Chapter 1: Vectors

Content Outline

I. Basic properties of vectors in two- and three dimensions

Include:

- addition and subtraction of vectors, multiplication of a vector by a scalar, and their geometrical interpretations
- position vectors, displacement vectors and direction vectors
- magnitude of a vector
- unit vectors
- distance between two points
- collinearity
- use of the ratio theorem in geometrical applications

II. Scalar and vector products in vectors

Include:

- concepts of scalar product and vector product of vectors and their properties
- calculation of the magnitude of a vector and the angle between two vectors
- geometrical meanings of $|\mathbf{a} \cdot \hat{\mathbf{n}}|$ and $|\mathbf{a} \times \hat{\mathbf{n}}|$, where $\hat{\mathbf{n}}$ is a unit vector

Exclude triple products $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.

III. Three-dimensional vector geometry

Include:

- vector and cartesian equations of lines and planes
- finding the foot of the perpendicular and distance from a point to a line or to a plane
- finding the angle between two lines, between a line and a plane, or between two planes
- relationships between (i) two lines (coplanar or skew)
  (ii) a line and a plane
  (iii) two planes

Exclude:

- finding the shortest distance between two skew lines
- finding an equation for the common perpendicular to two skew lines

References

- **Websites**
  (1) http://www.h2maths.site
  (2) https://www.mathsisfun.com/algebra/vectors.html
- **Books**
  (1) Ho Soo Thong, Tay Yong Chiang & Koh Khee Meng, "College Mathematics Syllabus C Volume 1", Pan Pacific Publications, Call Number: HO510
  (2) Pure Mathematics by Alan Sherlock, Elizabeth Roebuck, Timothy Heneage, Shirley Beck: Chapter 16
  (3) Pure Mathematics 2 by L Bostock, S Chandler: Chapter 5
  (4) My First Step in using TI-84 Plus CE for H1 and H2 Math by Attal Lam

---

Chapter 1(a): Vector Algebra and Scalar Product

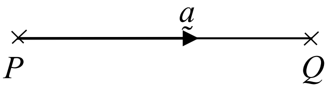
1. Introduction

Pre-requisite: GCE "O" Level Elementary Mathematics – 2-dimensional vectors

- A ***scalar*** quantity is one that is fully defined by **magnitude** alone, e.g. length, distance, speed.
- A ***vector*** quantity is one that has both **magnitude** and **direction**, e.g. force, displacement, velocity.

Geometrical Representation

A vector **a** (directed from P to Q) may be represented geometrically by a directed line $\overrightarrow{PQ}$.



Notation: $\overrightarrow{PQ}$, $\underset{\sim}{a}$ or **a** (in the direction from P to Q)

2. 2-Dimensional / 3-Dimensional Vectors

Column vector form

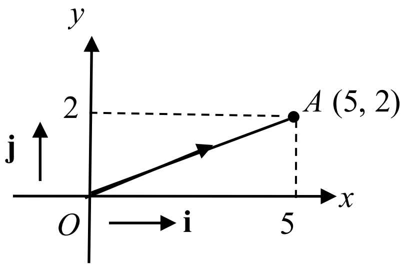
Examples of 2D vectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix}.$

Examples of 3D vectors: $\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}.$

Geometrical Representation of 2D Vectors

2D vectors can be represented on the Cartesian plane with the **i** value corresponding to the x-coordinate and the **j** value corresponding to the y-coordinate. The zero vector corresponds to the origin (0, 0), denoted by **0** or $\underset{\sim}{0}$.

$\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$, $\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$



For example,

$\overrightarrow{OA} = 5\mathbf{i}+2\mathbf{j} = \begin{pmatrix}5\\2\end{pmatrix}$

Magnitude/modulus of $\overrightarrow{OA}$ (= $\left|\overrightarrow{OA}\right|$ = length of vector $\overrightarrow{OA}$ = OA)

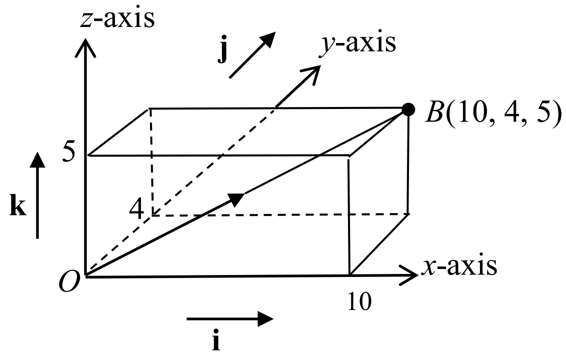
= $\sqrt{5^2+2^2}=\sqrt{29}$ (by Pythagoras theorem)

In general, the **magnitude/modulus** of a vector $\overrightarrow{PQ}=x\mathbf{i}+y\mathbf{j} = \begin{pmatrix}x\\y\end{pmatrix}$ is $\left|\overrightarrow{PQ}\right|=\sqrt{x^2+y^2}$

Geometrical Representation of 3D Vectors

3D vectors can be represented on the Cartesian space with the **i** value corresponding to the x-coordinate, the **j** value corresponding to the y-coordinate and the **k** value corresponding to the z-coordinate. **0** or $\underset{\sim}{0}$ corresponds to the origin (0, 0, 0).

$\mathbf{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$



For example,

$\overrightarrow{OB} = 10\mathbf{i}+4\mathbf{j}+5\mathbf{k} = \begin{pmatrix}10\\4\\5\end{pmatrix}$

Magnitude/modulus of $\overrightarrow{OB}$ (= $\left|\overrightarrow{OB}\right|$ = length of vector $\overrightarrow{OB}$ is OB)

$=\sqrt{10^2+4^2+5^2} = \sqrt{141}$

In general, the **magnitude/modulus** of a vector $\overrightarrow{PQ}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ is

$\left|\overrightarrow{PQ}\right|=\sqrt{x^2+y^2+z^2}$

Recall : In Chp 0A, $|\,x\,|$ is the distance of the real number x from zero on a number line.

<u>Example 1</u>

Find the magnitude of the following vectors, in exact form:

(i) $\overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}$, (ii) $\overrightarrow{OB} = 6\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, (iii) $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Solution

(i) $\left|\overrightarrow{OA}\right| =$

(ii) $\left|\overrightarrow{OB}\right| =$

(iii) $\left|\overrightarrow{OC}\right| =$

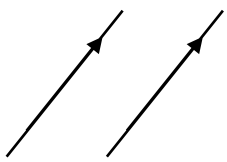
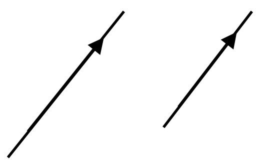
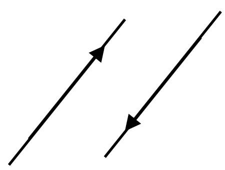
3. <u>General types of vectors</u>

3.1 Equal Vectors	**3.2 Negative Vectors**	**3.3 Parallel Vectors**
3.4 Unit Vectors	**3.5 Position Vectors**	**3.6 Displacement Vectors**

3.1 <u>Equal Vectors</u>

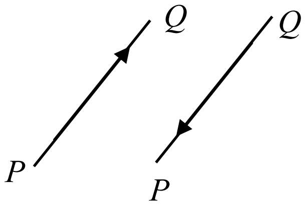
Two vectors are **<u>equal</u>** if and only if they have the **<u>same magnitude and direction</u>**.

i.e. $\mathbf{a} = \mathbf{b} \Leftrightarrow \quad |\mathbf{a}| = |\mathbf{b}|$ **<u>AND</u>** **a** and **b** are in the same direction

 Same magnitude, same direction, so **equal**	 Different magnitude, same direction, so **not equal**	 Same magnitude, different direction, so **not equal**

3.2 Negative Vectors

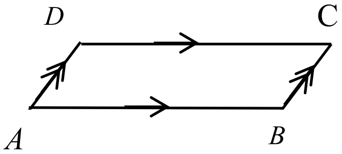
If two vectors **a** and **b** have the same magnitude but are parallel in opposite directions, then $\mathbf{a} = -\mathbf{b}$ or $-\mathbf{a} = \mathbf{b}$.



From the diagram, we can see that the directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ have the same magnitude but are parallel in opposite directions.

$$\therefore \overrightarrow{PQ} = -\overrightarrow{QP}$$

Given that $ABCD$ is a parallelogram, state all the pairs of equal vectors.

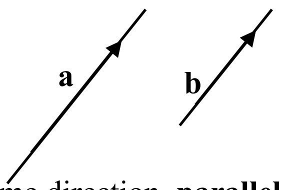
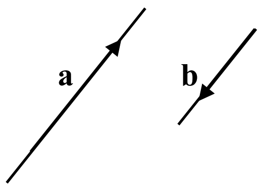
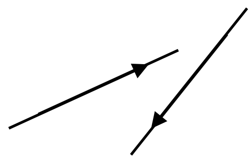


$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

Do you know why $\overrightarrow{AD} \neq \overrightarrow{CB}$ and $\overrightarrow{AB} \neq \overrightarrow{CD}$?

> **Note:** To prove $ABCD$ is a parallelogram, we need to prove <u>either</u> $\overrightarrow{AB} = \overrightarrow{DC}$ **OR** $\overrightarrow{AD} = \overrightarrow{BC}$.

3.3 Parallel (//) Vectors

 Same direction, **parallel** (e.g. $\mathbf{a} = 2\mathbf{b}$)	 Opposite direction, **parallel** (e.g. $\mathbf{a} = -2\mathbf{b}$)	 Different direction, **not parallel**

> If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\mathbf{a} \,/\!/\, \mathbf{b} \Leftrightarrow \mathbf{a} = \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$ where $\lambda \neq 0$.

In other words, if **a** and **b** are any two vectors and $\mathbf{a} = \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, then there are 2 possibilities . Either

(i) **a** and **b** are parallel,

(ii) $\mathbf{a} = \mathbf{b} = \mathbf{0}$

Example 2

Determine if the following pairs of vectors are parallel

(i) $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$,

(ii) $3\mathbf{i} - 5\mathbf{k}$ and $\begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix}$.

Solution

(i) Since $\Rightarrow$ they are ___________.

(ii) There exist ______________ such that

$\Rightarrow$ they are ______________.

Note: Observe that in (ii), when we try to solve for λ , there is no unique λ value found.

Example 3

Given that $2\mathbf{i} + h\mathbf{j} + 4\mathbf{k}$ is parallel to $5\mathbf{i} - 4\mathbf{j} + p\mathbf{k}$, find the values of h and p.

Solution

Since $\begin{pmatrix} 2 \\ h \\ 4 \end{pmatrix}$ is parallel to $\begin{pmatrix} 5 \\ -4 \\ p \end{pmatrix}$,

for some $\lambda \in \mathbb{R}$ where $\lambda \neq 0$.

From (1), $\lambda = \dfrac{2}{5}$

From (2),

From (3),

3.4 Unit Vector

The unit vector of $\mathbf{a}$, denoted by $\hat{\mathbf{a}}$, is the vector in the direction of $\mathbf{a}$ with magnitude 1, i.e.

$$\boxed{\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.}$$

Example 4

Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, find $\hat{\mathbf{a}}$.

Solution

$|\mathbf{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$. Therefore, $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} =$

Note (Self-check): $|\hat{\mathbf{a}}| = \sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{-1}{\sqrt{6}}\right)^2} = 1$

Example 5

Find the vector of length 6 in the direction of $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

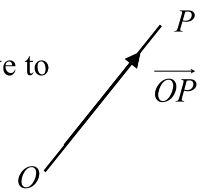
Solution
Let $\mathbf{a}$ be the required vector.

Then , $\hat{\mathbf{a}} =$

$\mathbf{a} =$

3.5 Position Vectors

To specify the position of a point P, we introduce a point of reference, which is the fixed point O (origin). The position vector of a point P relative to the point O is indicated by the directed line segment $\overrightarrow{OP}$.
The vector $\overrightarrow{OP}$ is called the position vector of P relative to O.



3.6 Displacement Vectors

The displacement vector **v** with initial point (x_1, y_1, z_1) and ending point (x_2, y_2, z_2) is given by

$$\mathbf{v} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}.$$

E.g. Given the points $A\,(-4, 1, 3)$ and $B(-2, -5, 2)$, then

the position vector $\overrightarrow{OA} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$ and the position vector $\overrightarrow{OB} = \begin{pmatrix} -2 \\ -5 \\ 2 \end{pmatrix}$.

The displacement vector $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix}$ and the displacement vector $\overrightarrow{BA} = \begin{pmatrix} -2 \\ 6 \\ 1 \end{pmatrix}$.

The distance between A and B =

4. Vector Algebra

4.1 Vector Addition and Subtraction

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

$$\mathbf{a} \pm \mathbf{b} = (a_1 \pm b_1)\mathbf{i} + (a_2 \pm b_2)\mathbf{j} + (a_3 \pm b_3)\mathbf{k} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix}.$$

Note: These results can be extended to more than 2 vectors.

Example 6

Given that **a** = 2**i** + 3**j** + 2**k**, **b** = 4**i** – 3**j** + **k** and **c** = 6**i** – 11**j**.

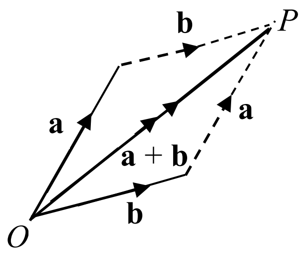
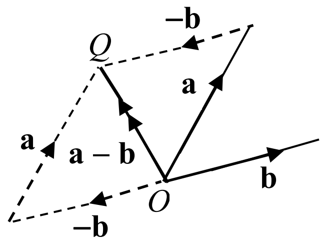
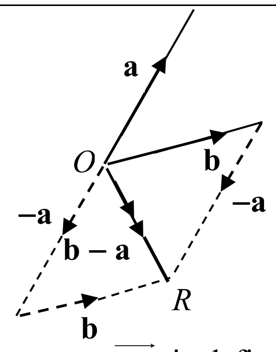
Find (i) **a** + **b**, (ii) **a** – **b** + **c**.

Solution

(i) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} =$

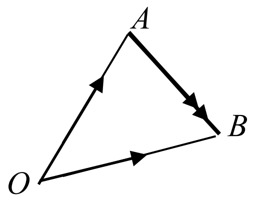
(ii) $\mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ -11 \\ 0 \end{pmatrix} =$

Geometrical Representation of Vector Addition and Subtraction

		
The vector $\overrightarrow{OP}$ is defined as the sum of **a** and **b** and is written as **a** + **b** or **b** + **a**.	The vector $\overrightarrow{OQ}$ is defined as the difference of **a** and **b** and is written as **a** − **b** or −**b** + **a**.	The vector $\overrightarrow{OR}$ is defined as the difference of **a** and **b** and is written as −**a** + **b** or **b** − **a**.

Similarly, for any vector $\overrightarrow{AB}$, it can be expressed as foll ws:

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO}+\overrightarrow{OB}\\ &= -\overrightarrow{OA}+\overrightarrow{OB}\\ &= \overrightarrow{OB}-\overrightarrow{OA}\end{aligned}$$



$\overrightarrow{AB}$ is called **displacement vector**. **(Refer to 3.6)**

$$\boxed{\text{In general, for any vector } \overrightarrow{AB},\ \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}}$$

How do you find the vector $\overrightarrow{PQ}$?

4.2 Multiplication of a Vector by a Scalar

If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$, then $\lambda\mathbf{a}=\lambda a_1\mathbf{i}+\lambda a_2\mathbf{j}+\lambda a_3\mathbf{k}=\begin{pmatrix}\lambda a_1\\\lambda a_2\\\lambda a_3\end{pmatrix}$.

Example 7

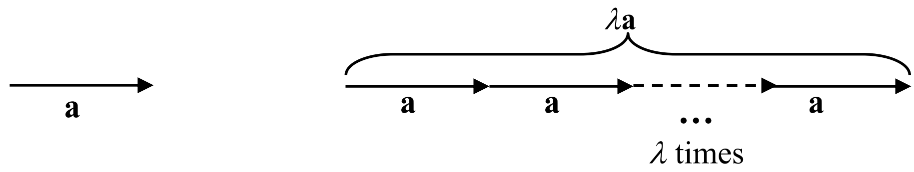
Given that **a** = 2**i** + 3**j** + 2**k** and **b** = 4**i** − 3**j** + **k**.

Find (i) **a** + 2**b** (ii) 3**a** − **b**

Solution

(i) $\mathbf{a}+2\mathbf{b}=\begin{pmatrix}2\\3\\2\end{pmatrix}+2\begin{pmatrix}4\\-3\\1\end{pmatrix}=$ (ii) $3\mathbf{a}-\mathbf{b}=3\begin{pmatrix}2\\3\\2\end{pmatrix}-\begin{pmatrix}4\\-3\\1\end{pmatrix}=$

Geometrical Representation of Multiplication of a Vector by a Scalar



$\lambda > 1$ in diagram

For any $\lambda \in \mathbb{R}$, $\mathbf{a}\lambda = \lambda\mathbf{a}$.   What is the magnitude of $\lambda\mathbf{a}$?

Important results of Vector Operations

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (ii) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

(iii) $\mathbf{a}\lambda = \lambda\mathbf{a}, \quad \lambda \in \mathbb{R}$ (iv) $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}, \quad \lambda, \mu \in \mathbb{R}$

(v) $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$

Collinear Points

If 3 points A, B and C are ***collinear***, i.e. A, B, and C all lie on a straight line, then
$\overrightarrow{AB} = \lambda\overrightarrow{BC}$,   for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Note: We can also say that $\overrightarrow{AB} = \mu\overrightarrow{AC}$ or $\overrightarrow{BC} = k\overrightarrow{AC}$ for some $\mu, k \in \mathbb{R}$, $\mu, k \neq 0$.

Example 8

The position vectors of points A, B and C relative to a fixed point O are $\mathbf{a} = 4\mathbf{i} - 9\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Prove that A, B and C are collinear.

Solution

Since

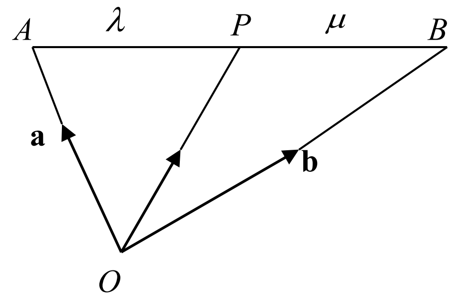
A, B and C are collinear.

Note: We can also prove $\overrightarrow{AB} = \lambda\overrightarrow{BC}$ or $\overrightarrow{BC} = \lambda\overrightarrow{AC}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, in the above example.

5. Ratio Theorem

Consider a point *P*, which divides the line segment *AB* in the ratio $\lambda : \mu$ as shown in the diagram below. The position vector of *P*,

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ ($write\ \overrightarrow{AP}\ in\ terms\ of$ **a** *and* **b**)



$$= \frac{(\lambda + \mu)\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})}{\lambda + \mu}$$

$$= \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\mu + \lambda}$$

Hence, the position vector of a point *P*, which divides the line segment *AB* in the ratio $\lambda : \mu$, is given by

$$\boxed{\overrightarrow{OP} = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\mu + \lambda}.}$$

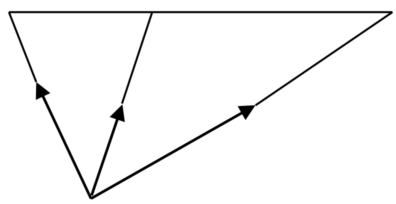
This formula is given in the formula list MF27

This is known as the **Ratio Theorem**.

Example 9

Points *A* and *B* have coordinates (2, 1, 4) and (3, 4, 2) respectively. Find the position vector of the point *P* that divides *AB* in the ratio 1:2.

Solution

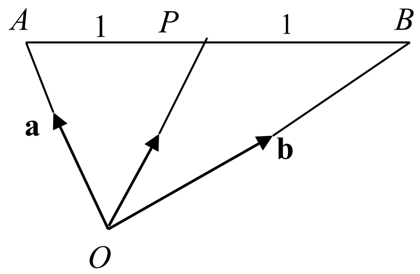


<u>Example 10</u>

Given that P is the mid point of AB, where A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, find the position vector of P.

Solution

By using Ratio Theorem,



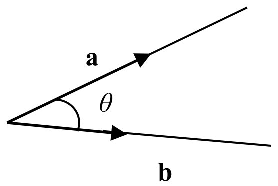
<u>Note</u>: $\overrightarrow{OP} \neq \frac{1}{2}\overrightarrow{AB}$. In fact, $\frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

6. <u>Scalar Product (also called Dot Product)</u>

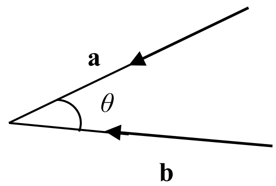
The ***scalar* product (dot product)** of two vectors, $\mathbf{a}$ and $\mathbf{b}$, whose directions are inclined at an angle θ to each other, is denoted by $\mathbf{a} \bullet \mathbf{b}$ and is defined as the *scalar quantity* $|\mathbf{a}||\mathbf{b}|\cos\theta$.

i.e. $\boxed{\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta}$

where $0 \le \theta \le 180^\circ$ and $\mathbf{a}$ and $\mathbf{b}$ must be either <u>*divergent*</u> or <u>*convergent*</u>.



Diverging vectors



Converging vectors

<u>Basic Properties of a Scalar Product</u>

(i) $\mathbf{a} \bullet \mathbf{b}$ is a scalar, not a vector.
(ii) $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ (commutative)
(iii) $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos 0^\circ = |\mathbf{a}|^2$
(iv) $(\lambda\mathbf{a}) \bullet \mathbf{b} = \lambda(\mathbf{a} \bullet \mathbf{b})$, $\lambda \in \mathbb{R}$
(v) $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$ (distributive)

<u>*Important Results</u>

(i) <u>Perpendicular Vectors</u>

If two vectors **a** and **b** are perpendicular, the angle θ between **a** and **b** is _____.

Then $\cos\theta$ = ______ $\Leftrightarrow \mathbf{a}\bullet\mathbf{b}$ = ____________.

$$\boxed{\text{i.e. } \mathbf{a}\perp\mathbf{b} \iff \mathbf{a}\bullet\mathbf{b}=0}$$

What is $\mathbf{i}\bullet\mathbf{j}$, $\mathbf{j}\bullet\mathbf{k}$, $\mathbf{i}\bullet\mathbf{k}$?

(ii) **<u>Parallel Vectors</u>**

If **a** and **b** are parallel and in the same direction, θ = ______.

$\cos\theta$ = ______ $\Leftrightarrow \mathbf{a}\bullet\mathbf{b}$ = ____________.

In particular, $\boxed{\mathbf{a}\bullet\mathbf{a}=|\mathbf{a}|^2}$

What is $\mathbf{i}\bullet\mathbf{i}$, $\mathbf{j}\bullet\mathbf{j}$, $\mathbf{k}\bullet\mathbf{k}$?

If they are in opposite directions, θ = ________.

$\cos\theta$ = ______ $\Leftrightarrow \mathbf{a}\bullet\mathbf{b}$ = ____________.

<u>Example 11</u>

Given that the vectors **a** and **b** are perpendicular to each other and $|\mathbf{a}|=2$ and $|\mathbf{b}|=3$, find $|\mathbf{a}+2\mathbf{b}|$.

Solution

$|\mathbf{a}+2\mathbf{b}|^2 = (\mathbf{a}+2\mathbf{b})\bullet(\mathbf{a}+2\mathbf{b})$

$=$

$=$

$= 2^2+4(3)^2=40$

$\therefore \quad |\mathbf{a}+2\mathbf{b}| = \sqrt{40}$

> **Note:**
> For this example, we make use of the property $\mathbf{a}\bullet\mathbf{a}=|\mathbf{a}|^2$ in both directions.

6.1 <u>Calculation of the Scalar Product</u>

Given vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then we have

$$\boxed{\mathbf{a}\bullet\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\bullet\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3}$$

Proof: (Read it yourself)

$\mathbf{a} \bullet \mathbf{b} = \left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right) \bullet \left(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}\right)$

$= a_1b_1(\mathbf{i} \bullet \mathbf{i}) + a_1b_2(\mathbf{i} \bullet \mathbf{j}) + a_1b_3(\mathbf{i} \bullet \mathbf{k})$

$+ a_2b_1(\mathbf{j} \bullet \mathbf{i}) + a_2b_2(\mathbf{j} \bullet \mathbf{j}) + a_2b_3(\mathbf{j} \bullet \mathbf{k})$

$+ a_3b_1(\mathbf{k} \bullet \mathbf{i}) + a_3b_2(\mathbf{k} \bullet \mathbf{j}) + a_3b_3(\mathbf{k} \bullet \mathbf{k})$

Obtained using important result (i).

$= a_1b_1 + a_2b_2 + a_3b_3$ since $(\mathbf{i} \bullet \mathbf{j}) = (\mathbf{i} \bullet \mathbf{k}) = (\mathbf{j} \bullet \mathbf{i}) = (\mathbf{j} \bullet \mathbf{k}) = (\mathbf{k} \bullet \mathbf{i}) = (\mathbf{k} \bullet \mathbf{j}) = 0$

$(\mathbf{i} \bullet \mathbf{i}) = (\mathbf{j} \bullet \mathbf{j}) = (\mathbf{k} \bullet \mathbf{k}) = 1$

Obtained using property (iii).

Example 12

Given that $\mathbf{a} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$, evaluate the scalar product of **a** and **b**.

Solution

$$\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -3 \\ -5 \end{pmatrix} =$$

Example 13

Find p given that the vectors $\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = p\mathbf{i} + \mathbf{j} - p\mathbf{k}$ are perpendicular to each other.

Solution

Since **a** and **b** are perpendicular to each other,

Hence,

6.2 Finding the Angle Between 2 Vectors

Example 14

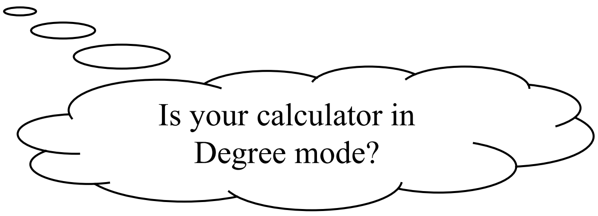
Find the angle between the vectors $3\mathbf{i} - 4\mathbf{j}$ and $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

Solution

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$\cos\theta = \frac{-6}{5\sqrt{17}}$$

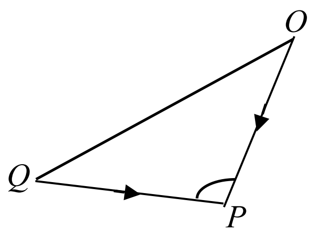
$\theta = 106.9°$ (to 1 d.p.)



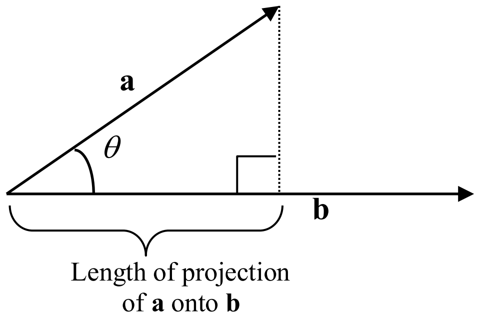
Example 15

Given $\overrightarrow{OP} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OQ} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, find $\angle OPQ$.

Solution



6.3 <u>Projection of a Vector onto Another Vector</u>



Length of the projection of $\mathbf{a}$ onto $\mathbf{b} = |\mathbf{a}||\cos\theta|$

From the definition of scalar product, we have

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

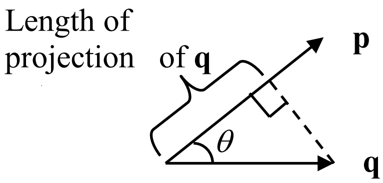
$\cos\theta = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$|\mathbf{a}||\cos\theta| = |\mathbf{a}|\dfrac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \dfrac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{b}|}$

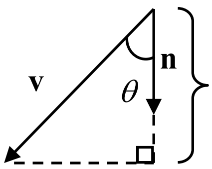
As scalar product may be negative, use modulus to consider positive value for length.

i.e. Length of projection of $\mathbf{a}$ onto $\mathbf{b} = \dfrac{|\mathbf{a} \bullet \mathbf{b}|}{|\mathbf{b}|} = |\mathbf{a} \bullet \hat{\mathbf{b}}|$

<u>Think</u>: Is the length of projection of **a** onto **b** the same as **b** onto **a** ?



What is the length of projection of **q** onto **p**?



What is the length of projection of **v** onto **n** ?

<u>Example 16</u>

Relative to the origin O, the points A, B and C have position vectors $-4\mathbf{i}+10\mathbf{j}-\mathbf{k}$, $-8\mathbf{i}+6\mathbf{j}-3\mathbf{k}$ and $-\mathbf{i}+8\mathbf{j}+3\mathbf{k}$ respectively.

(i) Find the length of the projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$,

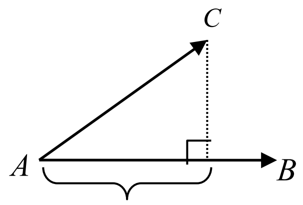
(ii) Hence, find the perpendicular distance from C to $\overrightarrow{AB}$.

Solution

(i) $\overrightarrow{AC}=\begin{pmatrix}-1\\8\\3\end{pmatrix}-\begin{pmatrix}-4\\10\\-1\end{pmatrix}=\begin{pmatrix}3\\-2\\4\end{pmatrix}$ $\quad \overrightarrow{AB}=\begin{pmatrix}-8\\6\\-3\end{pmatrix}-\begin{pmatrix}-4\\10\\-1\end{pmatrix}=\begin{pmatrix}-4\\-4\\-2\end{pmatrix}$

Length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$

$=\left|\overrightarrow{AC}\bullet\overrightarrow{AB}\right|$



Length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$

(ii)

Perpendicular distance from C to $\overrightarrow{AB}=$