RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Additional Practice Questions for Chapter 2A: Vectors I
Vector Algebra, Ratio Theorem, Scalar and Vector Products

1 Referred to an origin O, points A and B have position vectors given respectively by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k} \text{ and } \overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}.$$

The point P on AB is such that $AP : PB = \lambda : 1 - \lambda$. Show that

$$\overrightarrow{OP} = (1+\lambda)\mathbf{i} + (2-5\lambda)\mathbf{j} + (-2+8\lambda)\mathbf{k}.$$

(i) Find the value of λ for which OP is perpendicular to AB.

(ii) Find the value of λ for which angles AOP and POB are equal.

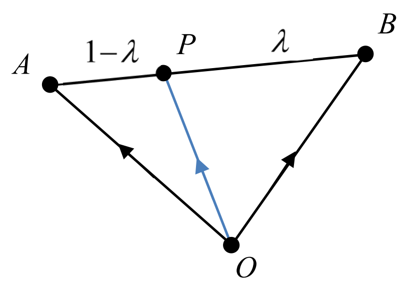
9205/1984/02/Q5

Solution:

By Ratio Theorem,

$$\overrightarrow{OP} = \frac{(1-\lambda)\overrightarrow{OA} + \lambda\overrightarrow{OB}}{(1-\lambda)+\lambda}$$

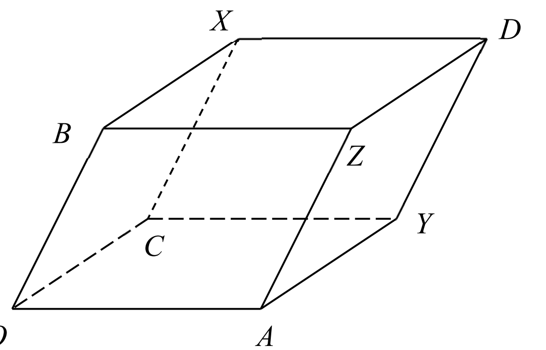
$$= (1-\lambda)\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = \begin{pmatrix} (1-\lambda)+2\lambda \\ 2(1-\lambda)-3\lambda \\ -2(1-\lambda)+6\lambda \end{pmatrix}$$



$$= \begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix}, \text{ that is, } \overrightarrow{OP} = (1+\lambda)\mathbf{i} + (2-5\lambda)\mathbf{j} + (-2+8\lambda)\mathbf{k} \text{ (shown)}$$

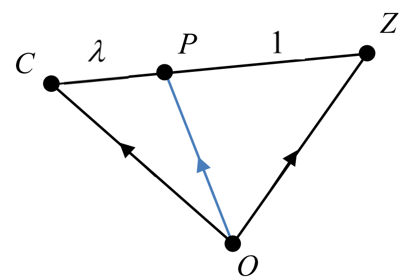
(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\-3\\6\end{pmatrix}-\begin{pmatrix}1\\2\\-2\end{pmatrix}=\begin{pmatrix}1\\-5\\8\end{pmatrix}$ $OP\perp AB\Rightarrow\overrightarrow{OP}\cdot\overrightarrow{AB}=0$ $\Rightarrow\begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\8\end{pmatrix}=0$ $\Rightarrow(1-10-16)+\lambda(1+25+64)=0$ $\Rightarrow\lambda=\frac{25}{90}=\frac{5}{18}$
(ii)	$\angle AOP=\angle POB \quad \Rightarrow\cos\angle AOP=\cos\angle POB$ $\Rightarrow\frac{\overrightarrow{OP}\cdot\overrightarrow{OA}}{\lvert\overrightarrow{OP}\rvert\lvert\overrightarrow{OA}\rvert}=\frac{\overrightarrow{OP}\cdot\overrightarrow{OB}}{\lvert\overrightarrow{OP}\rvert\lvert\overrightarrow{OB}\rvert}$ $\Rightarrow\frac{1}{3}\begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=\frac{1}{7}\begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}$ $\Rightarrow\frac{1}{3}\left[(1+4+4)+\lambda(1-10-16)\right]=\frac{1}{7}\left[(2-6-12)+\lambda(2+15+48)\right]$ $\Rightarrow 7(9-25\lambda)=3(-16+65\lambda)$ $\Rightarrow\lambda=\frac{3}{10}$ OR : Use the fact that in a rhombus, the diagonal is an angle bisector. We can form a rhombus with the vectors $\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}$ and $\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}$ which has diagonal $\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}+\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}$. So $OP\,//\,\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}+\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}$ ie. $\overrightarrow{OP}=k\left(\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}+\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}\right)$ for some $k\in\mathbb{R}$. By Ratio Theorem, since P on AB is such that $AP:PB=\lambda:1-\lambda$. $\overrightarrow{OP}=(1-\lambda)\mathbf{a}+\lambda\mathbf{b}$ $(1-\lambda)\mathbf{a}+\lambda\mathbf{b}=k\left(\frac{\mathbf{a}}{\lvert\mathbf{a}\rvert}+\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}\right)$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non parallel vectors, $(1-\lambda)=\frac{k}{\lvert\mathbf{a}\rvert}$ ---(1) and $\lambda=\frac{k}{\lvert\mathbf{b}\rvert}$---(2) $(2)\div(1)\Rightarrow\frac{\lambda}{1-\lambda}=\frac{\lvert\mathbf{a}\rvert}{\lvert\mathbf{b}\rvert}=\frac{3}{7}\Rightarrow\lambda=\frac{3}{10}$

2 Referring to an origin O, the position vectors of points A, B and C are given by $\overrightarrow{OA} = 7\mathbf{i}$, $\overrightarrow{OB} = \mathbf{i} + 4\mathbf{k}$ and $\overrightarrow{OC} = \mathbf{i} + 4\mathbf{j}$ respectively. A parallelepiped has OA, OB, OC as three edges, and the remaining vertices are X, Y, Z and D as shown in the diagram.



(i) Write down the position vector of Z in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [1]

(ii) The point P divides CZ such that $\frac{CP}{PZ} = \lambda$. Given that $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{CZ}$, find the value of λ and evaluate $\left|\overrightarrow{OP}\right|$. [6]

IJCPrelim9233/2006/01/Q14(i), (ii)	
Solution:	
(i)	$\overrightarrow{OZ} = \overrightarrow{OA} + \overrightarrow{AZ}$ $= \overrightarrow{OA} + \overrightarrow{OB} = 8\mathbf{i} + 4\mathbf{k}$
(ii)	Since P divides CZ such that $\frac{CP}{PZ} = \lambda$, by Ratio Theorem, $\overrightarrow{OP} = \frac{\overrightarrow{OC} + \lambda\overrightarrow{OZ}}{1+\lambda}$ $= \frac{1}{1+\lambda}\left[\begin{pmatrix}1\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}8\\0\\4\end{pmatrix}\right]$ 

Since $OP \perp CZ$, $\overrightarrow{OP} \cdot \overrightarrow{CZ} = 0$ where $\overrightarrow{CZ} = \overrightarrow{OZ} - \overrightarrow{OC} = \begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix}$

$$\frac{1}{1+\lambda}\left[\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix}\right] \cdot \begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix} = 0$$

$$7 - 16 + (56 + 16)\lambda = 0$$

$$\lambda = \frac{1}{8}$$

$$\therefore \overrightarrow{OP} = \frac{1}{1+\frac{1}{8}} \begin{pmatrix} 1+1 \\ 4 \\ 1/2 \end{pmatrix} = \frac{4}{9} \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$$

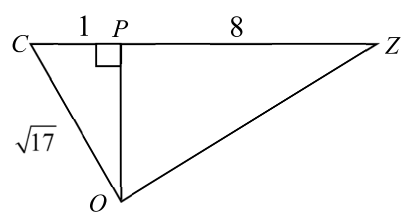
and $\left|\overrightarrow{OP}\right| = \frac{4}{9}\sqrt{4^2 + 8^2 + 1^2} = 4$

Alternative Method

From (i), $\overrightarrow{OZ} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix}$. So $\overrightarrow{CZ} = \overrightarrow{OZ} - \overrightarrow{OC} = \begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix}$

$CZ = \sqrt{7^2 + 2(4^2)} = 9$

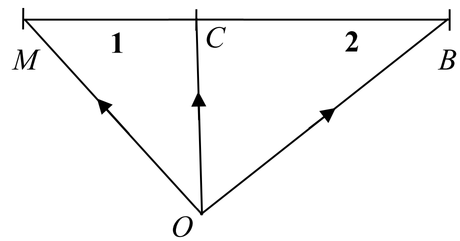
Since $\lambda = \frac{1}{8}$, $\frac{CP}{PZ} = \frac{1}{8} \Rightarrow CP = 1$ and $PZ = 8$



$\therefore \left|\overrightarrow{OP}\right| = \sqrt{17 - 1} = 4$

3 The position vectors of A and B with respect to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively. M is the mid-point of OA and C is the point on MB such that $\overrightarrow{CB} = 2\overrightarrow{MC}$.

Given that $|\mathbf{a}| = 6$, $|\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 45°, evaluate the exact value of the scalar product of $-\mathbf{b}$ and $\mathbf{c}$.

MI Prelim 9233/2006/01/Q1
Solution:
Given that M is the mid-point of OA, $\overrightarrow{OM} = \frac{1}{2}\mathbf{a}$ Given $\overrightarrow{CB} = 2\overrightarrow{MC}$ $\Rightarrow \frac{CB}{MC} = \frac{2}{1}$ $\Rightarrow CB : MC = 2:1$ By Ratio Theorem, $\mathbf{c} = \frac{2\overrightarrow{OM} + \overrightarrow{OB}}{3} = \frac{1}{3}(\mathbf{a}+\mathbf{b})$ 
$-\mathbf{b}\cdot\mathbf{c} = -\mathbf{b}\cdot\frac{1}{3}(\mathbf{a}+\mathbf{b})$ $= -\frac{1}{3}(\mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b})$ $= -\frac{1}{3}\left(|\mathbf{b}||\mathbf{a}|\cos 45^\circ + |\mathbf{b}|^2\right)$ $= -\frac{1}{3}\left[(5)(6)\frac{\sqrt{2}}{2} + 5^2\right]$ $= -\frac{1}{3}\left(15\sqrt{2} + 25\right)$ $= -\frac{5}{3}\left(5 + 3\sqrt{2}\right)$

4 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}\ .$$

(i) Find the size of angle OAB. [2]

The point C has position vector $\mathbf{c}$ given by $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, where λ and μ are positive constants. Given that the area of triangle OAC is twice that of triangle OBC,

(ii) find μ in terms of λ, [3]

(iii) hence, if $OC = \sqrt{118}$, find the position vector $\mathbf{c}$. [4]

JPJC Prelim 9758/2021/01/Q2	
Solution:	
(i)	$\overrightarrow{BA} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}0\\-1\\-1\end{pmatrix}$ $\cos\measuredangle OAB = \dfrac{\overrightarrow{OA}\bullet\overrightarrow{BA}}{\lvert\overrightarrow{OA}\rvert\lvert\overrightarrow{BA}\rvert} = \dfrac{\begin{pmatrix}1\\1\\1\end{pmatrix}\bullet\begin{pmatrix}0\\-1\\-1\end{pmatrix}}{\sqrt{3}\sqrt{2}} = \dfrac{-2}{\sqrt{6}}$ $\measuredangle OAB = 144.7°$ (1 d.p)
(ii)	Area of $\Delta OAC = \dfrac{1}{2}\lvert \underset{\sim}{a} \times (\lambda\underset{\sim}{a} + \mu\underset{\sim}{b})\rvert = \dfrac{\mu}{2}\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert$ since $\mu > 0$ Area of $\Delta OBC = \dfrac{1}{2}\lvert \underset{\sim}{b} \times (\lambda\underset{\sim}{a} + \mu\underset{\sim}{b})\rvert = \dfrac{\lambda}{2}\lvert\underset{\sim}{b}\times\underset{\sim}{a}\rvert$ since $\lambda > 0$ Given area of Δ OAC is twice that of Δ OBC, $\dfrac{\mu}{2}\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert = 2\dfrac{\lambda}{2}\lvert\underset{\sim}{b}\times\underset{\sim}{a}\rvert$ Since $\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert = \lvert\underset{\sim}{b}\times\underset{\sim}{a}\rvert$, $\mu = 2\lambda$

(iii)	$OC=\sqrt{118}$ $\lvert\lambda\underset{\sim}{a}+\mu\underset{\sim}{b}\rvert=\sqrt{118}$ $\lvert\lambda\underset{\sim}{a}+2\lambda\underset{\sim}{b}\rvert=\sqrt{118}$ since $\mu=2\lambda$ $\left\lvert\begin{pmatrix}\lambda+2\lambda\\ \lambda+4\lambda\\ \lambda+4\lambda\end{pmatrix}\right\rvert=\sqrt{118}$ $\sqrt{(3\lambda)^2+(5\lambda)^2+(5\lambda)^2}=\sqrt{118}$ $9\lambda^2+25\lambda^2+25\lambda^2=118$ $59\lambda^2=118$ $\lambda=\pm\sqrt{2}$ Since $\lambda>0$, $\lambda=\sqrt{2}$, $\underset{\sim}{c}=\begin{pmatrix}3\sqrt{2}\\ 5\sqrt{2}\\ 5\sqrt{2}\end{pmatrix}$ OR: $\left\lvert\begin{pmatrix}3\lambda\\ 5\lambda\\ 5\lambda\end{pmatrix}\right\rvert=\sqrt{118}$ $\lambda\left\lvert\begin{pmatrix}3\\ 5\\ 5\end{pmatrix}\right\rvert=\sqrt{118}$ since $\lambda>0$ $\lambda\sqrt{3^2+5^2+5^2}=\sqrt{118}$ $\lambda=\frac{\sqrt{118}}{\sqrt{59}}=\sqrt{2}$ $\underset{\sim}{c}=\begin{pmatrix}3\sqrt{2}\\ 5\sqrt{2}\\ 5\sqrt{2}\end{pmatrix}$

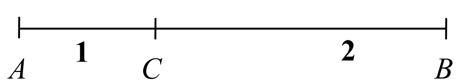
5 With respect to the origin O, the position vectors of the points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Point C lies on AB such that $AC:CB=1:2$. It is given that $\mathbf{a}$ is a unit vector and the length of OB is 2 units.

(i) Give a geometrical interpretation of $|\mathbf{a}\cdot\mathbf{c}|$. [1]

(ii) It is given that the angle AOB is $60°$. By considering $(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})$, find $|2\mathbf{a}-\mathbf{b}|$. [3]

(iii) Find $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(iv) Hence by considering cosine of angle AOC and cosine of angle COB, determine if the line segment OC bisects the angle AOB. [3]

<table>
<tr><td colspan="2">MI Prelim 9758/2021/01/Q4</td></tr>
<tr><td colspan="2">Solution:</td></tr>
<tr><td>(i)</td><td>$|\mathbf{a}\cdot\mathbf{c}|$ is the length of projection of $\mathbf{c}$ onto $\mathbf{a}$.</td></tr>
<tr><td>(ii)</td><td>$$\begin{aligned}(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b}) &= 4(\mathbf{a}\cdot\mathbf{a})-2(\mathbf{a}\cdot\mathbf{b})-2(\mathbf{b}\cdot\mathbf{a})+(\mathbf{b}\cdot\mathbf{b})\\ &= 4|\mathbf{a}|^2-4(\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2 \quad (\because\ \mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{b})\\ &= 4(1)^2-4|\mathbf{a}||\mathbf{b}|\cos 60°+(2)^2\\ &= 4-4(1)(2)\left(\frac{1}{2}\right)+4=4.\end{aligned}$$ $$(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})=|2\mathbf{a}-\mathbf{b}|^2=4 \Rightarrow |2\mathbf{a}-\mathbf{b}|=2.$$</td></tr>
<tr><td>(iii)</td><td>Since C lies on AB such that $AC:CB=1:2$, by Ratio Theorem,
$\mathbf{c}=\dfrac{\mathbf{b}+2\mathbf{a}}{3}$.
</td></tr>
<tr><td>(iv)</td><td>$$\begin{aligned}\cos\angle AOC &= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|}\\ &= \frac{\mathbf{a}\cdot\left(\frac{\mathbf{b}+2\mathbf{a}}{3}\right)}{|\mathbf{a}||\mathbf{c}|} \quad \text{[from (iii)]}\\ &= \frac{1}{3}\left(\frac{\mathbf{a}\cdot\mathbf{b}+2(1^2)}{|\mathbf{c}|}\right) \quad [\text{since } \mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2=1^2]\\ &= \frac{1}{3}\left(\frac{\mathbf{a}\cdot\mathbf{b}+2}{|\mathbf{c}|}\right).\end{aligned}$$</td></tr>
</table>

	$\cos\angle COB = \dfrac{\mathbf{b}\cdot\mathbf{c}}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert}$ $= \dfrac{\mathbf{b}\cdot\left(\dfrac{\mathbf{b}+2\mathbf{a}}{3}\right)}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert}$ [from **(iii)**] $= \dfrac{1}{3}\left(\dfrac{2^2+2(\mathbf{a}\cdot\mathbf{b})}{2\lvert\mathbf{c}\rvert}\right)$ [since $\mathbf{b}\cdot\mathbf{b} = \lvert\mathbf{b}\rvert^2 = 2^2$] $= \dfrac{1}{3}\left(\dfrac{2+\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{c}\rvert}\right) = \cos\angle AOC.$ $\therefore$ line OC bisects the angle AOB.

6 Relative to the origin O, the points A, B, and C, have non-zero position vectors $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a}$ respectively. D lies on AB such that $AD = \lambda AB$, where $0 < \lambda < 1$.

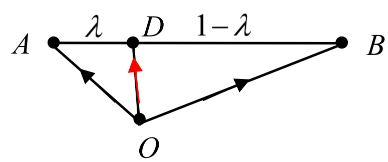
(i) Write down the position vector of the point D. [1]

(ii) The point E is the midpoint of BC. Find the value of λ if E lies on the line OD, and write down the ratio $OE : ED$. [4]

<table>
<tr><td colspan="2">NYJC JC2 Prelim 9758/2019/01/Q5 (modified)</td></tr>
<tr><td colspan="2">Solution:</td></tr>
<tr><td>(i)</td><td>

Since $AD = \lambda AB$,
by Ratio Theorem,

$\overrightarrow{OD} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}, \quad \lambda \in \mathbb{R}$



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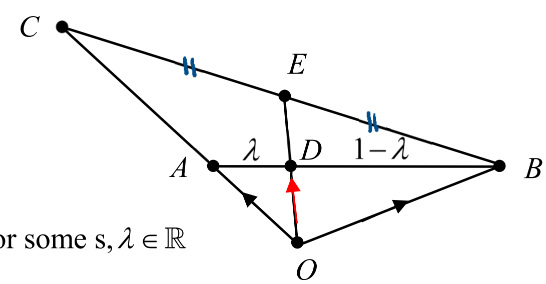
<tr><td>(ii)</td><td>

Since E is the midpoint of BC,
by Ratio Theorem,
$\overrightarrow{OE} = \frac{1}{2}(\mathbf{b}+\mathbf{c}) = \frac{1}{2}(\mathbf{b}+3\mathbf{a})$



Since E lies on OD,
$\overrightarrow{OE} = s\overrightarrow{OD} = s\left[\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right]$, for some $s, \lambda \in \mathbb{R}$

Hence, $s\left[\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right] = \frac{1}{2}(\mathbf{b}+3\mathbf{a})$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors ($\lambda > 0$),
$s\lambda = \frac{1}{2}$ and $s(1-\lambda) = \frac{3}{2}$

Solving, $\lambda = \frac{1}{4}$.

Since $\overrightarrow{OE} = \frac{1}{2}(\mathbf{b}+3\mathbf{a})$ and $\overrightarrow{OD} = \frac{1}{4}(\mathbf{b}+3\mathbf{a}) = \frac{1}{2}\overrightarrow{OE}$.
Hence, $OE : ED = 2 : 1$.

</td></tr>
</table>

7 Referred to origin O, the points P, Q, R and S have position vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ respectively.

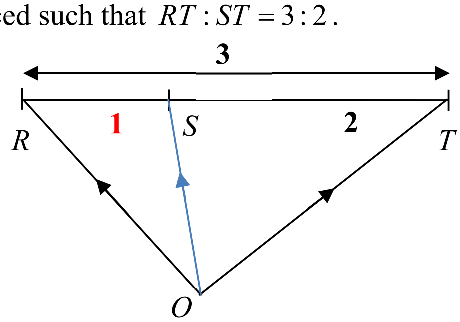
(i) Given that $\alpha\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$ and $\alpha+\beta+1=0$, where α and β are non-zero constants. Show that P, Q and R are collinear. [3]

It is given that the point R lies on PQ produced. The point T lies on line RS produced such that $RT:ST=3:2$.

(ii) Find the position vector of the point T in terms of $\mathbf{r}$ and $\mathbf{s}$. [2]

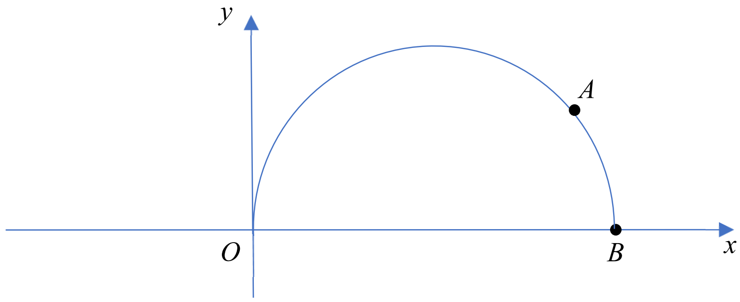
(iii) Give a geometrical interpretation of $\dfrac{|(\mathbf{p}-\mathbf{r})\cdot(\mathbf{s}-\mathbf{r})|}{|\mathbf{s}-\mathbf{r}|}$. [1]

(iv) Find the area of triangle PRT in the form $\gamma|\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}|$, where γ is a constant to be determined. [3]

TMJC Promo 9758/2021/Q10	
Solution:	
(i)	$\alpha+\beta+1=0\Rightarrow\alpha=-\beta-1$ Substitute into $\alpha\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$: $(-\beta-1)\mathbf{p}+\beta\mathbf{q}+\mathbf{r}=\mathbf{0}$ $\beta(\mathbf{q}-\mathbf{p})+(\mathbf{r}-\mathbf{p})=\mathbf{0}$ $\beta\overrightarrow{PQ}+\overrightarrow{PR}=\mathbf{0}$ $\overrightarrow{PQ}=k\overrightarrow{PR},\quad k=-\dfrac{1}{\beta},\ \beta\neq0$ Since $\overrightarrow{PQ}$ is parallel to $\overrightarrow{PR}$, and P is a common point, P, Q and R are collinear. (shown)
(ii)	Since T lies on line RS produced such that $RT:ST=3:2$. $\overrightarrow{OS}=\dfrac{\overrightarrow{OT}+2\overrightarrow{OR}}{3}$ $\mathbf{s}=\dfrac{\overrightarrow{OT}+2\mathbf{r}}{3}$ $\overrightarrow{OT}=3\mathbf{s}-2\mathbf{r}$ 
(iii)	$\dfrac{\lvert(\mathbf{p}-\mathbf{r})\cdot(\mathbf{s}-\mathbf{r})\rvert}{\lvert\mathbf{s}-\mathbf{r}\rvert}=\left\lvert(\mathbf{p}-\mathbf{r})\cdot\dfrac{(\mathbf{s}-\mathbf{r})}{\lvert\mathbf{s}-\mathbf{r}\rvert}\right\rvert=\left\lvert\overrightarrow{\mathbf{RP}}\cdot\dfrac{\overrightarrow{\mathbf{RS}}}{\lvert\overrightarrow{\mathbf{RS}}\rvert}\right\rvert$ is the length of projection of $\overrightarrow{RP}$ on $\overrightarrow{RS}$.
(iv)	Area of triangle $PRT=\dfrac{1}{2}\lvert\overrightarrow{RP}\times\overrightarrow{RT}\rvert$

	$=\frac{1}{2}\left\|(\mathbf{p}-\mathbf{r})\times(3\mathbf{s}-2\mathbf{r}-\mathbf{r})\right\|$ $=\frac{1}{2}\left\|(\mathbf{p}-\mathbf{r})\times(3\mathbf{s}-3\mathbf{r})\right\|$ $=\frac{3}{2}\left\|\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}+\mathbf{r}\times\mathbf{r}\right\|$ $=\frac{3}{2}\left\|\mathbf{p}\times\mathbf{s}-\mathbf{p}\times\mathbf{r}-\mathbf{r}\times\mathbf{s}\right\| \quad (\because \mathbf{r}\times\mathbf{r}=\mathbf{0})$

8 The diagram shows a semi-circle with diameter OB. Point A is on the circumference of the semi-circle and point C lies on chord AB such that $AC = 3CB$.



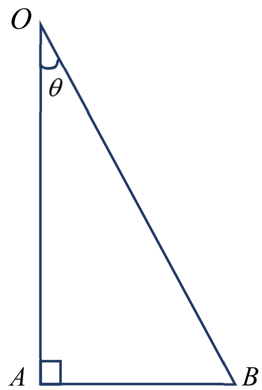
Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors.

(i) Show that $\mathbf{a}.\mathbf{b} = |\mathbf{a}|^2$. [2]

(ii) Find the position vector of C in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(iii) Hence find the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$ if $|\mathbf{a}| = \sqrt{3}$ and $|\mathbf{b}| = 2$. [4]

EJC Promo 9758/2021/01/Q5	
Solution:	
(i)	<u>Method 1</u>: $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ Since $\angle OAB = 90^\circ$ (angle in a semi circle), $\overrightarrow{AB} \perp \overrightarrow{OA} \Rightarrow \overrightarrow{AB} \cdot \overrightarrow{OA} = 0$ $\Rightarrow (\mathbf{b} - \mathbf{a}) \cdot \mathbf{a} = 0$ $\Rightarrow \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{a} = 0$ $\Rightarrow \mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ $\Rightarrow \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2$ ($\because \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$) [shown] <u>Method 2</u>: Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Note that triangle OAB is a right angled triangle.

	 Hence $OA = OB\cos\theta = \lvert\mathbf{b}\rvert\cos\theta$ Since $\mathbf{a}.\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta = \lvert\mathbf{a}\rvert\left(\lvert\mathbf{a}\rvert\right) = \lvert\mathbf{a}\rvert^2$ [shown]
(ii)	$AC = 3CB \Rightarrow \dfrac{AC}{CB} = \dfrac{3}{1}$ By Ratio Theorem, $\overrightarrow{OC} = \dfrac{\overrightarrow{OA} + 3\overrightarrow{OB}}{4} = \dfrac{\mathbf{a}+3\mathbf{b}}{4}$
(iii)	Length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$ $= \left\lvert \overrightarrow{OC} \cdot \dfrac{\overrightarrow{OB}}{\lvert\overrightarrow{OB}\rvert} \right\rvert$ $= \left\lvert \left(\dfrac{\mathbf{a}+3\mathbf{b}}{4}\right) \cdot \dfrac{\mathbf{b}}{\lvert\mathbf{b}\rvert} \right\rvert$ $= \left\lvert \left(\dfrac{\mathbf{a}\cdot\mathbf{b}+3\mathbf{b}\cdot\mathbf{b}}{4\lvert\mathbf{b}\rvert}\right) \right\rvert$ $= \left\lvert \left(\dfrac{\mathbf{a}\cdot\mathbf{b}+3\lvert\mathbf{b}\rvert^2}{4\lvert\mathbf{b}\rvert}\right) \right\rvert$ $= \left\lvert \left(\dfrac{\lvert\mathbf{a}\rvert^2+3\lvert\mathbf{b}\rvert^2}{4\lvert\mathbf{b}\rvert}\right) \right\rvert$ (from (i)) $= \left\lvert \left(\dfrac{3+3(4)}{4(2)}\right) \right\rvert = \dfrac{15}{8}$ since $\lvert\mathbf{a}\rvert = \sqrt{3}$ and $\lvert\mathbf{b}\rvert = 2$.

9 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point P is on the line AB such that $AP:PB = m:n$ where m and n are positive integers. Point C is on OP extended such that $OP:PC = 1:2$.

(i) Show that $\overrightarrow{AC} = \left(\frac{2n-m}{m+n}\right)\mathbf{a} + \left(\frac{3m}{m+n}\right)\mathbf{b}$. [3]

(ii) If $|\mathbf{a}| = |\mathbf{b}|$ and the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{3}$, find the area of the triangle ABC in terms of $|\mathbf{a}|$. [4]

(iii) Find the ratio $AP:PB$ such that AC is parallel to OB. [2]

EJC Prelim 9758/2021/02/Q5	
Solution:	
(i)	By Ratio Theorem, $\overrightarrow{OP} = \frac{m\mathbf{b}+n\mathbf{a}}{m+n}$. $\overrightarrow{OC} = 3\overrightarrow{OP} = 3\left(\frac{m\mathbf{b}+n\mathbf{a}}{m+n}\right)$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 3\left(\frac{m\mathbf{b}+n\mathbf{a}}{m+n}\right) - \mathbf{a} = \frac{3m}{m+n}\mathbf{b} + \frac{3n}{m+n}\mathbf{a} - \mathbf{a}$ $= \frac{3m}{m+n}\mathbf{b} + \left(\frac{3n}{m+n} - 1\right)\mathbf{a} = \frac{3m}{m+n}\mathbf{b} + \left(\frac{3n-n-m}{m+n}\right)\mathbf{a} = \frac{3m}{m+n}\mathbf{b} + \left(\frac{2n-m}{m+n}\right)\mathbf{a}$ (shown)
(ii)	Area of triangle ABC $= \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$ $= \frac{1}{2}\left\|(\mathbf{b}-\mathbf{a}) \times \left(\left(\frac{3m}{m+n}\right)\mathbf{b} + \left(\frac{2n-m}{m+n}\right)\mathbf{a}\right)\right\|$ $= \frac{1}{2}\left\|\frac{3m}{m+n}(\mathbf{b}\times\mathbf{a}) - \frac{2n-m}{m+n}(\mathbf{a}\times\mathbf{b})\right\|$ (since $\mathbf{a}\times\mathbf{a} = \mathbf{b}\times\mathbf{b} = 0$) $= \frac{1}{2}\left\|\frac{3m}{m+n}(\mathbf{b}\times\mathbf{a}) + \frac{2n-m}{m+n}(\mathbf{b}\times\mathbf{a})\right\|$ $= \frac{1}{2}\left\|\frac{2m+2n}{m+n}(\mathbf{b}\times\mathbf{a})\right\| = \left\|(\mathbf{b}\times\mathbf{a})\right\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\|\mathbf{a}\|^2$
(iii)	$\overrightarrow{AC} = \frac{3m}{m+n}\mathbf{b} + \left(\frac{2n-m}{m+n}\right)\mathbf{a}$ Since $\overrightarrow{AC}$ is parallel to $\overrightarrow{OB}$, $\frac{2n-m}{m+n} = 0$ Thus $2n - m = 0 \Rightarrow \frac{m}{n} = 2$ Thus, $AP:PB = 2:1$

10 Referred to the origin O, three distinct and non-collinear points A, B and C have position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively. Point L is the mid-point of BC. The position vector of a point P is given by $(1-k)\mathbf{a}+\frac{k}{2}(\mathbf{b}+\mathbf{c})$, where k is a non-zero constant and $k \neq 1$.

(i) Show that A, L and P are collinear. [3]

(ii) Show that $\frac{1}{2}\left|\overrightarrow{CP}\times\overrightarrow{CB}\right| = \frac{|1-k|}{2}\left|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\right|$. [3]

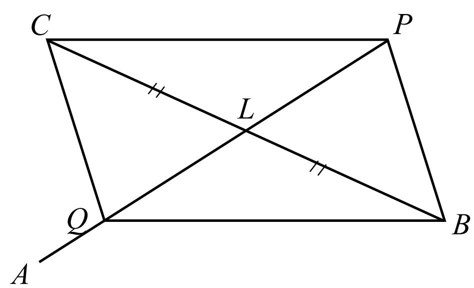
For the rest of the question, let $k=\frac{1}{2}$.

Let point Q be a point on the line passing through A and L. P and Q are distinct points and the areas of triangle CPB and triangle CQB are equal.

(iii) By considering part **(ii)**, find the position vector of Q in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. [4]

(iv) Given that $\left|\overrightarrow{BC}\right|=1$, interpret geometrically $\left|\overrightarrow{LP}\cdot\overrightarrow{BC}\right|$. [1]

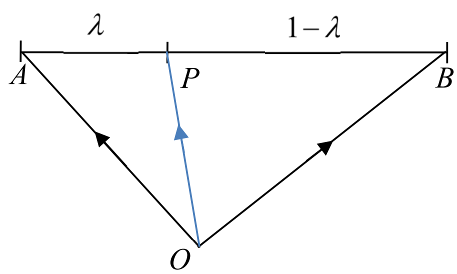
NJC Prelim 9758/2021/02/Q3 (modified)	
Solution:	
(i)	$\overrightarrow{AL}=\overrightarrow{OL}-\overrightarrow{OA}=\frac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}$ $\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=(1-k)\mathbf{a}+k\frac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}$ $=k\left(\frac{\mathbf{b}+\mathbf{c}}{2}-\mathbf{a}\right)$ $=k\overrightarrow{AL}$ Since $\overrightarrow{AP}$ is parallel to $\overrightarrow{AL}$ with a common point L, A, L and P are collinear. (shown)
(ii)	$\frac{1}{2}\left|\overrightarrow{CP}\times\overrightarrow{CB}\right|$ $=\frac{1}{2}\left|\left(\overrightarrow{OP}-\overrightarrow{OC}\right)\times\left(\overrightarrow{OB}-\overrightarrow{OC}\right)\right|$ $=\frac{1}{2}\left|\left((1-k)\mathbf{a}+k\frac{(\mathbf{b}+\mathbf{c})}{2}-\mathbf{c}\right)\times(\mathbf{b}-\mathbf{c})\right|$ $=\frac{1}{2}\left|(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c})+\frac{k}{2}(\mathbf{b}+\mathbf{c})\times(\mathbf{b}-\mathbf{c})-\mathbf{c}\times(\mathbf{b}-\mathbf{c})\right|$

	$=\frac{1}{2}\left\|(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c})+\frac{k}{2}(\mathbf{b}\times\mathbf{b}-\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{c})-\mathbf{c}\times\mathbf{b}+\mathbf{c}\times\mathbf{c}\right\|$ $=\frac{1}{2}\left\|(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c})+\frac{k}{2}(\mathbf{0}-\mathbf{b}\times\mathbf{c}-\mathbf{b}\times\mathbf{c}-\mathbf{0})+\mathbf{b}\times\mathbf{c}+\mathbf{0}\right\|$ $=\frac{1}{2}\left\|(1-k)\mathbf{a}\times(\mathbf{b}-\mathbf{c})+(1-k)(\mathbf{b}\times\mathbf{c})\right\|$ $=\frac{\|1-k\|}{2}\left\|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}\right\|$ (shown)
(iii)	 Since areas of triangle *CPB* and triangle *CQB* are equal, *CPQB* is a parallelogram. Given that $k=\frac{1}{2}$, $\overrightarrow{OP}=(1-k)\mathbf{a}+\frac{k}{2}(\mathbf{b}+\mathbf{c})=\frac{1}{2}\mathbf{a}+\frac{1}{4}(\mathbf{b}+\mathbf{c})$ Since *CPQB* is a parallelogram, $\overrightarrow{QC}=\overrightarrow{BP}$ $\overrightarrow{OC}-\overrightarrow{OQ}=\overrightarrow{OP}-\overrightarrow{OB}$ $\overrightarrow{OQ}=\overrightarrow{OB}+\overrightarrow{OC}-\overrightarrow{OP}$ $=\mathbf{b}+\mathbf{c}-\left[\frac{1}{2}\mathbf{a}+\frac{1}{4}(\mathbf{b}+\mathbf{c})\right]$ $=-0.5\mathbf{a}+0.75(\mathbf{b}+\mathbf{c})$
(iv)	Since $\left\|\overrightarrow{BC}\right\|=1$, $\left\|\overrightarrow{LP}\cdot\overrightarrow{BC}\right\|$ is the length of projection of $\overrightarrow{LP}$ onto $\overrightarrow{BC}$.

11 With reference to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. A point P lies on AB between A and B such that $AP:PB=\lambda:1-\lambda$, $0<\lambda<1$.

(i) Show that $\cos(\angle AOP)=\dfrac{(1-\lambda)|\mathbf{a}|}{|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}|}$. [4]

(ii) Prove that $\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]\bullet\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]=(1-\lambda)^2|\mathbf{a}|^2+\lambda^2|\mathbf{b}|^2$. Hence, given also that OP bisects $\angle AOB$, find the ratio of $\dfrac{|\mathbf{a}|}{|\mathbf{b}|}$, leaving your answer in terms of λ. [6]

SAJC Prelim 9758/2022/01/Q6	
Solution:	
(i)	Given that P lies on AB between A and B such that $AP:PB=\lambda:1-\lambda$, $0<\lambda<1$, by Ratio Theorem, $\overrightarrow{OP}=\dfrac{(1-\lambda)\mathbf{a}+\lambda\mathbf{b}}{1-\lambda+\lambda}$ $=(1-\lambda)\mathbf{a}+\lambda\mathbf{b}$  $\cos(\angle AOP)=\dfrac{\overrightarrow{OA}\cdot\overrightarrow{OP}}{\left\|\overrightarrow{OA}\right\|\left\|\overrightarrow{OP}\right\|}$ $=\dfrac{\mathbf{a}\cdot\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]}{\|\mathbf{a}\|\|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\|}$ $=\dfrac{(1-\lambda)\mathbf{a}\cdot\mathbf{a}+\lambda\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\|}$ $=\dfrac{(1-\lambda)\|\mathbf{a}\|^2+0}{\|\mathbf{a}\|\|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\|}$ since $\mathbf{a}\cdot\mathbf{b}=0$ as $\mathbf{a}\perp\mathbf{b}$ $=\dfrac{(1-\lambda)\|\mathbf{a}\|}{\|(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\|}$ (shown)
(ii)	$\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]\cdot\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]$ $=(1-\lambda)\mathbf{a}\cdot(1-\lambda)\mathbf{a}+(1-\lambda)\mathbf{a}\cdot\lambda\mathbf{b}+\lambda\mathbf{b}\cdot(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\cdot\lambda\mathbf{b}$ $=(1-\lambda)^2\|\mathbf{a}\|^2+\lambda(1-\lambda)\mathbf{a}\cdot\mathbf{b}+\lambda(1-\lambda)\mathbf{b}\cdot\mathbf{a}+\lambda^2\|\mathbf{b}\|^2$ $=(1-\lambda)^2\|\mathbf{a}\|^2+\lambda^2\|\mathbf{b}\|^2$, since $\mathbf{a}\cdot\mathbf{b}=0$ given that $\mathbf{a}\perp\mathbf{b}$ (proven)

	Given also that OP bisects $\angle AOB$, $\angle AOP = \frac{\pi}{4}$. $\therefore \cos\frac{\pi}{4} = \frac{(1-\lambda)\lvert\mathbf{a}\rvert}{\lvert(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\rvert}$ $\frac{1}{\sqrt{2}} = \frac{(1-\lambda)\lvert\mathbf{a}\rvert}{\lvert(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\rvert}$ $\frac{1}{2} = \frac{(1-\lambda)^2\lvert\mathbf{a}\rvert^2}{\lvert(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\rvert^2}$ $= \frac{(1-\lambda)^2\lvert\mathbf{a}\rvert^2}{\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]\cdot\left[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}\right]}$ $= \frac{(1-\lambda)^2\lvert\mathbf{a}\rvert^2}{(1-\lambda)^2\lvert\mathbf{a}\rvert^2+\lambda^2\lvert\mathbf{b}\rvert^2}$ $(1-\lambda)^2\lvert\mathbf{a}\rvert^2+\lambda^2\lvert\mathbf{b}\rvert^2 = 2(1-\lambda)^2\lvert\mathbf{a}\rvert^2$ Hence $(1-\lambda)^2\lvert\mathbf{a}\rvert^2 = \lambda^2\lvert\mathbf{b}\rvert^2$ $\frac{\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{b}\rvert^2} = \frac{\lambda^2}{(1-\lambda)^2}$ $\frac{\lvert\mathbf{a}\rvert}{\lvert\mathbf{b}\rvert} = \left\lvert\frac{\lambda}{(1-\lambda)}\right\rvert$ $\frac{\lvert\mathbf{a}\rvert}{\lvert\mathbf{b}\rvert} = \pm\frac{\lambda}{1-\lambda} = \frac{\lambda}{1-\lambda}$, reject $-\frac{\lambda}{1-\lambda}$ since $0<\lambda<1$ and ratio of length is positive.

12 Relative to the origin O, the position vectors of the points P, Q and R are $\mathbf{i}$, $2\mathbf{j} - t\mathbf{k}$ and $t\mathbf{k}$ respectively, where t is a fixed constant. The points A and B divides both line segments PQ and QR respectively in the same ratio of $\mu : 1 - \mu$, where μ is a parameter such that $0 < \mu < 1$.

(i) Find the vector $\overline{AB}$ in terms of t and μ. [3]

(ii) Determine whether the points O, A, B are collinear. [1]

(iii) Find the values of μ such that the length of projection of $\overline{AB}$ onto $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ is $\frac{1}{5}$ unit. [2]

(iv) Given that angle AOB is a right angle, find the set of possible values of t, justifying your answer clearly. [3]

DHS Prelim 9758/2022/02/Q4	
Solution:	
(i)	$\overline{OA} = \dfrac{\mu\overline{OQ} + (1-\mu)\overline{OP}}{\mu + (1-\mu)} = \mu\begin{pmatrix} 0 \\ 2 \\ -t \end{pmatrix} + (1-\mu)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-\mu \\ 2\mu \\ -t\mu \end{pmatrix}$ $\overline{OB} = \dfrac{\mu\overline{OR} + (1-\mu)\overline{OQ}}{\mu + (1-\mu)} = \mu\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} + (1-\mu)\begin{pmatrix} 0 \\ 2 \\ -t \end{pmatrix} = \begin{pmatrix} 0 \\ 2-2\mu \\ -t+2t\mu \end{pmatrix}$ $\therefore \overline{AB} = \begin{pmatrix} 0 \\ 2-2\mu \\ -t+2t\mu \end{pmatrix} - \begin{pmatrix} 1-\mu \\ 2\mu \\ -t\mu \end{pmatrix} = \begin{pmatrix} \mu-1 \\ 2-4\mu \\ -t+3t\mu \end{pmatrix}$
(ii)	From (i), $\overline{OA} \neq k\overline{OB}$. $\therefore$ the points O, A, B are not collinear.
(iii)	Given that the length of projection of $\overrightarrow{AB}$ onto $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ is $\frac{1}{5}$ unit,

	$\dfrac{\left\lvert \overrightarrow{AB} \cdot \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert}{\left\lvert \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert} = \dfrac{1}{5}$ $\dfrac{\left\lvert \begin{pmatrix} \mu - 1 \\ 2 - 4\mu \\ -t + 3t\mu \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert}{\sqrt{3^2 + 4^2}} = \dfrac{1}{5}$ $\lvert 3\mu - 3 + 8 - 16\mu \rvert = 1$ $\lvert -13\mu + 5 \rvert = 1$ $\lvert 13\mu - 5 \rvert = 1$ $13\mu = 1 + 5 \quad \text{or} \quad -1 + 5$ $\mu = \dfrac{4}{13} \quad \text{or} \quad \dfrac{6}{13}$
(iv)	If angle *AOB* is a right angle, then $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$ $\begin{pmatrix} 1 - \mu \\ 2\mu \\ -t\mu \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 - 2\mu \\ -t + 2t\mu \end{pmatrix} = 0$ $4\mu - 4\mu^2 + t^2\mu - 2t^2\mu^2 = 0$ $\mu = 0 \quad \text{or} \quad 4 - 4\mu + t^2 - 2t^2\mu = 0$ (reject $\because 0 < \mu < 1$) $\quad \mu = \dfrac{4 + t^2}{4 + 2t^2}$ Clearly, $\dfrac{4 + t^2}{4 + 2t^2} > 0$ since $4 + t^2 > 0$ and $4 + 2t^2$ for all $t \in \mathbb{R}$. Since $0 < \mu < 1$, $\dfrac{4 + t^2}{4 + 2t^2} < 1$ $4 + t^2 < 4 + 2t^2$ $t^2 > 0$ Hence $t \in \mathbb{R} \setminus \{0\}$