrac{1}{3}\left[2\begin{pmatrix} 0 \\ 10 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ \frac{20}{3} \\ 3 \end{pmatrix}$

Also, $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ and $\overrightarrow{NM} = \begin{pmatrix} 2 \\ 10 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{20}{3} \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{10}{3} \\ -3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ 10 \\ -9 \end{pmatrix}$

(i) $l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 10 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 10 \\ -9 \end{pmatrix}$, $\mu \in \mathbb{R}$

Since $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \neq m \begin{pmatrix} 6 \\ 10 \\ -9 \end{pmatrix}$ for any $m \in \mathbb{R}$, l_1 and l_2 are not parallel.

Let $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 10 \\ -9 \end{pmatrix} \Rightarrow \Rightarrow \begin{cases} 4\lambda - 6\mu = 2 & ..... \quad (1) \\ 10\mu = -10 & ..... \quad (2) \\ -3\lambda + 9\mu = -3 & ..... \quad (3) \end{cases}$

Solving (1) and (2), we have $\lambda = -1$ and $\mu = -1$.

When $\lambda = -1$ and $\mu = -1$, we have from (3), $\text{LHS} = -3(-1) + 9(-1) = -6 \neq \text{ RHS}$.

Hence l_1 and l_2 do not intersect,

Since l_1 and l_2 are non-parallel and non-intersecting, they are skew lines.

(ii) Let θ be the angle between $\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 10 \\ -9 \end{pmatrix}$.

$$\cos\theta = \frac{\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 10 \\ -9 \end{pmatrix}}{\sqrt{4^2 + 0^2 + (-3)^2}\sqrt{6^2 + 10^2 + (-9)^2}} = \frac{24 + 27}{5\sqrt{217}} = \frac{51}{5\sqrt{217}}$$

$\Rightarrow \quad \theta = 46.2°$

Hence required acute angle between l_1 and l_2 is 46.2° (nearest $0.1°$)

(iii) Let F be the foot of perpendicular from point E to the line BC.

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \\ 0 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$

$l_{BC} : \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$, $\alpha \in \mathbb{R}$

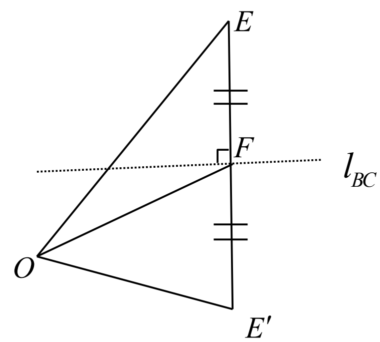
F lies on l_{BC} $\quad \Rightarrow \overrightarrow{OF} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$, for some $\alpha \in \mathbb{R}$.

$\Rightarrow \overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\alpha \\ -10 + 5\alpha \\ 0 \end{pmatrix}$

Since $\overrightarrow{EF} \perp l_{BC} \quad \Rightarrow \overrightarrow{EF} \cdot \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\alpha \\ -10+5\alpha \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} = 29\alpha - 50 = 0$

$\Rightarrow \alpha = \dfrac{50}{29}$

Hence $\overrightarrow{OF} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \dfrac{50}{29}\begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} = \dfrac{1}{29}\begin{pmatrix} 16 \\ 250 \\ 0 \end{pmatrix}$, i.e. coordinates of $F = \left(\dfrac{16}{29}, \dfrac{250}{29}, 0\right)$



(iv) Let E' be the mirror image of E about BC.

Then $\overrightarrow{OF} = \dfrac{\overrightarrow{OE} + \overrightarrow{OE'}}{2}$

$\Rightarrow \overrightarrow{OE'} = 2\overrightarrow{OF} - \overrightarrow{OE} = 2\begin{pmatrix} \frac{16}{29} \\ \frac{250}{29} \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{84}{29} \\ \frac{210}{29} \\ 0 \end{pmatrix}$,

i.e. coordinates of $E' = \left(-\dfrac{84}{29}, \dfrac{210}{29}, 0\right)$

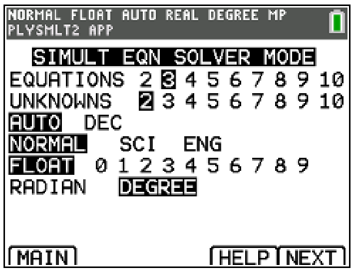
Appendix A: Guide on Solving Simultaneous Equations with/without a GC

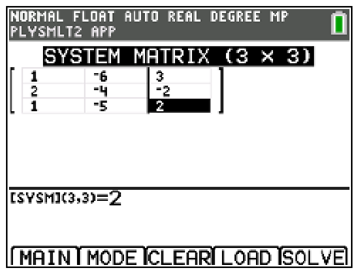
Scan the QR code on the right to access a YouTube video by *Texas Instrument Education* on using the **Plysmlt2** app in the GC. Watch the video to learn the steps for activating the **Plysmlt2** app.

Recall that in Example **6(b)**, we have the following simultaneous equations:

$$\alpha - 6\beta = 3$$
$$2\alpha - 4\beta = -2$$
$$\alpha - 5\beta = 2$$

Try using the app to solve the simultaneous equations. Below are the GC screenshots.





Can we still solve without a GC?
Most scientific calculators approved for use in Secondary School have the *simultaneous equations solver* feature, but they are limited to solving 2 equations with 2 unknowns, or 3 equations with 3 unknowns. However, most of the questions in Chap 2B involves 3 equations with 2 unknowns. For **Example 6(b)**, you can use your scientific calculator (or solve by hand) to find the solution for the first two equations simultaneously:

$$\left.\begin{aligned} \alpha - 6\beta &= 3 \\ 2\alpha - 4\beta &= -2 \end{aligned}\right\} \ \alpha = -3, \beta = -1$$

After that, you will check that it works for the 3rd equation.

Checking consistency in the 3rd equation: $\alpha - 5\beta = -3 - 5(-1) = 2$
we obtain a unique solution $\alpha = -3$ and $\beta = -1$.

In Example **6(c)**, we have the following simultaneous equations:

$$2s - t = 1$$
$$s + 2t = -1$$
$$s = -2$$

Solve the first 2 equations simultaneously: $\left.\begin{aligned} 2s - t &= 1 \\ s + 2t &= -1 \end{aligned}\right\} \quad s = \frac{1}{5}, t = -\frac{3}{5}$

This is NOT consistent with the 3rd equation where $s = -2$, while $s = \frac{1}{5}$ from the 1st 2 equations. So, we say that there is no solution for *s* and *t*.

SUMMARY