H3 Physics Topic 5 - Capacitors and Inductors (Oscillatory)

Key Ideas

- In an oscillating LC circuit, energy is shuttled periodically between the electric field of the capacitor and the magnetic field of the inductor; instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2},$$

where q is the instantaneous charge on the capacitor and i is the instantaneous current through the inductor.

- The total energy $U\ (= U_E + U_B)$ remains constant.
- The principle of conservation of energy leads to

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0 \quad (LC \text{ oscillations})$$

as the differential equation of LC oscillations (with no resistance).

- The solution of this differential equation is

$$q = Q\cos(\omega t + \phi) \quad \text{(charge)},$$

in which Q is the charge amplitude (maximum charge on the capacitor) and the angular frequency ω of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}.$$

- The phase constant ϕ is determined by the initial conditions (at $t = 0$) of the system.
- The current i in the system at any time t is

$$i = -\omega Q\sin(\omega t + \phi) \quad \text{(current)},$$

in which ωQ is the current amplitude I.

11 A series circuit consists of a capacitor of capacitance C, an inductor of inductance L and a switch, as shown in Fig. 11.1.

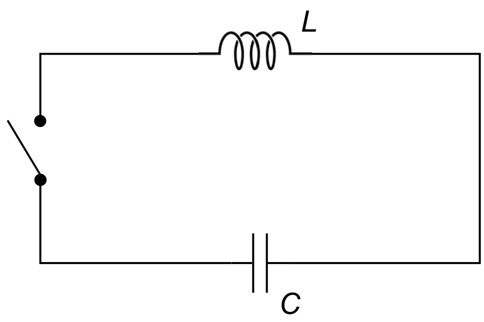

Fig. 11.1

Initially, the switch is open and the capacitor holds a charge Q_o.

(a) Compare how the energy is stored in a capacitor and in an inductor.

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(b) The switch is closed at time $t = 0$ and the capacitor begins to discharge through the inductor.

Show that the charge q on the capacitor obeys the equation

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0\ .$$

(c) **(i)** Deduce an expression for the natural frequency f_o of the circuit.

(ii) Determine an expression for the charge q as a function of time, $q(t)$.

(iii) On Fig. 11.2, draw two separate graphs to show the variation with time of the charge q on the capacitor and the current I in the circuit.

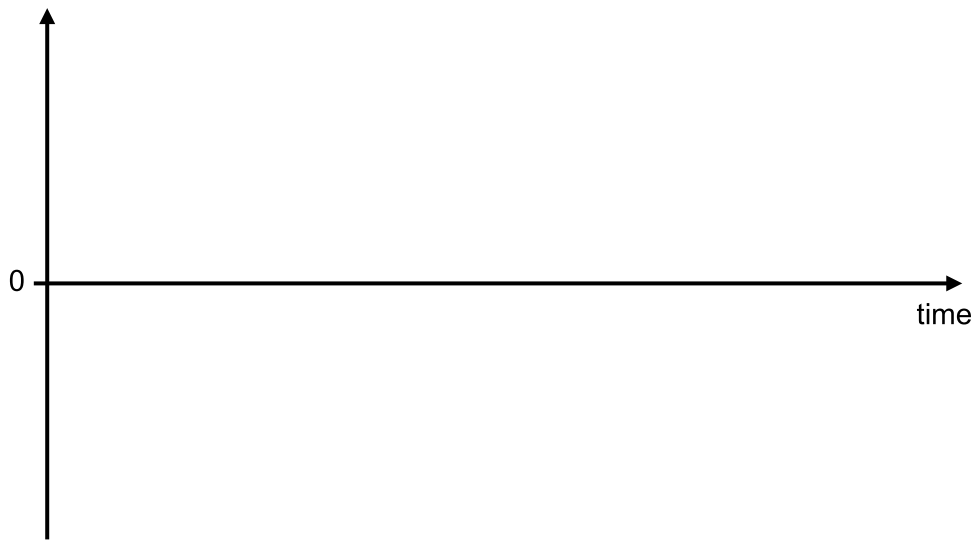

Fig. 11.2

(iv) Show that the total energy E_T of the circuit for $t > 0$ is given by the expression

$$E_T = \frac{Q_o^2}{2C}$$

(d) A resistor of resistance R is now added in series with the capacitor and inductor as shown in Fig. 9.3.

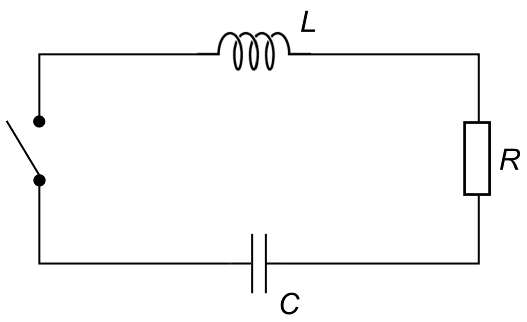

Fig. 11.1

The capacitor is again fully charged and the switch is closed at time $t = 0$.

The charge q on the capacitor obeys the equation

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

Assuming that $R^2 << \frac{L}{C}$, an expression of the form $q = Q_o e^{-\gamma t}\cos(\omega t + \varphi)$ is a general solution of this equation.

(i) Determine the expressions for the constants γ in terms of R and L, and ω in terms of γ and ω_o where $\omega_o = 2\pi f_o$ and f_o is the natural frequency of the circuit in **(c)(i)**.

(ii) Derive an expression for the half-life time τ of the circuit, when the energy of the system drops to half of its initial value.

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Name: ______________________________ **Class:** ____________

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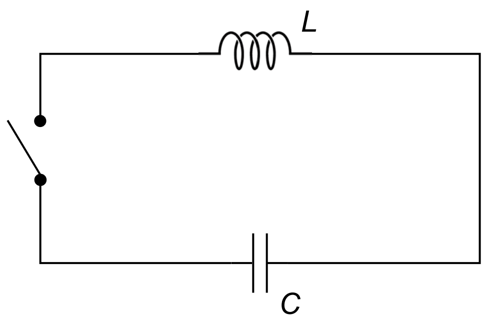

Fig. 11.1

Initially, the switch is open and the capacitor holds a charge Q_o.

(a) Compare how the energy is stored in a capacitor and in an inductor.

In a capacitor, the energy is stored in the electric field produced by the oppositely charged plates.

In an inductor, the energy is stored in the magnetic field produced by the current flowing through it.

(b) The switch is closed at time $t = 0$ and the capacitor begins to discharge through the inductor.

Show that the charge q on the capacitor obeys the equation

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 .$$

The total energy stored by the capacitor and inductor at any time $U_T = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2$

Since there are no energy losses:

$$\frac{dU_T}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2\right) = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}L\left(\frac{dq}{dt}\right)^2\right) = 0$$

$$\frac{q}{C}\frac{dq}{dt} + L\left(\frac{dq}{dt}\right)\frac{d^2q}{dt^2} = 0$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

(c) **(i)** Deduce an expression for the natural frequency f_o of the circuit.

By comparing to the defining equation for a simple harmonic motion:

$$\frac{d^2x}{dt^2} = -\omega_o^{\,2}x \Rightarrow \omega_o = \sqrt{\frac{1}{LC}}$$

$$\therefore \quad f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

(ii) Determine an expression for the charge q as a function of time, $q(t)$.

Since at t = 0, q = Q_o, $\therefore \quad q = Q_o \cos(\omega_o t)$

$$= Q_o \cos\left(\sqrt{\frac{1}{LC}}t\right)$$

(iii) On Fig. 9.2, draw two separate graphs to show the variation with time of the charge q on the capacitor and the current I in the circuit.

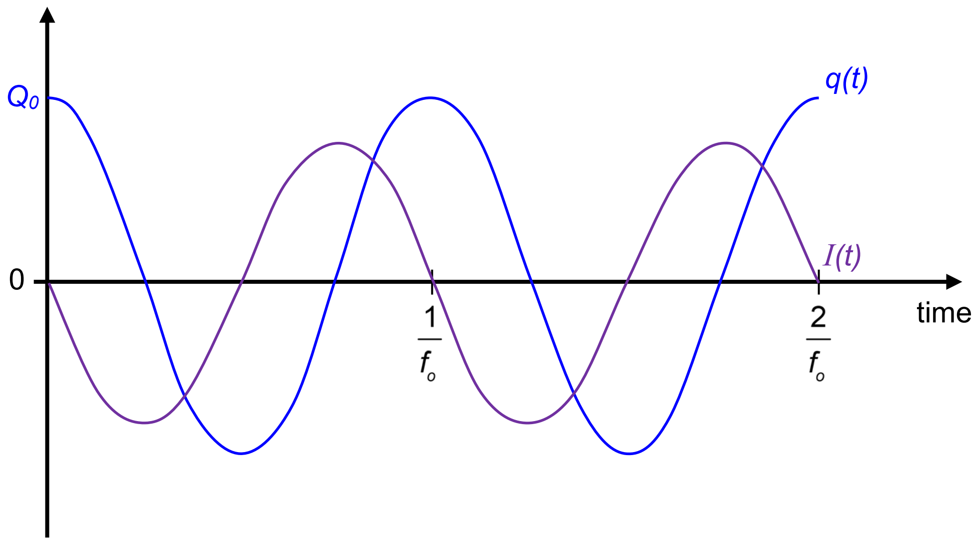

Fig. 9.2

(iv) Show that the total energy E_T of the circuit for $t > 0$ is given by the expression

$$E_T = \frac{Q_o^2}{2C}$$

The energy stored in the capacitor $U_C = \frac{q^2}{2C} = \frac{1}{2C}\left(Q_o \cos(\omega t)\right)^2 = \frac{Q_o^2}{2C}\cos^2(\omega t),\ \omega = \sqrt{\frac{1}{LC}}$

The energy stored in the inductor $U_L = \frac{LI^2}{2} = \frac{L}{2}\left(\omega Q_o \sin(\omega t)\right)^2 = \frac{Q_o^2}{2C}\sin^2(\omega t)$

$$\begin{aligned} U_T &= U_C + U_L \\ &= \frac{Q_o^2}{2C}\cos^2(\omega t) + \frac{Q_o^2}{2C}\sin^2(\omega t) \\ &= \frac{Q_o^2}{2C}\left(\cos^2(\omega t) + \sin^2(\omega t)\right) \\ &= \frac{Q_o^2}{2C} \quad \text{(shown)} \end{aligned}$$

$E = \frac{1}{2}\frac{q^2}{C}$.

(d) A resistor of resistance R is now added in series with the capacitor and inductor as shown in Fig. 9.3.

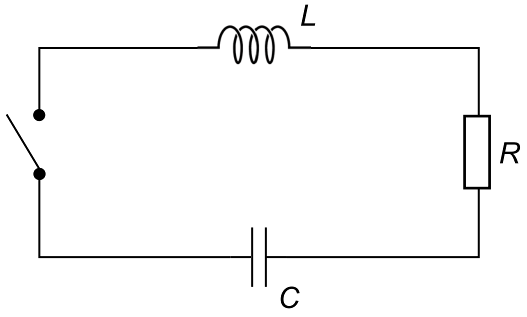

Fig. 9.3

The capacitor is again fully charged and the switch is closed at time $t = 0$.

The charge q on the capacitor obeys the equation

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

Assuming that $R^2 << \frac{L}{C}$, an expression of the form $q = Q_o e^{-\gamma t}\cos(\omega t + \varphi)$ is a general solution of this equation.

(i) Determine the expressions for the constants γ in terms of R and L, and ω in terms of γ and ω_o where $\omega_o = 2\pi f_o$ and f_o is the natural frequency of the circuit in **(c)(i)**.

Since at t = 0, q = Q_o $\Rightarrow$ $\varphi = 0$

$$\therefore \quad q = Q_o e^{-\gamma t}\cos(\omega t)$$

$$\frac{dq}{dt} = -\gamma Q_o e^{-\gamma t}\cos(\omega t) - \omega Q_o e^{-\gamma t}\sin(\omega t)$$

$$\begin{aligned}\frac{d^2q}{dt^2} &= \gamma^2 Q_o e^{-\gamma t}\cos(\omega t) + \omega\gamma Q_o e^{-\gamma t}\sin(\omega t) - \left(-\gamma\omega Q_o e^{-\gamma t}\sin(\omega t) + \omega^2 Q_o e^{-\gamma t}\cos(\omega t)\right)\\ &= \gamma^2 Q_o e^{-\gamma t}\cos(\omega t) + 2\omega\gamma Q_o e^{-\gamma t}\sin(\omega t) - \omega^2 Q_o e^{-\gamma t}\cos(\omega t)\\ &= \left(\gamma^2 Q_o e^{-\gamma t} - \omega^2 Q_o e^{-\gamma t}\right)\cos(\omega t) + 2\omega\gamma Q_o e^{-\gamma t}\sin(\omega t)\end{aligned}$$

Since $\dfrac{d^2q}{dt^2} + \dfrac{R}{L}\dfrac{dq}{dt} + \dfrac{1}{LC}q = 0$

$$\left[\left(\gamma^2 Q_o e^{-\gamma t} - \omega^2 Q_o e^{-\gamma t}\right)\cos(\omega t) + 2\omega\gamma Q_o e^{-\gamma t}\sin(\omega t)\right] +$$
$$\frac{R}{L}\left[-\gamma Q_o e^{-\gamma t}\cos(\omega t) - \omega Q_o e^{-\gamma t}\sin(\omega t)\right] + \frac{1}{LC}\left[Q_o e^{-\gamma t}\cos(\omega t)\right] = 0$$

Omitting the common factors $Q_o e^{-\gamma t}$, and collecting terms:

$$\Rightarrow \cos(\omega t)\times\left[\gamma^2 - \omega^2 - \frac{R}{L}\gamma + \frac{1}{LC}\right] + \sin(\omega t)\times\left[2\omega\gamma - \frac{R}{L}\omega\right] = 0$$

Since the RHS is identically 0, the coefficients of the sine and cosine functions must be 0:

$$2\omega\gamma - \frac{R}{L}\omega = 0 \Rightarrow \gamma = \frac{R}{2L}$$

$$\gamma^2 - \omega^2 - \frac{R}{L}\gamma + \frac{1}{LC} = 0 \Rightarrow \omega^2 = \gamma^2 - \frac{R}{L}\gamma + \frac{1}{LC} = \left(\frac{R}{2L}\right)^2 - \frac{R}{L}\frac{R}{2L} + \frac{1}{LC} = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2 = {\omega_0}^2 - \gamma^2$$

$$\Rightarrow \omega = \sqrt{{\omega_0}^2 - \gamma^2}$$

OR

For $\omega t = \dfrac{\pi}{2}$: $2\omega\gamma Q_o e^{-\gamma\frac{\pi}{2\omega}} - \dfrac{R}{L}\omega Q_o e^{-\gamma\frac{\pi}{2\omega}} = 0 \qquad \Rightarrow \gamma = \dfrac{R}{2L}$

For $\omega t = 0$: $\left(\gamma^2 Q_o - \omega^2 Q_o\right) + \dfrac{R}{L}(-\gamma Q_o) + \dfrac{1}{LC}Q_o = 0$

$$\left(\gamma^2 - \omega^2\right) - \gamma\frac{R}{L} + \frac{1}{LC} = 0$$

Substituting $\gamma = \dfrac{R}{2L}$: $\left(\left(\dfrac{R}{2L}\right)^2 - \omega^2\right) - \left(\dfrac{R}{2L}\right)\dfrac{R}{L} + \dfrac{1}{LC} = 0$

$$\begin{aligned}\omega^2 &= \frac{1}{LC} - \frac{R^2}{4L^2}\\ &= \omega_o^2 - \gamma^2\\ \omega &= \sqrt{\omega_o^2 - \gamma^2}\end{aligned}$$

(ii) Derive an expression for the half-life time τ of the circuit, when the energy of the system drops to half of its initial value.

Since $q = Q_o e^{-\frac{R}{2L}t} \cos\left(\sqrt{\omega_o^2 - \frac{R^2}{4L^2}}t\right)$

$\Rightarrow$ maximum charge stored on the capacitor at any time $= Q_o e^{-\frac{R}{2L}t}$

Since the initial energy in the circuit is the energy stored in the capacitor $= \dfrac{Q_o^2}{2C}$

When energy of system drops to half its initial value, maximum energy stored in the capacitor $= \dfrac{Q_o^2}{4C}$

This occurs when the maximum charge stored on the capacitor $= \dfrac{Q_o}{\sqrt{2}}$

Hence,
$$\frac{Q_o}{\sqrt{2}} = Q_o e^{-\frac{R}{2L}\tau}$$
$$\ln\frac{1}{\sqrt{2}} = -\frac{R}{2L}\tau$$
$$\tau = \frac{L}{R}\ln 2$$