H3 Current of Electricity and Direct Current

Maximum Power Transfer Theorem

1 **(a)** A battery of e.m.f. E and internal resistance r is connected to a variable load resistor, as shown in Fig. 1.1.

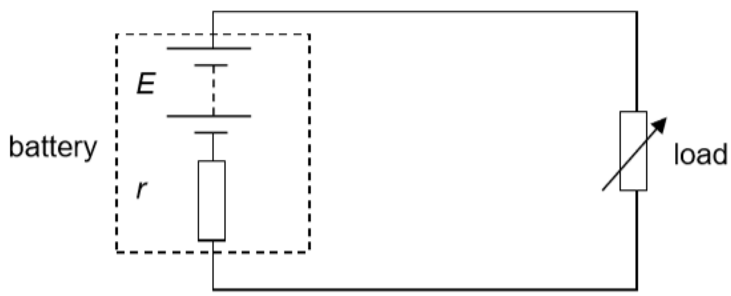


Fig. 1.1

Consider the circuit when the resistance of the load is R.

(i) Obtain expressions for the total power P_{total} dissipated in the circuit and the power P_{load} delivered to the load. Give your answers in terms of E, R and r.

[2]

(ii) Show that the condition for P_{load} to be a maximum is $R = r$.

[3]

(b) Fig. 1.2 shows a battery of e.m.f. E and internal resistance r connected to a load of fixed resistance r through a component known as a T-section attenuator. The function of the attenuator is to adjust the power delivered to the load.

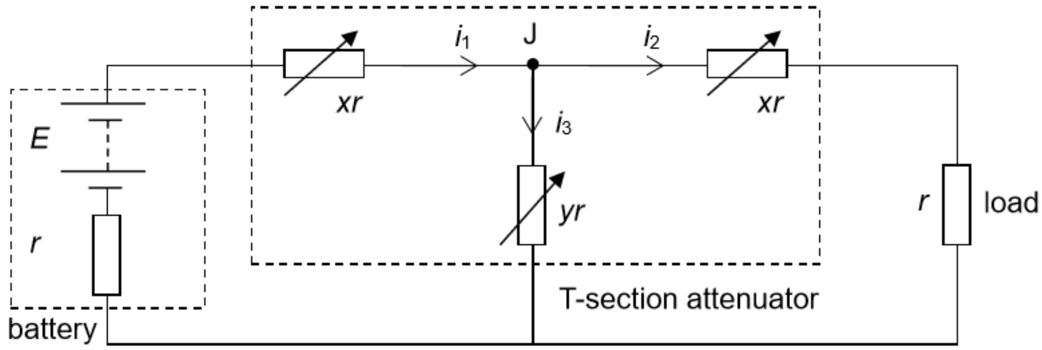


Fig. 1.2

The attenuator consists of three variable resistors connected in the arms of a T-shaped network. The resistors in the horizontal arms of the T-shape each have resistance xr, where x is a fraction between 0 and 1. The resistor in the vertical arm has resistance yr, where y is again a fraction between 0 and 1.

The circuit in Fig. 1.2 may be considered as a battery of e.m.f. E and internal resistance r connected to a single resistor of resistance R_{equiv} (Fig. 1.3).

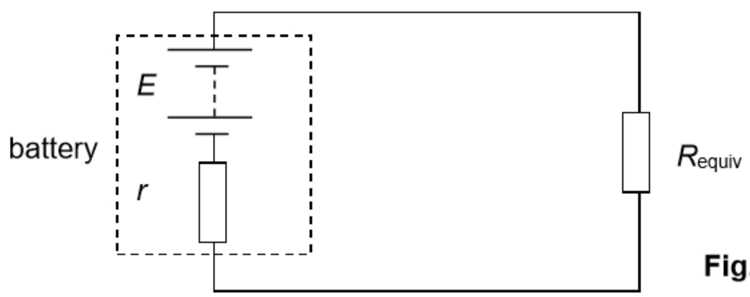


Fig. 1.3

This single resistor produces an effect at the battery terminals equivalent to the combination of the T-section attenuator and the load resistor.

(i) Show that

$$R_{equiv} = \frac{r\left(x^2 + 2xy + x + y\right)}{1 + x + y}$$

[2]

(ii) The attenuator is adjusted by varying the values of x and y. This changes the value of R_{equiv}. Show that the condition for maximum power to be delivered from the battery terminals is

$$1 - x^2 = 2xy$$

[2]

(iii) The current into junction J of the attenuator is i_1, and the currents out are i_2 and i_3 (see Fig. 1.2). By applying Kirchhoff's laws, show that

$$\frac{i_1}{i_2} = \frac{1 + x + y}{y}$$

[3]

(iv) The power delivered by the battery to the combination of attenuator and load is P_{input}. The power dissipated in the load is P_{load}.

Show that, when maximum power is delivered from the battery terminals,

$$\frac{P_{\text{input}}}{P_{\text{load}}} = \frac{(1+x)^2}{(1-x)^2}$$

[4]

(v) The ratio of powers in **(iv)** is measured by a quantity called the attenuation A of the attenuator. The defining equation for A is

$$A = 10\lg\frac{P_{\text{input}}}{P_{\text{load}}}.$$

The unit of attenuation is the decibel (dB).

Calculate the value of the fraction x required to give an attenuation of 15 dB. [3]

x = ... [3]

2 A car battery has e.m.f. 12 V and internal resistance 0.50 Ω. It is connected to a parallel arrangement of four lamps, as shown in Fig. 2.1.

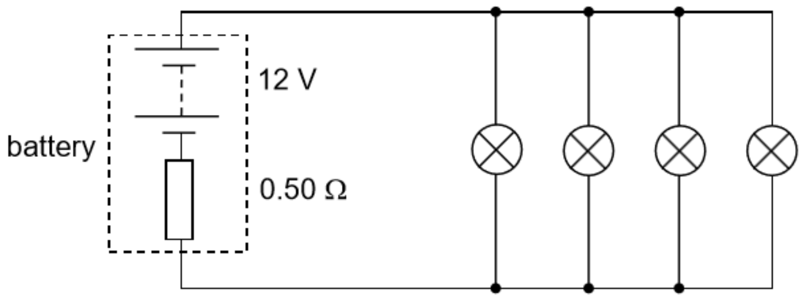


Fig. 2.2

Each lamp acts as a pure resistor of constant resistance 30 Ω.

(a) Calculate the total power dissipated in the lamps. [4]

Total power = ... W [4]

(b) The owner of the car thinks that the brightness of the lamps would be increased by connecting an additional resistor in the circuit, placed so as to extract the maximum power from the battery.

(i) State and explain where the additional resistor should be connected so as to extract the maximum power from the battery.

....

....

....

.... [2]

(ii) Calculate the resistance of this resistor.

resistance = ... Ω [2]

(iii) With this resistor in the circuit, calculate the consequent change in the power dissipated in the lamps.

change in power = ... [3]

(iv) Explain whether or not the car owner achieved his aim.

....

....

....

.... [2]

Kirchoff's Rule

3 **(a)** State Kirchhoff's laws of d.c. circuits.

....

....

....

.... [2]

(b) Two resistors, of resistance R_1 and R_2 respectively, are connected in parallel.

Derive a formula for the combined resistance R_p of the resistors. Point out how you have used Kirchoff's laws in your derivation.

[5]

Fig. 3.1 shows an infinitely long network made up of resistors each of resistance *r*. the resistance of the network, measured between the terminals C and D, is *R*.

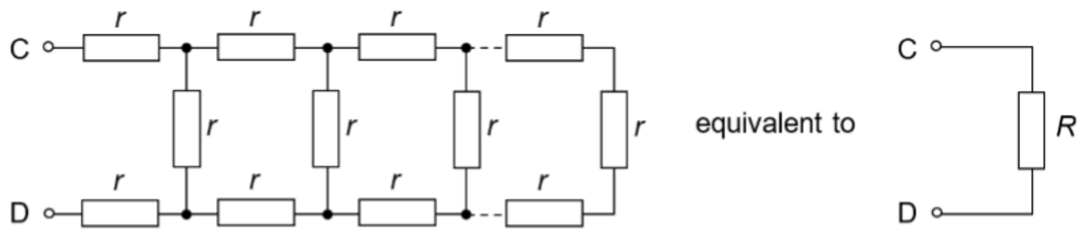


Fig. 3.1

An extra segment of three resistors is then added to the left-hand end of the network, as shown in Fig. 3.2. the combination of the extra segment and the infinite network is equivalent to the circuit shown in Fig. 3.3.

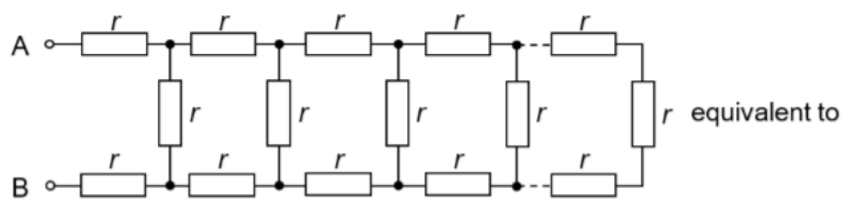


Fig. 3.2

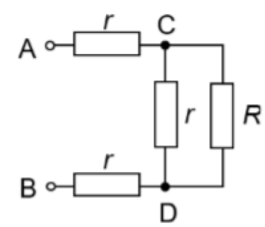


Fig. 3.3

(i) Find, in terms of *r* and *R*, the resistance of the circuit shown Fig. 3.3 when measured between the terminals A and B.

[3]

(ii) Because the network is infinitely long, the addition of the extra segment makes no difference to the overall resistance. Use this fact to show that

$$R = \left(1 + \sqrt{3}\right) r$$

4 A resistance network is formed by connecting twelve identical constantan wires, each of length 20 mm and cross-sectional area 0.66 mm^2, as the edges of a cube, as shown in Fig. 4.1.

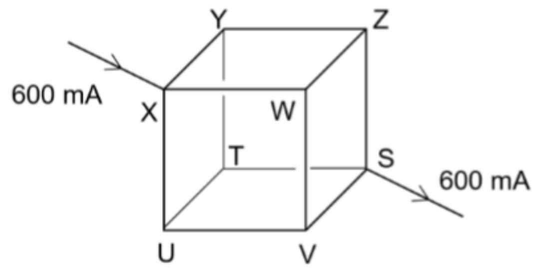


Fig. 4.1

(a) Find the resistance of one of the constantan wires, given that the resistivity of constantan is 4.9×10^{-7} Ω m.

resistance = …………………………… Ω [3]

(b) A current of 600 mA enters at the corner X and leaves at the diagonally opposite corner S. By considering the symmetry of the cube, find the currents in each of the wires.

[2]

(c) The potential difference between X and S is the algebraic sum of the potential differences across each of the wires making up any continuous path between X and S. Hence find the resistance of the network between X and S.

resistance = …………………………… Ω [5]

Answer Key

1 (b)(v) $x = 0.70$
2 (a) 16.9 W **(b)(ii)** 0.536 Ω **(b)(iii)** 12.1W
4 (a) 1.48×10^{-2} Ω **(b)** XW,XU,XY = 200 mA ; WV,WZ,YT,YZ,UT,UV = 100mA ; ZS,TS,VS = 200 mA **(c)** 0.012 Ω

H3 Current of Electricity and Direct Current

Maximum Power Transfer Theorem

1 **(a)** A battery of e.m.f. E and internal resistance r is connected to a variable load resistor, as shown in Fig. 1.1.

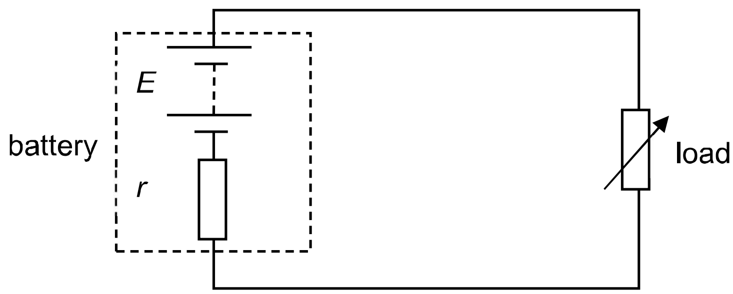


Fig. 1.1

Consider the circuit when the resistance of the load is R.

(i) Obtain expressions for the total power P_{total} dissipated in the circuit and the power P_{load} delivered to the load. Give your answers in terms of E, R and r.

$$E = I(R+r)$$

$$P_{\text{total}} = \frac{E^2}{R+r}$$

$$P_{\text{load}} = I^2 R$$

$$= \left(\frac{E}{R+r}\right)^2 R$$

[2]

(ii) Show that the condition for P_{load} to be a maximum is $R = r$.

$$P_{\text{load}} = \left(\frac{E}{R+r}\right)^2 R$$

To maximise P_{load} while verying R means $\frac{dP_{\text{load}}}{dR} = 0$

$$\frac{E^2(R+r)^2 - 2(R+r)E^2R}{(R+r)^4} = 0$$

$$\frac{E^2(R+r) - 2E^2R}{(R+r)^3} = 0$$

$$E^2(R+r) - 2E^2R = 0$$

$$R = r$$

Can use either 1st Derivative test or 2nd Derivative test to show max point.

(b) Fig. 1.2 shows a battery of e.m.f. E and internal resistance r connected to a load of fixed resistance r through a component known as a T-section attenuator. The function of the attenuator is to adjust the power delivered to the load.

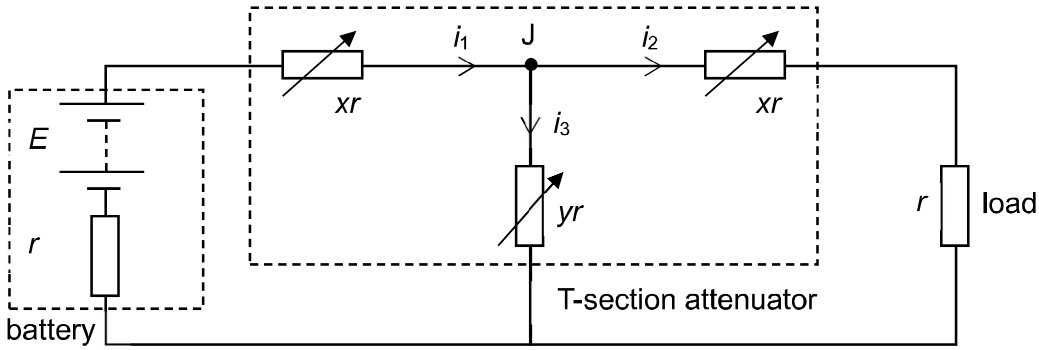


Fig. 1.2

The attenuator consists of three variable resistors connected in the arms of a T-shaped network. The resistors in the horizontal arms of the T-shape each have resistance xr, where x is a fraction between 0 and 1. The resistor in the vertical arm has resistance yr, where y is again a fraction between 0 and 1.

The circuit in Fig. 1.2 may be considered as a battery of e.m.f. E and internal resistance r connected to a single resistor of resistance R_{equiv} (Fig. 1.3).

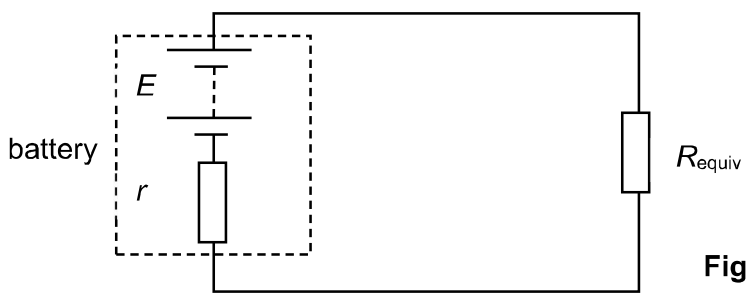


Fig. 1.3

This single resistor produces an effect at the battery terminals equivalent to the combination of the T-section attenuator and the load resistor.

(i) Show that

$$R_{\text{equiv}} = \frac{r\left(x^2 + 2xy + x + y\right)}{1 + x + y}$$

$$R_{\text{equiv}} = \left(\frac{1}{xr + r} + \frac{1}{yr}\right)^{-1} + xr$$

$$= \frac{yr\left(xr + r\right)}{yr + xr + r} + xr$$

$$= \frac{xyr + yr + xr\left(y + x + 1\right)}{y + x + 1}$$

$$= \frac{2xyr + yr + xr + x^2 r}{y + x + 1}$$

$$= \frac{r\left(x^2 + 2xy + x + y\right)}{1 + x + y} \quad \text{(shown)}$$

(ii) The attenuator is adjusted by varying the values of x and y. This changes the value of R_{equiv}. Show that the condition for maximum power to be delivered from the battery terminals is

$$1-x^2=2xy$$

For maximum power, $R_{equiv} = r$

$$r=\frac{r\left(x^2+2xy+x+y\right)}{1+x+y}$$

$$1+x+y=x^2+2xy+x+y$$

$$1-x^2=2xy \qquad \text{(shown)}$$

[2]

(iii) The current into junction J of the attenuator is i_1, and the currents out are i_2 and i_3 (see Fig. 1.2). By applying Kirchhoff's laws, show that

$$\frac{i_1}{i_2}=\frac{1+x+y}{y}$$

Applyinf Kirchoff's current Law,

$$i_1=i_2+i_3 \quad \text{---(1)}$$

Applyinf Kirchoff's Voltage Law to the right loop,

$$i_2xr+i_2r-i_3yr=0 \quad \text{---(2)}$$

Sub (1) into (2),

$$i_2xr+i_2r-(i_1-i_2)yr=0$$

$$\frac{i_1}{i_2}=\frac{1+x+y}{y}$$

[3]

(iv) The power delivered by the battery to the combination of attenuator and load is P_{input}. The power dissipated in the load is P_{load}.

Show that, when maximum power is delivered from the battery terminals,

$$\frac{P_{\text{input}}}{P_{\text{load}}} = \frac{(1+x)^2}{(1-x)^2}$$

When maximum power is delivered from battery terminals, $R_{\text{equiv}} = r$.

$\therefore P_{\text{input}} = i_1^2 R_{\text{equiv}} = i_1^2 r$ and $P_{\text{load}} = i_2^2 r$

$$\frac{P_{\text{input}}}{P_{\text{load}}} = \frac{i_1^2 r}{i_2^2 r}$$

$$= \left(\frac{i_1}{i_2}\right)^2$$

$$= \left(\frac{1+x+y}{y}\right)^2 \quad \text{(from (b)(iii))}$$

substituting $y = \dfrac{1-x^2}{2x}$ from (b)(ii)

$$= \left(\frac{1+x+\frac{1-x^2}{2x}}{\frac{1-x^2}{2x}}\right)^2$$

$$= \left(\frac{2x(1+x)+1-x^2}{1-x^2}\right)^2$$

$$= \left(\frac{x^2+2x+1}{1-x^2}\right)^2$$

$$= \left(\frac{(x+1)^2}{(1-x)(1+x)}\right)^2$$

$$= \frac{(1+x)^2}{(1-x)^2} \quad \text{(shown)}$$

(v) The ratio of powers in **(iv)** is measured by a quantity called the attenuation A of the attenuator. The defining equation for A is

$$A = 10\lg\frac{P_{\text{input}}}{P_{\text{load}}}.$$

The unit of attenuation is the decibel (dB).

Calculate the value of the fraction x required to give an attenuation of 15 dB. [3]

$$A = 10\lg\frac{P_{\text{input}}}{P_{\text{load}}}$$

$$15 = 10\lg\frac{(1+x)^2}{(1-x)^2}$$

$$10^{1.5} = \frac{(1+x)^2}{(1-x)^2}$$

$$x = 0.70 \quad \text{(2 s.f.)}$$

x = ... [3]

2 A car battery has e.m.f. 12 V and internal resistance 0.50 Ω. It is connected to a parallel arrangement of four lamps, as shown in Fig. 2.1.

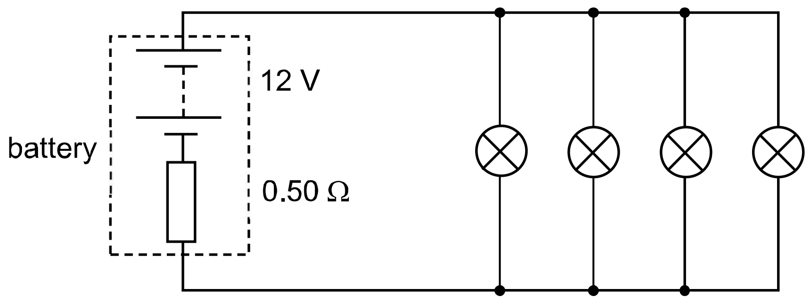


Fig. 2.2

Each lamp acts as a pure resistor of constant resistance 30 Ω.

(a) Calculate the total power dissipated in the lamps. [4]

$$R_{eff} = \frac{30}{4} = 7.5\ \Omega$$

$$P_{load} = \frac{E^2 R}{(R+r)^2} = \frac{12^2(7.5)}{(7.5+0.50)^2} = 16.9\text{ W}$$

Total power = ... W [4]

(b) The owner of the car thinks that the brightness of the lamps would be increased by connecting an additional resistor in the circuit, placed so as to extract the maximum power from the battery.

(i) State and explain where the additional resistor should be connected so as to extract the maximum power from the battery.

It should be placed parallel to the rest of the lamps so that the effective resistance is equal to the internal resistance of the battery i.e. R_{eff} = 0.50 Ω.

....

....

.... [2]

(ii) Calculate the resistance of this resistor.

$$\left(\frac{1}{R}+\frac{1}{7.5}\right)^{-1}=0.50$$

$$R=0.536\ \Omega$$

resistance = ………………………… Ω [2]

(iii) With this resistor in the circuit, calculate the consequent change in the power dissipated in the lamps.

When R = r, potential difference across the lamps = 12 / 2 = 6.0 V.

$$P=\frac{V^2}{R}=\frac{6.0^2}{7.5}=4.8\text{ W}$$

Change in power = 16.9 – 4.8 = 12.1 W

change in power = …………………………… [3]

(iv) Explain whether or not the car owner achieved his aim.

Power dissipated in the lamps decreased from 16.9 W to 4.8 W. He did not achieve his aim because while more power is dissipated in the load, most of the power is dissipated in the new resistor.

………………………………………………………………………………………… [2]

Kirchoff's Rule

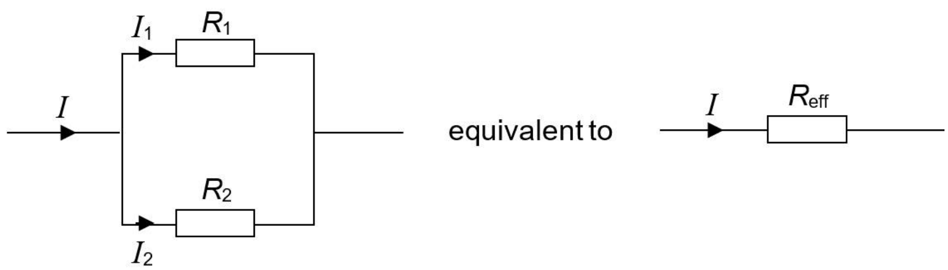
3 **(a)** State Kirchhoff's laws of d.c. circuits.

Kirchhoff's 1st Law (Current Law) states that the total current entering any junction of a circuit is equal to the total current leaving that same junction i.e. the algebraic sum of currents entering or leaving any junction of a circuit is zero.

Kirchhoff's 2nd Law (Ampere Law) states that the sum of electromotive forces (potential gains) around any closed loop in a circuit is equal to the sum of the potential drops around that same loop.

(b) Two resistors, of resistance R_1 and R_2 respectively, are connected in parallel.

Derive a formula for the combined resistance R_p of the resistors. Point out how you have used Kirchoff's laws in your derivation.



Consider a resistor with R_{eff} that has the same current I and potential difference V across the two parallel resistors as shown.

Applying Kirchhoff's 1st Law,

$$I = I_1 + I_2$$

$$\frac{V}{R_{eff}} = \frac{V}{R_1} + \frac{V}{R_2}$$

Since V is the same,

$$R_{eff} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

Fig. 3.1 shows an infinitely long network made up of resistors each of resistance r. the resistance of the network, measured between the terminals C and D, is R.

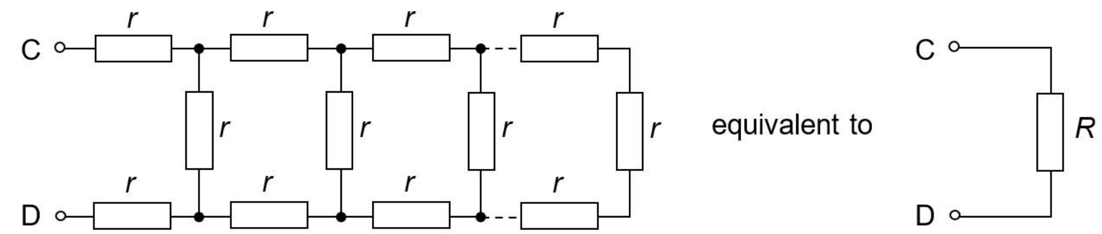


Fig. 3.1

An extra segment of three resistors is then added to the left-hand end of the network, as shown in Fig. 3.2. the combination of the extra segment and the infinite network is equivalent to the circuit shown in Fig. 3.3.

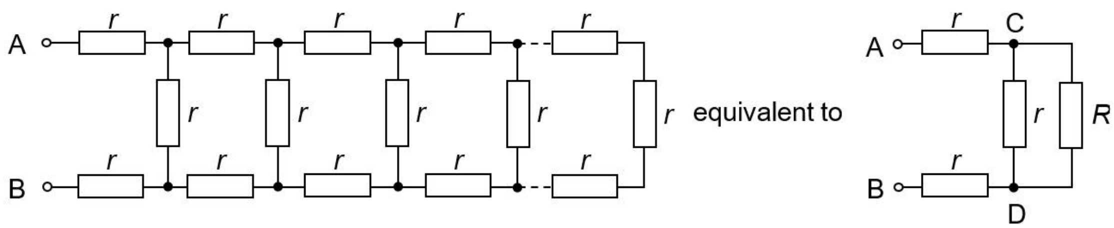


Fig. 3.2 **Fig. 3.3**

(i) Find, in terms of r and R, the resistance of the circuit shown Fig. 3.3 when measured between the terminals A and B.

$$R_{AB} = r + r + \left(\frac{1}{r} + \frac{1}{R}\right)^{-1}$$
$$= 2r + \frac{rR}{r+R}$$

[3]

(ii) Because the network is infinitely long, the addition of the extra segment makes no difference to the overall resistance. Use this fact to show that

$$R = \left(1+\sqrt{3}\right)r$$

$$R = 2r + \frac{rR}{r+R}$$
$$R(r+R) = 2r(r+R) + rR$$
$$rR + R^2 = 2r^2 + 3rR$$
$$R^2 - 2rR - 2r^2 = 0$$
$$R = \frac{-(-2r) \pm \sqrt{(-2r)^2 - 4(-2r^2)}}{2}$$
$$= r \pm r\sqrt{3}$$
$$= \left(1+\sqrt{3}\right)r \qquad \text{(shown)}$$

4 A resistance network is formed by connecting twelve identical constantan wires, each of length 20 mm and cross-sectional area 0.66 mm^2, as the edges of a cube, as shown in Fig. 4.1.

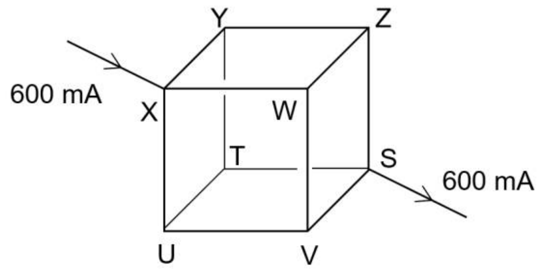


Fig. 4.1

(a) Find the resistance of one of the constantan wires, given that the resistivity of constantan is 4.9×10^{-7} Ω m.

$$R = \frac{\rho L}{A}$$
$$= \frac{(4.9 \times 10^{-7})(0.020)}{0.66 \times 10^{-6}}$$
$$= 1.48 \times 10^{-2}\ \Omega$$

resistance = ... Ω [3]

(b) A current of 600 mA enters at the corner X and leaves at the diagonally opposite corner S. By considering the symmetry of the cube, find the currents in each of the wires.

Wire XW, XU, XY will have a current of $\frac{600}{3} = 200$ mA

Wire WV, WZ, YT, YZ, UT, UV will have a current of $\frac{200}{2} = 100$ mA

Wire ZS, TS, VS will have a current of $100 \times 2 = 200$ mA

[2]

(c) The potential difference between X and S is the algebraic sum of the potential differences across each of the wires making up any continuous path between X and S. Hence find the resistance of the network between X and S.

Considering one of the paths: XW → WV → VS,

Potential difference $= I_{\text{XW}}R + I_{\text{WV}}R + I_{\text{VS}}R$

$= \left(200\times10^{-3} + 100\times10^{-3} + 200\times10^{-3}\right)\left(1.48\times10^{-2}\right)$

$= 7.4242\times10^{-3}$ V

$V = IR_{\text{eff}}$

$R_{\text{eff}} = \dfrac{7.4242\times10^{-3}}{600\times10^{-3}} = 0.012\ \Omega$

[5]

Answer Key

1 (b)(v) x = 0.70
2 (a) 16.9 W **(b)(ii)** 0.536 Ω **(b)(iii)** 12.1W
4 (a) 1.48×10^{-2} Ω **(b)** XW,XU,XY = 200 mA ; WV,WZ,YT,YZ,UT,UV = 100mA ; ZS,TS,VS = 200 mA **(c)** 0.012 Ω