RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 2A: Vectors I
Vector Algebra, Ratio Theorem, Scalar and Vector Products

Section A (Basic Questions)

1 (a) The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a}-2\mathbf{b}+\mathbf{c}=\mathbf{0}$. If $\mathbf{a}=4\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-2\mathbf{k}$, find a unit vector in the direction of $\mathbf{c}$.

(b) Find a vector $\mathbf{a}$ such that $\mathbf{a}$ is parallel to the vector $8\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and is equal in magnitude to the vector $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$.

(c) Find the position vector of P if OP is of length 5 units and is in the direction $2\mathbf{i}-\mathbf{j}+4\mathbf{k}$.

[**(a)** $\frac{1}{\sqrt{30}}\begin{pmatrix}-2\\1\\-5\end{pmatrix}$ **(b)** $\frac{1}{3}\begin{pmatrix}8\\1\\4\end{pmatrix}$ or $-\frac{1}{3}\begin{pmatrix}8\\1\\4\end{pmatrix}$ **(c)** $\frac{5}{\sqrt{21}}\begin{pmatrix}2\\-1\\4\end{pmatrix}$]

Solution:	
(a)	$\mathbf{a}=\begin{pmatrix}4\\-1\\1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$ Given that $\mathbf{a}-2\mathbf{b}+\mathbf{c}=\mathbf{0}$, we have $\mathbf{c}=2\mathbf{b}-\mathbf{a} =2\begin{pmatrix}1\\0\\-2\end{pmatrix}-\begin{pmatrix}4\\-1\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\-5\end{pmatrix}$ Unit vector in direction of $\mathbf{c}$ $=\hat{\mathbf{c}}=\frac{\mathbf{c}}{\lvert\mathbf{c}\rvert}=\frac{1}{\sqrt{(-2)^2+1^2+(-5)^2}}\begin{pmatrix}-2\\1\\-5\end{pmatrix}=\frac{1}{\sqrt{30}}\begin{pmatrix}-2\\1\\-5\end{pmatrix}$ [**Note:** The question states that the unit vector and **c** must be in the same direction]
(b)	Given $\mathbf{a}\ //\begin{pmatrix}8\\1\\4\end{pmatrix} \Rightarrow \mathbf{a}=\lambda\begin{pmatrix}8\\1\\4\end{pmatrix}=\begin{pmatrix}8\lambda\\\lambda\\4\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}\setminus\{0\}$. and $\lvert\mathbf{a}\rvert=\left\lvert\begin{pmatrix}1\\2\\-2\end{pmatrix}\right\rvert=\sqrt{1^2+2^2+(-2)^2}=\sqrt{9}=3$

	$\therefore \quad \sqrt{(8\lambda)^2+\lambda^2+(4\lambda)^2}=3$ $81\lambda^2=9$ $\lambda^2=\frac{1}{9}$ $\lambda=\pm\frac{1}{3}$ Hence, $\mathbf{a}=\pm\frac{1}{3}\begin{pmatrix}8\\1\\4\end{pmatrix}$ **OR :** $\mathbf{a}=\lvert\mathbf{a}\rvert\hat{\mathbf{a}}=3\times\left[\pm\frac{1}{\sqrt{8^2+1^2+4^2}}\begin{pmatrix}8\\1\\4\end{pmatrix}\right]=\pm\frac{3}{\sqrt{81}}\begin{pmatrix}8\\1\\4\end{pmatrix}=\pm\frac{1}{3}\begin{pmatrix}8\\1\\4\end{pmatrix}$ [**Note:** The question states that **a** and $8\mathbf{i}+\mathbf{j}+4\mathbf{k}$ are parallel to each other. i.e. they could be in the same direction or opposite direction. That is the reason for **2** possible answers. On the other hand, the question ask for a vector **a**, so you can just give any of the 2 possible answers.]
(c)	Given $\lvert\overrightarrow{OP}\rvert=5$ and $\overrightarrow{OP}=\lambda\begin{pmatrix}2\\-1\\4\end{pmatrix}$ $\Rightarrow \sqrt{(2\lambda)^2+(-\lambda)^2+(4\lambda)^2}=5$ $21\lambda^2=25$ $\lambda=\pm\frac{5}{\sqrt{21}}$ But since $\overrightarrow{OP}$ is in the direction of $\begin{pmatrix}2\\-1\\4\end{pmatrix}$, $\lambda=\frac{5}{\sqrt{21}}$ i.e. $\overrightarrow{OP}=\frac{5}{\sqrt{21}}\begin{pmatrix}2\\-1\\4\end{pmatrix}$ **OR :** $\overrightarrow{OP}=\lvert\overrightarrow{OP}\rvert\times$ unit vector in the direction of $\overrightarrow{OP}=5\times\frac{1}{\sqrt{4+1+16}}\begin{pmatrix}2\\-1\\4\end{pmatrix}=\frac{5}{\sqrt{21}}\begin{pmatrix}2\\-1\\4\end{pmatrix}$

2 The points P and Q have position vectors $-3\mathbf{i}-\mathbf{j}+17\mathbf{k}$ and $8\mathbf{i}+9\mathbf{j}+2\mathbf{k}$ respectively when referred from the origin O.

(a) Evaluate $\overrightarrow{OP}\cdot\overrightarrow{OQ}$ and $\overrightarrow{OP}\times\overrightarrow{OQ}$.

(b) Find the position vector of the point R if

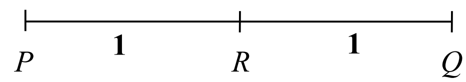
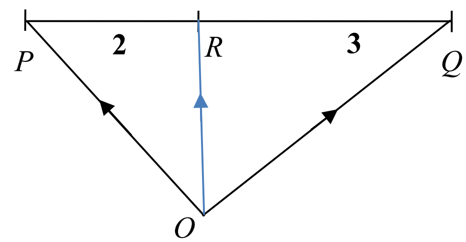
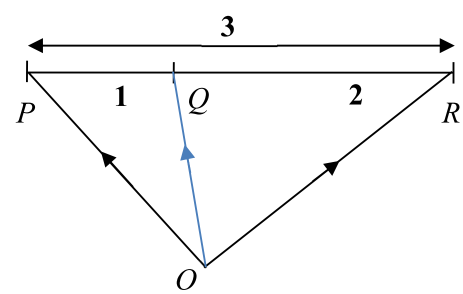
(i) R is the mid-point of PQ;

(ii) R lies on the line segment PQ such that $PR:RQ=2:3$;

(iii) R lies on PQ produced so that $PR:QR=3:2$.

[**(a)** 1, $\begin{pmatrix}-155\\142\\-19\end{pmatrix}$ **(b)(i)** $\frac{1}{2}\begin{pmatrix}5\\8\\19\end{pmatrix}$ **(ii)** $\begin{pmatrix}7/5\\3\\11\end{pmatrix}$ **(iii)** $\begin{pmatrix}30\\29\\-28\end{pmatrix}$]

Solution:	
(a)	$\overrightarrow{OP}=\begin{pmatrix}-3\\-1\\17\end{pmatrix}$ and $\overrightarrow{OQ}=\begin{pmatrix}8\\9\\2\end{pmatrix}$ **Dot/Scalar Product** $\overrightarrow{OP}\cdot\overrightarrow{OQ}=\begin{pmatrix}-3\\-1\\17\end{pmatrix}\cdot\begin{pmatrix}8\\9\\2\end{pmatrix}=-24-9+34=1$ **Cross/Vector Product** $\overrightarrow{OP}\times\overrightarrow{OQ}=\begin{pmatrix}-3\\-1\\17\end{pmatrix}\times\begin{pmatrix}8\\9\\2\end{pmatrix}=\begin{pmatrix}-2-153\\-(-6-136)\\-27+8\end{pmatrix}=\begin{pmatrix}-155\\142\\-19\end{pmatrix}$ **Note :** If $\mathbf{a}\times\mathbf{b}=\mathbf{c}$, then $\mathbf{c}\perp\mathbf{a}$ and $\mathbf{c}\perp\mathbf{b}$. Hence, after performing a cross product, you can check your answer by checking that $\begin{pmatrix}-155\\142\\-19\end{pmatrix}\cdot\begin{pmatrix}-3\\-1\\17\end{pmatrix}=\ldots=0$ & $\begin{pmatrix}-155\\142\\-19\end{pmatrix}\cdot\begin{pmatrix}8\\9\\2\end{pmatrix}=\ldots=0$
(b)	$\overrightarrow{OP}=\begin{pmatrix}-3\\-1\\17\end{pmatrix}$ and $\overrightarrow{OQ}=\begin{pmatrix}8\\9\\2\end{pmatrix}$

<table>
<tr><td>(b)
(i)</td><td>R is the mid-point of PQ;
By Ratio Theorem,
$$\overrightarrow{OR} = \frac{\overrightarrow{OP}+\overrightarrow{OQ}}{2} = \frac{1}{2}\begin{pmatrix}5\\8\\19\end{pmatrix}$$
</td></tr>
<tr><td>(b)
(ii)</td><td>R lies on the line segment PQ such that $PR:RQ=2:3$.
By Ratio Theorem,
$$\overrightarrow{OR} = \frac{3\overrightarrow{OP}+2\overrightarrow{OQ}}{5} = \begin{pmatrix}\frac{7}{5}\\3\\11\end{pmatrix}$$
</td></tr>
<tr><td>(b)
(iii)</td><td>R lies on PQ produced so that $PR:QR=3:2$.
By Ratio Theorem,
$$\overrightarrow{OQ} = \frac{2\overrightarrow{OP}+\overrightarrow{OR}}{3}$$
$$\overrightarrow{OR} = 3\overrightarrow{OQ}-2\overrightarrow{OP} = \begin{pmatrix}30\\29\\-28\end{pmatrix}$$
</td></tr>
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3 **(a)** Find the value of p so that the vectors $2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $p\mathbf{i}+\mathbf{j}+p\mathbf{k}$ are perpendicular.

(b) If $\mathbf{a}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=\mathbf{j}+\mathbf{k}$, find a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

[(a) $p=-\frac{1}{3}$ **(b)** $\frac{1}{\sqrt{11}}\begin{pmatrix}3\\-1\\1\end{pmatrix}$ or $\frac{1}{\sqrt{11}}\begin{pmatrix}-3\\1\\-1\end{pmatrix}$ **]**

Solution:	
(a)	Given that $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}p\\1\\p\end{pmatrix}$ are perpendicular, $\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}p\\1\\p\end{pmatrix}=0$ $2p+1+p=0$ $p=-\frac{1}{3}$
(b)	$\mathbf{a}=\begin{pmatrix}1\\2\\-1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ **Using cross product :** A vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}=\mathbf{a}\times\mathbf{b}$ $=\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}3\\-1\\1\end{pmatrix}$ So, a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{\sqrt{11}}\begin{pmatrix}-3\\1\\-1\end{pmatrix}$ or $\frac{1}{\sqrt{11}}\begin{pmatrix}3\\-1\\1\end{pmatrix}$ [**Note:** the cross product of 2 vectors **a** and **b** will give a vector that is perpendicular to the 2 vectors **a** and **b**]