ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1 **16 September 2025**

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 The curve C_1 and C_2 have equations $r = 3 - 2\cos\theta$ and $r = 2$ respectively for $0 \le \theta \le 2\pi$.

(i) Sketch C_1. [1]

(ii) Find the exact area of the region that is inside both C_1 and C_2. [4]

2 The function f is defined by $\mathrm{f}(x) = \mathrm{e}^{-x^2} - \frac{1}{2}$. The equation $\mathrm{f}(x) = 0$ has only one real root α in the interval $(0.75, 1)$.

(a) Carry out linear interpolation once on the interval $(0.75, 1)$ to find an approximation x_1 for α, giving your answer to 4 decimal places. By considering $\mathrm{f}'(x)$ and $\mathrm{f}''\ x$, explain with an aid of a diagram whether x_1 is an overestimate or underestimate of the root α. [4]

(b) Using the value of x_1 found in part **(i)** as the initial value, apply the Newton-Raphson method to find α correct to 6 decimal places. Justify the accuracy of your answer. [3]

3 **(i)** Use de Moivre's theorem to prove that

$$\sum_{r=0}^{n-1} x^r \sin r\theta = \frac{x^{n+1}\sin(n-1)\theta - x^n \sin n\theta + x\sin\theta}{1 - 2x\cos\theta + x^2}$$

where x is a real number not equal to ± 1, and θ is a real number not equal to $k\pi$ for any integer k. [4]

(ii) By substituting $\theta = \frac{1}{3}\pi$ in part **(i)**, find the sum of the first $4m$ terms of the series

$$x + x^2 - x^4 - x^5 + x^7 + x^8 - x^{10} - x^{11} + \ldots,$$

expressing your answer in the form $\frac{\mathrm{p}(x)}{\mathrm{q}(x)}$, where $\mathrm{p}(x)$ and $\mathrm{q}(x)$ are polynomials in x with degrees $6m + 1$ and 2 respectively. [3]

4 The variables x and y are related by the differential equation

$$(\sin x)\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x)\,y + 1,$$

with $y = 2$ when $x = \frac{1}{2}\pi$.

(a) Use two steps of the Euler method to find an approximation to the value of y when $x = \frac{1}{2}\pi + 1$. Give your answer to 3 decimal places. [3]

(b) Find the solution of the given differential equation and obtain the value of y when $x = \frac{1}{2}\pi + 1$. Give your answer to 3 decimal places [5]

(c) Find the percentage error in the approximation. [1]

(d) Explain why the answer in part **(a)** is not a good approximation to the value of y when $x = \frac{1}{2}\pi + 1$. [1]

5 The space-transformation T is given by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

The eigenvalues of the matrix $\mathbf{A}$ are denoted by λ_1, λ_2 and λ_3 where $\lambda_1 \le \lambda_2 \le \lambda_3$, and their corresponding eigenvectors are denoted by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

(i) Find the eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ of $\mathbf{A}$. [5]

(ii) Give a geometrical interpretation of each eigenvalue's set of eigenvectors in relation to T. [2]

(iii) Find matrices $\mathbf{Q}$ and $\mathbf{D}$ such that

$$\left(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4 + \mathbf{A}^5\right)^{-1} = \mathbf{QDQ}^{-1}$$

where $\mathbf{D}$ is a diagonal matrix. [3]

6 Let $\{x_n\}$ be a sequence of positive terms defined by

$$x_{n+1} = \frac{x_n\, x_{n-1}}{x_n + x_{n-1}}\text{ , for } n \geq 2,$$

with $x_1 = 1$ and $x_2 = 1$.

(a) By letting $y_n = \dfrac{1}{x_n}$, show that $\{y_n\}$ satisfies a second order linear homogeneous recurrence relation and find this recurrence relation. [2]

(b) Hence, find x_n in terms of n. [6]

(c) Using the expression derived in part (**b**), describe the long-term behaviour of the sequence $\{x_n\}$ as $n \to \infty$. Justify your answer. [2]

7 **(a)** The surface S_1 has equation $z = \mathrm{f}(x, y)$ where

$$\mathrm{f}(x, y) = x^2 + x - 3xy + y^3 - 5\,.$$

Find the exact coordinates of the stationary points of S_1 and determine their nature. [7]

(b) The surface S_2 has equation $z = \mathrm{g}(x, y)$ where

$$\mathrm{g}(x, y) = \tan^{-1}(x + 2y).$$

The quadratic approximation for S_2 in the region of the point $\left(1, 0, \tfrac{1}{4}\pi\right)$ is given by $z = \mathrm{Q}(x, y)$. Find $\mathrm{Q}(x, y)$. [5]

8 The linear transformation $\mathrm{T}:\mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix

$$\begin{pmatrix} -1 & \alpha & -7 \\ 5 & \alpha & 11 \\ \alpha+1 & 1 & 7 \end{pmatrix}.$$

(a) Find the set of values of α such that the nullity of T is zero. [4]

It is now given that $\alpha = 2$.

(b) Find a basis for the range space of T. State a geometrical interpretation of the range space of T. [3]

(c) State a basis for the null space of T and its nullity. Hence, find the subset P of $\mathbb{R}^3$ whose image under T is the line

$$\mathbf{r} = \begin{pmatrix} 5 \\ 11 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$$ [5]

9 **(a)** The curve C has equation $y = x^2$, for $0 \le x \le 2$. Show that the length of the curve C is given by

$$\sqrt{17} + \frac{1}{2}\int_0^2 \frac{1}{\sqrt{1+4x^2}}\,\mathrm{d}x.$$

Use Simpson's rule with 5 ordinates to find an approximation to the value of

$$\int_0^2 \frac{1}{\sqrt{1+4x^2}}\,\mathrm{d}x$$

and hence, find an approximation to the length of the curve C, giving your answer to three decimal places. [8]

(b) The ellipse with equation $\frac{x^2}{4} + y^2 = 1$ is rotated by π radians about the x-axis to form a solid. Find the exact surface area of the solid by integration using the substitution $x = \frac{4}{\sqrt{3}}\sin\theta$. [7]

10 The population of a new breed of fish in a farm is being studied. In an initial survey, the number, P (in thousands), of the fish at time t months is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P) - n,$$

where n is a positive real constant.

(a) Find the condition for n so that the population of fish in the farm is sustainable in the long run. [2]

After a new survey, it is found that P is better modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P) - \frac{kP}{4+P},$$

where k is a positive constant.

(b) Given that H is the non-zero equilibrium point, find H in terms of k and give the necessary condition on k for H to exist. [3]

(c) Suppose that $H > P$. Show that

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P\left(H^2 - P^2\right)}{4+P}.$$

Show that the general solution of this differential equation is

$$P^A (H-P)^B (H+P)^C = D\mathrm{e}^t,$$

where D is a positive real constant and A, B and C are to be determined in terms of H. [7]