Electromagnetic Induction

1 Fig. 1.1 below shows a metal bar of length L moving at a constant velocity v parallel to a long, straight wire carrying a steady current I.

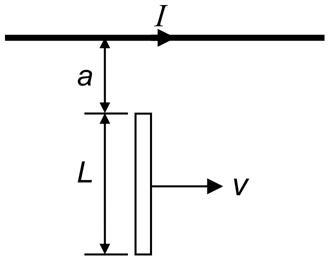


Fig. 1.1

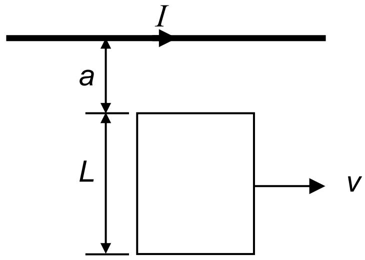


Fig. 1.2

(a) Show that the magnitude of the emf induced is

$$\varepsilon = \frac{\mu_0 I v}{2\pi} \ln\left(1 + \frac{L}{a}\right)$$

(b) If the bar is replaced by a rectangular wire loop of resistance R, as shown in Fig. 1.2, what is the magnitude of the current induced in the loop?

2 An induction furnace uses electromagnetic induction to produce eddy currents in a conductor, thereby raising the conductor's temperature. Commercial units operate at frequencies ranging from 60 Hz to about 1 MHz and deliver powers from a few watts to several megawatts. Induction heating can be used for warming a metal pan on a kitchen stove. By creating an induced current for a short time interval at an appropriately high frequency, one can heat a sample down to a controlled depth.

To explore induction heating, consider a flat conducting disk of radius R, thickness b, and resistivity ρ. A sinusoidal magnetic field $B_0\cos(\omega t)$ is applied perpendicular to the disk. Also assume the eddy currents occur in circles concentric with the disk.

(a) Calculate the average power delivered to the disk.

(b) By what factor does the power change when the amplitude of the field doubles? When the frequency doubles? When the radius of the disk doubles?

3 A long, straight wire of negligible resistance is bent into a V shape, its two arms making an angle α with each other, and placed horizontally in a vertical, homogeneous magnetic field of strength B. A rod of total mass m, and resistance r per unit length, is placed on the V-shaped conductor, at a distance x_0 from its vertex A, and perpendicular to the bisector of the angle α, as shown in Fig. 3.1.

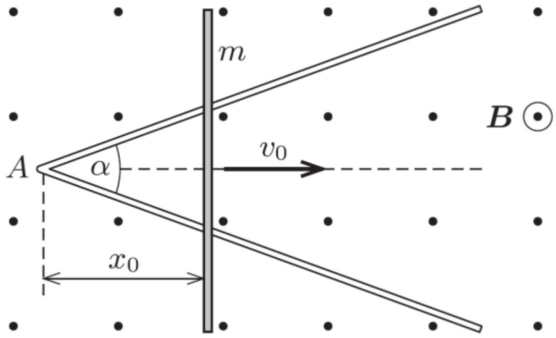


Fig. 3.1

The rod is started off with an initial velocity v_0 along the bisector, and away from A. The rod is long enough not to fall off the wire during the subsequent motion, and the electrical contact between the two is good – although the friction between them is negligible. T is the area of the triangle formed by the rod and the wire at that instant.

(a) Derive the magnetic force acting on the rod.

State the direction of the force relative to the velocity of the rod.

(b) Apply Newton's Second Law to show that

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{B^2}{r}\frac{\mathrm{d}T}{\mathrm{d}t}.$$

State the meaning of the negative sign.

(c) Let the distance between the rod and A at an instant in time be x.

Express the area T in terms of x and α.

(d) Hence, determine the distance at which the rod will stop moving in terms of m, r, B, α, v_0 and x_0.

Alternating Current

4 **(a)** A transformer connected to a 50 Hz AC supply may emit a humming sound. A student relates the sound to the windings of each coil. Suggest an explanation to support the student's claim, and deduce the frequency of the hum.

(b) When connected to the 50 Hz AC supply, the brightness of a light bulb changes periodically. Deduce the frequency of this change.

5 An electrical transmission line connects a generator to a load. The line has resistance X, giving rise to power loss in the line. The constant load resistance is R.

(a) Show that, for a given power P delivered by the generator, the power loss in the line is inversely proportional to V^2, where V is the output voltage of the generator. [3]

(b) This system is to transmit power of 200 kW from the generator. The line resistance is 0.60 Ω. It is required that the power loss in the line should not exceed 0.015 % of the total power. Find the minimum output voltage to achieve this. [3]

Answers

1 (b) 0

2 (a) $\dfrac{[B_0\omega]^2 \pi R^4 b}{16\rho}$, (b) 4, 4, 16

3 (a) $F = \dfrac{B^2}{r}\dfrac{\mathrm{d}T}{\mathrm{d}t}$ (c) $x^2 \tan\dfrac{\alpha}{2}$, (d) $\sqrt{\dfrac{mv_0 r}{B^2 \tan(\alpha/2)} + x_0^2}$

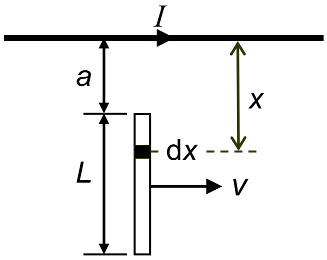
4 (a) 100 Hz, (b) 100 Hz

5 (b) 28.3 kV

Suggested Solutions to H3 Tutorial on Electromagnetic Induction & Alternating Currents

Electromagnetic Induction

1 (a) Consider an infinitesimal section of the rod, of length dx, at a distance x from the wire.



The magnetic flux density at a distance x from the wire is

$$B = \frac{\mu_0 I}{2\pi x}.$$

Hence the infinitesimal emf induced in dx is given by

$$\text{infinitesimal emf } = B \times dx \times v = \frac{\mu_0 I}{2\pi x} \times dx \times v .$$

The total emf is obtained by summing up all the infinitesimal emfs:

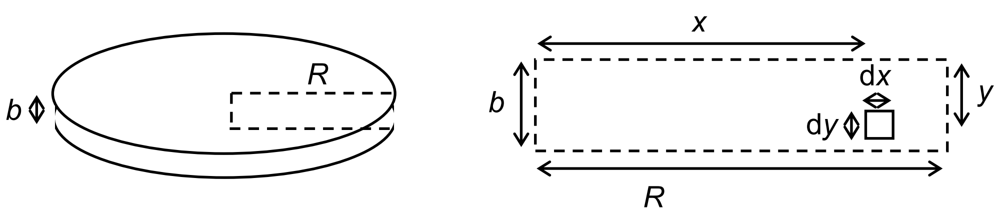
$$\text{total emf } = \Sigma\text{infinitesimal emf } = \frac{\mu_0 I v}{2\pi}\int_a^{a+L}\frac{\mathrm{d}x}{x} = \frac{\mu_0 I v}{2\pi}\ln\frac{a+L}{a} = \frac{\mu_0 I v}{2\pi}\ln\left(1+\frac{L}{a}\right)$$

(b) The current in the circuit is zero.

One way to understand this is to consider the flux linkage in the loop. It is constant in time. Hence, according to Faraday's law, the emf induced in the loop is zero. Consequently, the current is zero.

The other way to understand this is to consider the two sides of the loop that are cutting field lines as two sources of motional emf. They are of the same lengths, moving at the same velocity. Hence, the emf induced in them are of the same magnitude (as calculated in **(a)**) and in the same direction. Hence, the emf due to the two sides cancel each other.

2 (a)



Consider the cross-section of the disk denoted by the dashed rectangle, of height b and length R. dxdy is an infinitesimal area that is a distance x from the centre of the disk, and a distance y from the top of the disk. It should be clear that, as variables, x varies from 0 to R, and y varies from 0 to b.

Consider the infinitesimal ring shape, for which dxdy is a cross-section. The radius of the ring is x. The resistance of the ring shape (for the current flowing along the ring) is

$$R_{\text{ring}} = \rho \frac{2\pi x}{\text{d}x\text{d}y}$$

The flux linkage in the ring is

$$\Phi = BA = B_0 \cos(\omega t) \times \pi x^2$$

Hence the emf induced in the ring is

$$\text{emf} = -\frac{\text{d}\Phi}{\text{d}t} = -B_0 \pi x^2 \frac{\text{d}}{\text{d}t}\cos(\omega t) = B_0 \pi x^2 \omega \sin(\omega t)$$

The infinitesimal power dissipated in the ring is then

$$\text{d}p = \frac{\text{emf}^2}{R_{\text{ring}}} = \left[B_0 \pi x^2 \omega \sin(\omega t)\right]^2 \left[\rho \frac{2\pi x}{\text{d}x\text{d}y}\right]^{-1} = \frac{\left[B_0 \omega \sin(\omega t)\right]^2 \pi}{2\rho} x^3 \text{d}x\text{d}y$$

The power at time t is obtained by integrating over x and y:

$$\text{total power at } t = \int \text{d}p = \frac{\left[B_0 \omega \sin(\omega t)\right]^2 \pi}{2\rho} \int_0^R x^3 \text{d}x \int_0^b \text{d}y$$

$$= \frac{\left[B_0 \omega \sin(\omega t)\right]^2 \pi}{2\rho} \frac{R^4}{4} b$$

The average power is obtained by averaging over time:

$$\text{average power} = \frac{\left[B_0 \omega\right]^2 \pi}{2\rho} \frac{R^4}{4} b \left[\frac{1}{T}\int_0^T \sin^2(\omega t)\text{d}t\right]$$

$$= \frac{\left[B_0 \omega\right]^2 \pi}{2\rho} \frac{R^4}{4} b \times \frac{1}{2}$$

$$= \frac{\left[B_0 \omega\right]^2 \pi R^4 b}{16\rho}$$

(b) When the amplitude B_0 of the field doubles, the average power increases by a factor of 4; when the frequency doubles, the average power increases by a factor of 4; when the radius of the disk doubles, the average power increases by a factor of 16.

3 (a) magnitude of emf $= \dfrac{\text{d}\Phi}{\text{d}t} = \dfrac{\text{d}}{\text{d}t}(BT) = B\dfrac{\text{d}T}{\text{d}t}$

Let the length of the rod between the two points of contact be l.
The resistance in the circuit is then $R = rl$.
The current I in the circuit is

$$I = \frac{\text{emf}}{R} = \frac{B}{rl}\frac{\text{d}T}{\text{d}t}.$$

The magnetic force is given by

$$F = IlB = \frac{B}{rl}\frac{\text{d}T}{\text{d}t} lB = \frac{B^2}{r}\frac{\text{d}T}{\text{d}t}$$

The direction of the force is opposite to that of the velocity.

(b) Newton's Second Law for an object of constant mass:

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Hence

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{B^2}{r}\frac{\mathrm{d}T}{\mathrm{d}t},$$

where the negative sign means that the acceleration (to the left) is opposite to the direction of increasing area T (to the right) [or that while dv/dt is negative (since v is decreasing), dT/dt is positive (since the area T is increasing)].

(c) $$T = \frac{1}{2}x \times x\tan\frac{\alpha}{2} \times 2 = x^2\tan\frac{\alpha}{2}$$

(d) $$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{B^2}{r}\frac{\mathrm{d}T}{\mathrm{d}t}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{B^2}{r}\frac{\mathrm{d}T}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mv + \frac{B^2}{r}T\right) = 0$$

Hence $mv + \frac{B^2}{r}T$ is constant in time.

$$\left(mv + \frac{B^2}{r}T\right)_{\text{fin}} = \left(mv + \frac{B^2}{r}T\right)_{\text{ini}}$$

At the final position, $v = 0$, hence

$$\frac{B^2}{r}T_{\text{fin}} = mv_0 + \frac{B^2}{r}T_{\text{ini}}$$

$$\frac{B^2}{r}x_{\text{fin}}^2\tan\frac{\alpha}{2} = mv_0 + \frac{B^2}{r}x_0^2\tan\frac{\alpha}{2}$$

$$x_{\text{fin}} = \sqrt{\frac{mv_0 r}{B^2\tan(\alpha/2)} + x_0^2}$$

Alternating Currents

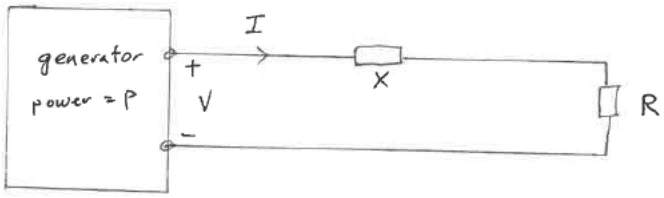
4 (a) The humming sound comes from the windings of the transformer knocking against one another. Adjacent windings in a transformer are wound parallel to each other. The current flows through adjacent windings in the same direction. The attractive force between parallel currents causes the windings to come together and knock into one another, producing a hum.

When the alternating current drops to zero momentarily before changing direction, this force disappears so that the windings return to their original position. When the current reverses direction, the current in adjacent windings are still in the same direction, the attractive force due to which produces another hum.

Since there are two hums produced in one period of the alternating current, the frequency of the hum will be twice that of the frequency of the alternating current. The frequency of the hum is thus 100 Hz.

(b) The brightness of the bulb depends on the magnitude of the current passing through it, but not on the direction of the current. Thus, in one period of the alternating current, the bulb reaches its peak brightness twice (when the magnitude of the current reaches its peak value in either direction). Thus the brightness of the bulb changes with a frequency of 100 Hz.

5 (a)



Power loss $= I^2 X$

Current delivered by generator $I = \frac{P}{V}$

$\therefore$ power loss $= \left(\frac{P}{V}\right)^2 X \propto \frac{1}{V^2}$

(b) power loss $= \frac{P^{\cancel{2}}}{V^2} X = \frac{0.015}{100} \cancel{P}$

$V^2 = \frac{100}{0.015}(200,000)(0.60)$

$V = 28.3\ kV$