RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Additional Practice Questions for Chapter 2B: Vectors II – Equation of Straight Lines (Solutions)

1 Referred to the origin O, the position vector of the point A is $2\mathbf{i}+2\mathbf{j}-6\mathbf{k}$ and the cartesian equation of the line l is $x-1=2-y=z+6$. Find

(i) the position vector of the foot of the perpendicular from A to l, [3]

(ii) the perpendicular distance from A to l, [2]

TPJC Prelim 9740/2013/01/Q10 (modified)
(i) $l:\mathbf{r}=\begin{pmatrix}1\\2\\-6\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix},\lambda\in\mathbb{R}$ Let B be the foot of perpendicular from A to l. $\overrightarrow{OB}=\begin{pmatrix}1+\lambda\\2-\lambda\\-6+\lambda\end{pmatrix}\therefore\overrightarrow{AB}=\begin{pmatrix}-1+\lambda\\-\lambda\\\lambda\end{pmatrix}$ $\overrightarrow{AB}.\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$ $\lambda=\frac{1}{3}\therefore\overrightarrow{OB}=\frac{1}{3}\begin{pmatrix}4\\5\\-17\end{pmatrix}$
(ii) Perpendicular distance from A to l is $\left\lvert\overrightarrow{AB}\right\rvert=\left\lvert\frac{1}{3}\begin{pmatrix}-2\\-1\\1\end{pmatrix}\right\rvert=\frac{1}{3}\sqrt{6}$

2 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 1+2\lambda \\ 1-2\lambda \\ -3\lambda \end{pmatrix}$, where λ is a parameter.

(i) Find the exact shortest distance between the origin O and l_1. [3]

Another line l_2 has equation $\dfrac{x-1}{2} = \dfrac{3-y}{2} = -\dfrac{z}{9}$.

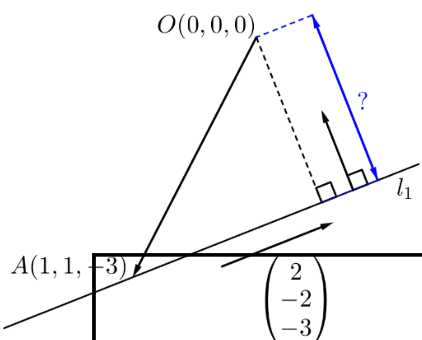
(ii) Write down a vector equation of l_2. [1]

(iii) Show that l_1 and l_2 are skew lines. [3]

(iv) Find a vector that is perpendicular to both l_1 and l_2. [1]

The points P and Q lie on l_1 and l_2 respectively, such that the line PQ is perpendicular to both l_1 and l_2.

(v) Find a vector equation of the line PQ. [3]

ASRJC Promo 9758/2021/Q8	
(i)	Shortest distance from O to l_1 $= \dfrac{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} \right\|}{\sqrt{4+4+9}}$ $= \dfrac{\left\| \begin{pmatrix} -3 \\ 3 \\ -4 \end{pmatrix} \right\|}{\sqrt{4+4+9}}$ $= \dfrac{\sqrt{34}}{\sqrt{17}}$ $= \sqrt{2}$ units 
(ii)	$l_2 : \dfrac{x-1}{2} = \dfrac{3-y}{2} = -\dfrac{z}{9} \Rightarrow \dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z-0}{-9}$ $l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix}, \quad \mu \in \mathbb{R}$

(iii)	$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ $l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix}, \quad \mu \in \mathbb{R}$ Clearly, $\begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix}$ are <u>not scalar multiples</u> of each other, hence both lines are not parallel. $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix}$ $\Rightarrow \begin{cases} 1+2\lambda = 1+2\mu & \text{---} \quad (1) \\ 1-2\lambda = 3-2\mu & \text{---} \quad (2) \\ -3\lambda = -9\mu & \text{---} \quad (3) \end{cases}$ From (3), we have $\lambda = 3\mu$ --- (4) Substituting (4) into (2), we get $1-2(3\mu) = 3-2\mu$ $\Rightarrow 1-6\mu = 3-2\mu$ $\Rightarrow 4\mu = -2$ $\Rightarrow \mu = -\frac{1}{2}$ Hence $\lambda = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$. Substituting $\mu = -\frac{1}{2}$ and $\lambda = -\frac{3}{2}$ into (1), LHS of (1) $= 1+2\left(-\frac{3}{2}\right) = -2$ RHS of (1) $= 1+2\left(-\frac{1}{2}\right) = 0 \neq$ LHS of (1) Hence both lines do not intersect. Therefore, both lines are skew. (shown)

(iv)	$\begin{pmatrix}2\\-2\\-3\end{pmatrix}\times\begin{pmatrix}2\\-2\\-9\end{pmatrix}$ $=\begin{pmatrix}18-6\\-(-18+6)\\-4+4\end{pmatrix}$ $=\begin{pmatrix}12\\12\\0\end{pmatrix}=12\begin{pmatrix}1\\1\\0\end{pmatrix}$ Thus, $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is a vector that is perpendicular to both l_1 and l_2.
(v)	Since $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is parallel to $\overrightarrow{PQ}$, $\overrightarrow{PQ}=k\begin{pmatrix}1\\1\\0\end{pmatrix}$ for some $k\in\mathbb{R}$ Since P lies on l_1, $\overrightarrow{OP}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ Since Q lies on l_2, $\overrightarrow{OQ}=\begin{pmatrix}1\\3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\-9\end{pmatrix}$ for some $\mu\in\mathbb{R}$. Thus, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$ $=\left[\begin{pmatrix}1\\3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\-9\end{pmatrix}\right]-\left[\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]$ $=\begin{pmatrix}1\\3\\0\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\-9\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}-\lambda\begin{pmatrix}2\\-2\\-3\end{pmatrix}$ $=\begin{pmatrix}2\mu-2\lambda\\2-2\mu+2\lambda\\-9\mu+3\lambda\end{pmatrix}$

	Comparing, $\begin{pmatrix} 2\mu-2\lambda \\ 2-2\mu+2\lambda \\ -9\mu+3\lambda \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\Rightarrow \begin{cases} 2\mu-2\lambda-k=0 \\ -2\mu+2\lambda-k=-2 \\ -9\mu+3\lambda=0 \end{cases}$ Using GC, we obtain $\mu=-\frac{1}{4},\ \lambda=-\frac{3}{4},\ k=1$ Therefore, $\overrightarrow{OP}=\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}-\frac{3}{4}\begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}=\begin{pmatrix} -0.5 \\ 2.5 \\ 2.25 \end{pmatrix}$. Hence an equation of the desired line is $\mathbf{r}=\begin{pmatrix} -0.5 \\ 2.5 \\ 2.25 \end{pmatrix}+t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \ t\in\mathbb{R}$

3 The line l_1 has equation $\dfrac{x-5}{3}=\dfrac{z-5}{m}$, $y=2$ and the line l_2 has equation $\dfrac{x}{-5}=y$, $z=0$, where m is a constant. It is given that l_1 and l_2 intersect at point A.

(i) Find the value of m, and the coordinates of A. [4]

(ii) Find the position vector of the point P on l_1 such that OP is perpendicular to l_1, where O is the origin. [3]

(iii) Find a vector equation of the line which is a reflection of l_2 in l_1. [3]

PJC JC2CT2 9740/2014/02/Q5

(i) Since $y = 2$, from $\dfrac{x}{-5}=y$ we have $x = -10$ and substituting into

$\dfrac{x-5}{3}=\dfrac{z-5}{m}$ (together with $z = 0$) we have $\dfrac{-15}{3}=\dfrac{-5}{m}$ and thus $m = 1$.

We also have therefore $A(-10, 2, 0)$.

Alternatively,

(i) Let $\lambda=\dfrac{x-5}{3}=\dfrac{z-5}{m}$

$x=5+3\lambda$

$z=5+m\lambda$

$$l_1:\mathbf{r}=\begin{pmatrix}5\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\m\end{pmatrix},\ \lambda\in\mathbb{R}$$

Let $\mu=\dfrac{x}{-5}=y$

$x=-5\mu$

$y=\mu$

$z=0$

$$l_2:\mathbf{r}=\mu\begin{pmatrix}-5\\1\\0\end{pmatrix},\ \mu\in\mathbb{R}$$

Given that l_1 and l_2 intersect,

$$\begin{pmatrix}5\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\m\end{pmatrix}=\mu\begin{pmatrix}-5\\1\\0\end{pmatrix}$$

$\mu=2$

$5+3\lambda=-5\mu\Rightarrow\lambda=-5$

$5+m\lambda=0\Rightarrow m=1$

	Coordinates of A is $(-10,2,0)$
(ii)	$\overrightarrow{OP}\cdot\begin{pmatrix}3\\0\\1\end{pmatrix}=0$ $\left[\begin{pmatrix}5\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\1\end{pmatrix}\right]\cdot\begin{pmatrix}3\\0\\1\end{pmatrix}=0$ $15+9\lambda+5+\lambda=0$ $\lambda=-2$ $\overrightarrow{OP}=\begin{pmatrix}5\\2\\5\end{pmatrix}-2\begin{pmatrix}3\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\2\\3\end{pmatrix}$
(iii)	Observe that O is on l_2. Let P' be the point of reflection of O in l_1. Then the line of reflection of l_2 in l_1 passes through A and P'. $\overrightarrow{OP}'=2\overrightarrow{OP}=\begin{pmatrix}-2\\4\\6\end{pmatrix}$ $\overrightarrow{AP}'=\overrightarrow{OP}'-\overrightarrow{OA}=\begin{pmatrix}8\\2\\6\end{pmatrix}=2\begin{pmatrix}4\\1\\3\end{pmatrix}$ Line of reflection: $\mathbf{r}=\begin{pmatrix}-10\\2\\0\end{pmatrix}+\gamma\begin{pmatrix}4\\1\\3\end{pmatrix},\ \gamma\in\mathbb{R}$

4 The points A and B have position vectors $\begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ respectively.

(i) Show that $AB = 2\sqrt{26}$. [1]

(ii) Find the cartesian equation for the line AB. [2]

(iii) The line l has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}$.

Find the length of the projection of AB onto l. [2]

(iv) Calculate the acute angle between AB and l, giving your answer correct to the nearest degree. [2]

(v) Find the position vector of the foot N of the perpendicular from A to l. Hence find the position vector of the image of A in the line l. [4]

CJC Prelim 9233/2006/01/Q14
(i) $\overrightarrow{OA} = \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix}$ $\therefore\ AB = \sqrt{(-10)^2 + 0^2 + 2^2} = \sqrt{104} = 2\sqrt{26}$ (shown)
(ii) $l_{AB} : \mathbf{r} = \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ Cartesian equation is $\frac{x-8}{-10} = \frac{z-2}{2}$, $y = 3$, i.e. $\frac{8-x}{10} = \frac{z-2}{2}$, $y = 3$
(iii) $l : \mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}$ Length of projection of AB onto l $= \dfrac{\left\lvert \begin{pmatrix} -10 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix} \right\rvert}{\sqrt{2^2 + 6^2 + 5^2}} = \dfrac{\lvert -20 + 0 + 10 \rvert}{\sqrt{65}} = \dfrac{10}{\sqrt{65}}$
(iv) Let θ be the angle between AB and l.

Then $\cos\theta = \dfrac{\begin{pmatrix}-10\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\6\\5\end{pmatrix}}{\sqrt{100+4}\sqrt{4+36+25}} = \dfrac{-20+0+10}{\sqrt{104}\sqrt{65}} = \dfrac{-10}{\sqrt{104}\sqrt{65}}$

$\Rightarrow\ \theta = 96.99^\circ$ (2dp) (obtuse)

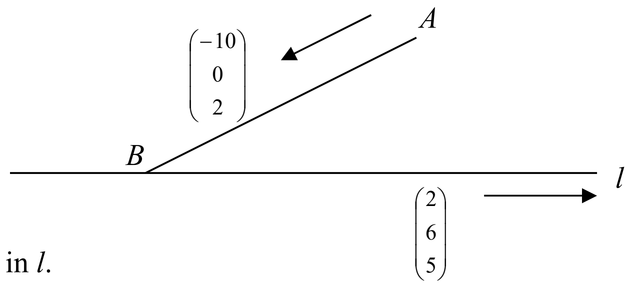
Hence required acute angle $= 180^\circ - 96.99^\circ = 83^\circ$ (nearest degree)

(v) N lies on l $\quad \Rightarrow\ \overrightarrow{ON} = \begin{pmatrix}-2\\3\\4\end{pmatrix} + t\begin{pmatrix}2\\6\\5\end{pmatrix}$, for some $t \in \mathbb{R}$

$$\Rightarrow\ \overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix}-2\\3\\4\end{pmatrix} + t\begin{pmatrix}2\\6\\5\end{pmatrix} - \begin{pmatrix}8\\3\\2\end{pmatrix} = \begin{pmatrix}-10\\0\\2\end{pmatrix} + t\begin{pmatrix}2\\6\\5\end{pmatrix}$$

Since $\overrightarrow{AN} \perp l \quad \Rightarrow\ \overrightarrow{AN}\cdot\begin{pmatrix}2\\6\\5\end{pmatrix} = \begin{pmatrix}-10\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\6\\5\end{pmatrix} + t\begin{pmatrix}2\\6\\5\end{pmatrix}\cdot\begin{pmatrix}2\\6\\5\end{pmatrix} = -10 + 65t = 0 \quad \Rightarrow\ t = \dfrac{2}{13}$

Hence $\overrightarrow{ON} = \dfrac{1}{13}\begin{pmatrix}-22\\51\\62\end{pmatrix}$

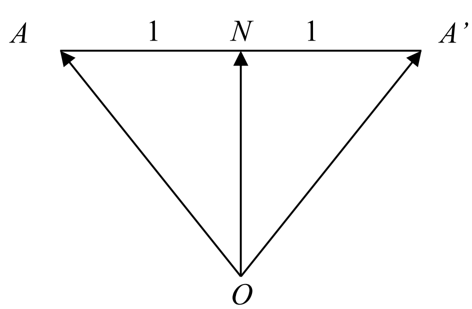


Let A' be the image of A in l.

By Ratio Theorem,

$$\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$$

$$\overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix}-44/13\\102/13\\124/13\end{pmatrix} - \begin{pmatrix}8\\3\\2\end{pmatrix} = \begin{pmatrix}-148/13\\63/13\\98/13\end{pmatrix}$$



5 With reference to the origin O, the points A and B have position vectors $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + 5\mathbf{k}$ respectively.

(i) Find a vector equation of the line l_1 that passes through point A and is parallel to the vector **a**. [1]

(ii) Find the exact length of projection of **b** on l_1. Hence find d, the exact perpendicular distance from the point B to l_1. [4]

(iii) Using the value of d found in part **(ii)**, find the position vector of the point C, the foot of perpendicular from the point B to l_1. [3]

(iv) The line l_2 passes through point B and is parallel to vector **b**. Find a cartesian equation of l_3 which is the reflection of l_2 in l_1. [3]

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<tr><td>4(i)</td><td>

$l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + k\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \lambda\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$

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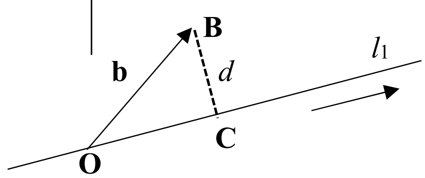
<tr><td>(ii)</td><td>

length of projection of **b** on l_1

$= \left| \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \cdot \dfrac{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{(-1)^2 + (1)^2 + (2)^2}} \right|$

$= \dfrac{12}{\sqrt{6}} = 2\sqrt{6}$



$|\mathbf{b}| = \left| \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \right| = \sqrt{29}$

By Pythagoras' Theorem,

$d = \sqrt{|\mathrm{b}|^2 - \left(2\sqrt{6}\right)^2} = \sqrt{29 - 4(6)} = \sqrt{5}$

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<tr><td>(iii)</td><td>

Since C lies on l_1, $\mathbf{c} = \lambda\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$\overrightarrow{BC} = \lambda\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -\lambda \\ \lambda - 2 \\ 2\lambda - 5 \end{pmatrix}$

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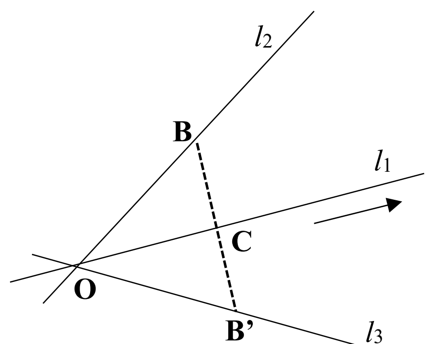
	$\left\|\overrightarrow{BC}\right\| = \left\|\begin{pmatrix} -\lambda \\ \lambda-2 \\ 2\lambda-5 \end{pmatrix}\right\| = \sqrt{5}$ $\sqrt{(-\lambda)^2+(\lambda-2)^2+(2\lambda-5)^2} = \sqrt{5}$ $6\lambda^2-24\lambda+24=0$ $\lambda^2-4\lambda+4=0$ $(\lambda-2)^2=0$ $\lambda=2$ $\mathbf{c} = 2\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$
(iv)	Let B' be the point of reflection of B about the line l_1. By mid-point theorem,  $\overrightarrow{OC} = \frac{\overrightarrow{OB}+\overrightarrow{OB'}}{2}$ $\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} + \overrightarrow{OB'}\right]$ $\overrightarrow{OB'} = 2\begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}$ Since l_3 parallel to $\overrightarrow{OB'}$ and the origin O lies on l_3, so $l_3 : \boldsymbol{r} = \gamma\begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}, \ \gamma \in \mathbb{R}$ So Cartesian equation of l_3 is : $-\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$

6 The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \text{ respectively, where } \lambda,\ \mu \in \mathbb{R}.$$

(i) Find the acute angle between l_1 and l_2. [2]

(ii) The points P and R lie on l_1 and l_2 respectively such that P is the reflection of the point R in the line l. The 3 lines intersect at the point $Q(0,3,-6)$.

Find a possible pair of vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$ such that $\angle PQR$ is acute and $\left|\overrightarrow{QP}\right| = 5$.

Hence, or otherwise, find the vector equation of the line l. [4]

DHS JC2CT2 9740/2012/01/Q8
(i) Acute angle between l_1 and l_2 $= \cos^{-1} \dfrac{\left| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} \right|}{\sqrt{3}\sqrt{27}}$ $= \cos^{-1} \dfrac{\lvert 1-1-5 \rvert}{\sqrt{3^4}}$ $= \cos^{-1} \dfrac{\lvert -5 \rvert}{9} = 56.3°$ (1dp)
(ii) From (i), since $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} < 0$, angle between $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $-\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ or $-\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ is acute. Since $\left|\overrightarrow{QP}\right| = \left|\overrightarrow{QR}\right| = 5$ and $\angle PQR$ is acute, i.e. $\overrightarrow{QP} \cdot \overrightarrow{QR} > 0$ **Either** $\overrightarrow{QP} = \dfrac{5}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\overrightarrow{QR} = \dfrac{-5}{\sqrt{27}} \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ **Or** $\overrightarrow{QP} = -\dfrac{5}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\overrightarrow{QR} = \dfrac{5}{\sqrt{27}} \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$

A direction vector of l is $\mathbf{d} = \overrightarrow{QP} + \overrightarrow{QR}$

$$= \frac{5}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix} - \frac{5}{3\sqrt{3}}\begin{pmatrix}1\\-1\\5\end{pmatrix}$$

$$= \frac{5}{3\sqrt{3}}\left[3\begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\5\end{pmatrix}\right]$$

$$= \frac{5}{3\sqrt{3}}\begin{pmatrix}2\\4\\-8\end{pmatrix} = \frac{10}{3\sqrt{3}}\begin{pmatrix}1\\2\\-4\end{pmatrix}$$

Since $Q(0,3,-6)$ lies on line l,

thus, $l:\ \mathbf{r} = \begin{pmatrix}0\\3\\-6\end{pmatrix} + s\begin{pmatrix}1\\2\\-4\end{pmatrix},\ s \in \mathbb{R}$

7 Relative to the origin O, the points A, B and C have position vectors $\mathbf{i}+2\mathbf{j}+5\mathbf{k}$, $\mathbf{i}-\mathbf{k}$ and $2\mathbf{i}+\mathbf{j}$ respectively.
The line l_1 passes through A and is parallel to the vector $3\mathbf{i}+2\mathbf{j}$. The line l_2 passes through the points B and C.

(i) Find a vector equation of the line l_1. [1]

(ii) Show that the lines l_1 and l_2 intersect and that the position vector $\overrightarrow{OX}$ of the point of intersection is $7\mathbf{i}+6\mathbf{j}+5\mathbf{k}$. [3]

The vector $a\mathbf{i}+b\mathbf{j}+\mathbf{k}$ is perpendicular to both l_1 and l_2. Find the values of a and b.
Hence give an equation of the line l_3 that is concurrent with l_1 and l_2 and is perpendicular to both l_1 and l_2.
If V is a point on l_3 such that $VABX$ is a tetrahedron with base ABX and height $5\sqrt{14}$ units, find the coordinates of V. [7]

AJC Prelim 9233/2006/01/Q13
(i) Equation of line l_1 is $\mathbf{r}=\begin{pmatrix}1\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\0\end{pmatrix},\quad \lambda\in\mathbb{R}$.
(ii) $\overrightarrow{BC}=\begin{pmatrix}2\\1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ Equation of line l_2 is $\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\beta\begin{pmatrix}1\\1\\1\end{pmatrix},\quad \beta\in\mathbb{R}$. Consider $\begin{pmatrix}1\\2\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\0\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\beta\begin{pmatrix}1\\1\\1\end{pmatrix}$ $1+3\lambda=1+\beta$ ----- (1) $2+2\lambda=\beta$ ----- (2) $5=-1+\beta$ ----- (3) From (3), $\beta=6$ Subst into (2), $\lambda=2$ Check using (1): LHS = 1 + 3(2) = 7 RHS = 1 + 6 = 7 = LHS Since there is unique solution for λ and β, therefore the lines intersect and

$$\overrightarrow{OX}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+6\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}7\\6\\5\end{pmatrix}\text{(shown)}$$

$$\begin{pmatrix}a\\b\\1\end{pmatrix}\cdot\begin{pmatrix}3\\2\\0\end{pmatrix}=0 \quad\Rightarrow\quad 3a+2b=0 \qquad \text{----- (1)}$$

$$\begin{pmatrix}a\\b\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=0 \quad\Rightarrow\quad a+b+1=0 \qquad \text{----- (2)}$$

Solving, $a=2,\ b=-3$

Equation of line l_3 is $\mathbf{r}=\begin{pmatrix}7\\6\\5\end{pmatrix}+s\begin{pmatrix}2\\-3\\1\end{pmatrix},\quad s\in\mathbb{R}$.

Since V lies on l_3, $\overrightarrow{OV}=\begin{pmatrix}7\\6\\5\end{pmatrix}+s\begin{pmatrix}2\\-3\\1\end{pmatrix}$ for some $s\in\mathbb{R}$.

$$\left|\overrightarrow{VX}\right|=5\sqrt{14}$$

$$\Rightarrow\quad \left|\begin{pmatrix}7\\6\\5\end{pmatrix}-\begin{pmatrix}7\\6\\5\end{pmatrix}-s\begin{pmatrix}2\\-3\\1\end{pmatrix}\right|=5\sqrt{14}$$

$$\Rightarrow\quad \left|s\begin{pmatrix}2\\-3\\1\end{pmatrix}\right|=5\sqrt{14}$$

$$\Rightarrow\quad |s|\sqrt{14}=5\sqrt{14}$$

$$\Rightarrow\quad |s|=5$$

$$\Rightarrow\quad s=\pm5$$

$\therefore$ Coordinates of $V=(17,-9,10)$ or $(-3,21,0)$

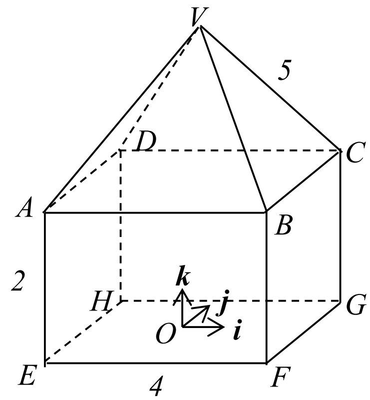
Alternatively,

Since $\left|\overrightarrow{XV}\right|=5\sqrt{14}$ and unit vector along l_3 is $\dfrac{1}{\sqrt{14}}\begin{pmatrix}2\\-3\\1\end{pmatrix}$,

$$\overrightarrow{XV}=\pm\frac{5\sqrt{14}}{\sqrt{14}}\begin{pmatrix}2\\-3\\1\end{pmatrix}\Rightarrow\overrightarrow{OV}=\overrightarrow{OX}\pm5\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}7\\6\\5\end{pmatrix}\pm\begin{pmatrix}2\\-3\\1\end{pmatrix}$$

$\therefore$ Coordinates of $V=(17,-9,10)$ or $(-3,21,0)$

8 The figure shows a right pyramid *VABCD* with a square base *ABCD*, standing horizontally on a cuboid *ABCDEFGH*. It is given that $VA = VB = VC = VD = 5$ cm, $EF = FG = 4$ cm and $AE = 2$ cm, as shown in the diagram. *O* is the centre of the square base *EFGH*. Perpendicular unit vectors **i**, **j**, **k** are parallel to *EF*, *FG*, *EA* respectively.



(i) Show that the height of the figure, *OV* is $2+\sqrt{17}$. [2]

(ii) State the geometrical meaning of $\dfrac{\left|\overrightarrow{VA}\times\overrightarrow{OV}\right|}{2+\sqrt{17}}$. [1]

(iii) Find the equation of the line passing through *B* and *V*. [2]

SAJC Prelim 9740/2013/01/Q5 (modified)
(i) $OE=\dfrac{1}{2}\sqrt{4^2+4^2}=\dfrac{1}{2}\sqrt{32}=\sqrt{8}$ Drop a perpendicular line from *V* to plane *ABCD*, and call the foot of perpendicular *M*. Consider the triangle *AVM*. $VM=\sqrt{5^2-8}=\sqrt{17}$ Thus, height of figure *OM* is $2+\sqrt{17}$ **<u>Alternative</u>** If we drop a perpendicular line down from *V* to *BC*, call the foot of the perpendicular *X*. $VX=\sqrt{5^2-2^2}=\sqrt{21}$. The height of the cone portion = $\sqrt{21-2^2}=\sqrt{17}$. Thus the height of the figure is $2+\sqrt{17}$ .G
(ii) $\dfrac{\left\lvert\overrightarrow{VA}\times\overrightarrow{OV}\right\rvert}{2+\sqrt{17}}=\dfrac{\left\lvert\overrightarrow{VA}\times\overrightarrow{OV}\right\rvert}{\left\lvert\overrightarrow{OV}\right\rvert}$. Thus it represents the length *OE*. Alternative answer: It represents the shortest distance of *A* to *OV*.

(iii) $\overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 2+\sqrt{17} \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$

$\overrightarrow{VB} = \overrightarrow{OB} - \overrightarrow{OV} = \begin{pmatrix} 2 \\ -2 \\ -\sqrt{17} \end{pmatrix}$

Thus equation of line VB: $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2+\sqrt{17} \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -\sqrt{17} \end{pmatrix}$, $\lambda \in \mathbb{R}$

9 Relative to an origin O, points A and B have position vectors $\begin{pmatrix}3\\4\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\2\\0\end{pmatrix}$ respectively.

The line l has vector equation $\mathbf{r}=\begin{pmatrix}6\\a\\0\end{pmatrix}+t\begin{pmatrix}1\\3\\a\end{pmatrix}$, where t is a real parameter and a is a constant. The line m passes through the point A and is parallel to the line OB.

(i) Find the position vector of the point P on m such that OP is perpendicular to m. [4]

(ii) Show that the two lines l and m have no common point. [3]

(ii) If the acute angle between the line l and the $z-$axis is $60°$, find the exact values of the constant a. [3]

MJC Prelim 9233/2006/02/Q5
(i) P lies on m $\Rightarrow \overrightarrow{OP}=\begin{pmatrix}3\\4\\1\end{pmatrix}+s\begin{pmatrix}-1\\2\\0\end{pmatrix}$ for some $s\in\mathbb{R}$ Since OP is perpendicular to m, $\overrightarrow{OP}\cdot\begin{pmatrix}-1\\2\\0\end{pmatrix}=0$ $\Rightarrow \begin{pmatrix}3\\4\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\0\end{pmatrix}+s\begin{pmatrix}-1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\0\end{pmatrix}=0$ $\Rightarrow 5+5s=0$ $\Rightarrow s=-1$ $\therefore \overrightarrow{OP}=\begin{pmatrix}3\\4\\1\end{pmatrix}+\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}4\\2\\1\end{pmatrix}$
(ii) Let $\begin{pmatrix}3\\4\\1\end{pmatrix}+s\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}6\\a\\0\end{pmatrix}+t\begin{pmatrix}1\\3\\a\end{pmatrix}$ $\Rightarrow \begin{cases}-s-t=3 & ---(1)\\2s-3t=a-4 & ---(2)\\at=1 & ---(3)\end{cases}$ From (1), $s=-3-t$ Subst into (2), we get $2(-3-t)-3t=a-4 \Rightarrow 5t=-2-a$ Subst (3), we get $\Rightarrow a(-2-a)=5 \Rightarrow a^2+2a+5=0 \Rightarrow (a+1)^2+4=0$ Since, there are no real solutions for the equation $(a+1)^2+4=0$, there does not exist $a\in\mathbb{R}$ such that the l and m intersect. Hence, the 2 lines have no common point.

(iii) $\cos 60^\circ = \dfrac{\left|\begin{pmatrix}1\\3\\a\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{1+9+a^2}\sqrt{1}}$

$\Rightarrow \dfrac{1}{2} = \dfrac{|a|}{\sqrt{10+a^2}}$

$\Rightarrow 2|a| = \sqrt{10+a^2}$

$\Rightarrow 4a^2 = 10 + a^2$

$\Rightarrow 3a^2 = 10$

$\Rightarrow a = \pm\sqrt{\dfrac{10}{3}}$

10 Referred to an origin O, the position vectors of two points A and B are **a** and **b** respectively. A line l has vector equation given by $\mathbf{r} = \frac{1}{3}\mathbf{a} + \lambda(\mathbf{2b} - \mathbf{a})$, where $\lambda \in \mathbb{R}$.

The point N is the foot of perpendicular from A to l. It is given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$ and **a** is perpendicular to **b**.

Find the position vector of N in terms of **a** and **b**. [5]

JPJC Promo 9758/2021/Q6(b)(i)	
(b)(i) **5m**	Since N lies on l, $\overrightarrow{ON} = \frac{1}{3}\mathbf{a} + \lambda(\mathbf{2b} - \mathbf{a})$ for some $\lambda \in \mathbb{R}$ Since AN is perpendicular to l, $\overrightarrow{AN} \bullet (2\mathbf{b} - \mathbf{a}) = 0$ $\left(\left(-\frac{2}{3} - \lambda\right)\mathbf{a} + 2\lambda\mathbf{b}\right) \bullet (2\mathbf{b} - \mathbf{a}) = 0$ $\left(-\frac{2}{3} - \lambda\right)\mathbf{a} \bullet 2\mathbf{b} + \left(\frac{2}{3} + \lambda\right)\mathbf{a} \bullet \mathbf{a} + 4\lambda\mathbf{b} \bullet \mathbf{b} - \mathbf{2}\lambda\mathbf{a} \bullet \mathbf{b} = 0$ Since **a** and **b** are perpendicular, $\therefore \mathbf{a} \bullet \mathbf{b} = 0$ $\left(\frac{2}{3} + \lambda\right)\mathbf{a} \bullet \mathbf{a} + 4\lambda\mathbf{b} \bullet \mathbf{b} = 0$ $\left(\frac{2}{3} + \lambda\right)\|\mathbf{a}\|^2 + 4\lambda\|\mathbf{b}\|^2 = 0$ $\left(\frac{2}{3} + \lambda\right)(4) + 4\lambda(1) = 0$ $\lambda = -\frac{1}{3}$ $\overrightarrow{ON} = \frac{1}{3}\mathbf{a} - \frac{1}{3}(2\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} - \frac{2}{3}\mathbf{b}$

11 The four points A, B, C, D have position vectors $p\mathbf{i}$, $\mathbf{j}+q\mathbf{k}$, $\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ respectively, where p and q are positive real numbers. Find vector equations for the lines AB and CD.

(i) Given that the lines AB and CD intersect at a point R, express q in terms of p, and show that the two lines cannot be perpendicular. Show also that $\dfrac{AR}{AB}=\dfrac{CR}{CD}$.

(ii) Given that AB and CD are perpendicular, express q in terms of p, and show that the angle between AC and BD is equal to the angle between AD and BC.

$$\overrightarrow{OA}=\begin{pmatrix}p\\0\\0\end{pmatrix} \quad \overrightarrow{OB}=\begin{pmatrix}0\\1\\q\end{pmatrix} \quad \overrightarrow{OC}=\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \overrightarrow{OD}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\overrightarrow{AB}=\begin{pmatrix}0\\1\\q\end{pmatrix}-\begin{pmatrix}p\\0\\0\end{pmatrix}=\begin{pmatrix}-p\\1\\q\end{pmatrix},\ \overrightarrow{CD}=\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Line AB : $\mathbf{r}=\begin{pmatrix}p\\0\\0\end{pmatrix}+t\begin{pmatrix}-p\\1\\q\end{pmatrix}, t\in\mathbb{R}$ Line CD: $\mathbf{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+s\begin{pmatrix}1\\1\\-1\end{pmatrix}, s\in\mathbb{R}$

(i) $$\begin{pmatrix}p\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-p\\1\\q\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\mu\begin{pmatrix}1\\1\\-1\end{pmatrix}\Rightarrow\begin{cases}p-\lambda p=\mu\\ \lambda=\mu\\ \lambda q=1-\mu\end{cases}\Rightarrow\begin{cases}p-\mu p=\mu\\ \mu q=1-\mu\end{cases}\Rightarrow\frac{1}{\mu}=\frac{1+p}{p}=\frac{q+1}{1}\Rightarrow$$

$$\begin{pmatrix}-p\\1\\q\end{pmatrix}\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}=-p+1-q=-p+1-\frac{1}{p}=-\frac{p^2-p+1}{p}=-\frac{(p-0.5)^2+0.75}{p}<0\quad\forall p>0$$

, so the lines are not perpendicular.

(Alternatively, since $\text{discriminant}=(-1)^2-4(1)(1)<0,$ then there is no p such that

$\begin{pmatrix}-p\\1\\ \frac{1}{p}\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=0$. Hence lines AB and CD cannot be perpendicular.)

$l_{AB}:\underset{\sim}{r}=\overrightarrow{OA}+\lambda\overrightarrow{AB}$

At point of intersection R, $\overrightarrow{OR}=\overrightarrow{OA}+\lambda\overrightarrow{AB}$ where $\lambda=\dfrac{p}{1+p}>0$

$$\Rightarrow\overrightarrow{AR}=\lambda\overrightarrow{AB}\Rightarrow\left|\overrightarrow{AR}\right|=\lambda\left|\overrightarrow{AB}\right|\Rightarrow\frac{AR}{AB}=\lambda$$

$l_{CD}: \underset{\sim}{r} = \overrightarrow{OC} + \mu\overrightarrow{CD}$

At point of intersection R, $\overrightarrow{OR} = \overrightarrow{OC} + \mu\overrightarrow{CD}$ where $\mu = \dfrac{p}{1+p} > 0$

$$\Rightarrow \overrightarrow{CR} = \mu\overrightarrow{CD} \Rightarrow \left|\overrightarrow{CR}\right| = \mu\left|\overrightarrow{CD}\right| \Rightarrow \frac{CR}{CD} = \mu$$

Since $\lambda = \mu$, $\dfrac{AR}{AB} = \dfrac{CR}{CD}$.

(Alternatively,

$$\overrightarrow{OR} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{p}{p+1}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{p}{p+1} \\ \frac{p}{p+1} \\ \frac{1}{p+1} \end{pmatrix} = \frac{1}{p+1}\begin{pmatrix} p \\ p \\ 1 \end{pmatrix}.$$

$$\overrightarrow{AR} = \frac{1}{p+1}\begin{pmatrix} p-p^2-p \\ p \\ 1 \end{pmatrix} = \frac{1}{p+1}\begin{pmatrix} -p^2 \\ p \\ 1 \end{pmatrix},\ \overrightarrow{CR} = \frac{1}{p+1}\begin{pmatrix} p \\ p \\ -p \end{pmatrix} = \frac{p}{p+1}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

$$\overrightarrow{AB} = \begin{pmatrix} -p \\ 1 \\ \frac{1}{p} \end{pmatrix} = \frac{1}{p}\begin{pmatrix} -p^2 \\ p \\ 1 \end{pmatrix},\ \overrightarrow{CD} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Clearly, $\dfrac{AR}{AB} = \dfrac{p}{p+1}$ and $\dfrac{CR}{CD} = \dfrac{p}{p+1}$ (shown))

(ii)

$$\begin{pmatrix} -p \\ 1 \\ q \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \Rightarrow -p+1-q = 0.$$

$\therefore q = 1-p$

Let θ be the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

$$\overrightarrow{AC} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -p \\ 0 \\ 1 \end{pmatrix},\ \overrightarrow{BD} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ p-1 \end{pmatrix}.$$

$$\cos\theta = \frac{\begin{pmatrix} -p \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ p-1 \end{pmatrix}}{\sqrt{(-p)^2+1^2} \cdot \sqrt{1^2+(p-1)^2}} = \frac{-1}{\sqrt{\left(1+p^2\right)\left(1+(p-1)^2\right)}}.$$

Let ϕ be the angle between $\overrightarrow{AD}$ and $\overrightarrow{BC}$.

$$\overrightarrow{AD}=\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}p\\0\\0\end{pmatrix}=\begin{pmatrix}1-p\\1\\0\end{pmatrix},\ \overrightarrow{BC}=\begin{pmatrix}0\\0\\1\end{pmatrix}-\begin{pmatrix}0\\1\\1-p\end{pmatrix}=\begin{pmatrix}0\\-1\\p\end{pmatrix}.$$

$$\cos\phi=\frac{\begin{pmatrix}1-p\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\p\end{pmatrix}}{\sqrt{(1-p)^2+1^2}\cdot\sqrt{(-1)^2+(p)^2}}=\frac{-1}{\sqrt{\left(p^2+1\right)\left(1+\left(p-1\right)^2\right)}}.$$

$$\cos\theta=\cos\phi$$

Since $0^\circ\le\theta,\phi\le180^\circ$, then $\theta=\phi$ (shown)

Alternative (finding acute angle between the lines):

Since $\cos\left(180^\circ-\theta\right)=-\cos\theta$, numerical values of $\cos\left(180^\circ-\theta\right)$ and $\cos\theta$ are the same.

If we define θ to be the acute angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$, and

ϕ to be the acute angle between $\overrightarrow{AD}$ and $\overrightarrow{BC}$, then

$$\cos\theta=\frac{\left|\begin{pmatrix}-p\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\p-1\end{pmatrix}\right|}{\sqrt{(-p)^2+1^2}\cdot\sqrt{1^2+(p-1)^2}}=\frac{1}{\sqrt{\left(1+p^2\right)\left(1+\left(p-1\right)^2\right)}}\text{ and}$$

$$\cos\phi=\frac{\left|\begin{pmatrix}1-p\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\p\end{pmatrix}\right|}{\sqrt{(1-p)^2+1^2}\cdot\sqrt{(-1)^2+(p)^2}}=\frac{1}{\sqrt{\left(p^2+1\right)\left(1+\left(p-1\right)^2\right)}}.$$

Since $\cos\theta=\cos\phi$ and both θ and ϕ are acute, $\theta=\phi$ (shown).

Comment: Part (i) says that

AB and CD intersect at one point. $\Rightarrow$ AB and CD cannot be perpendicular.

Hence the contrapositive is also true, that is

AB and CD are perpendicular $\Rightarrow$ AB and CD do not intersect at any point (they are skew).

THE END