Additional Problems for Special Relativity

1 **(a)** With reference to the rectangular coordinates system (x,y,z,t), write down the Galilean Transformation equations.

(b) Show that the Galilean Transformation equations are not consistent with the second postulate of the Special Theory of Relativity.

(c) Consider two objects that interact through a force that is dependent on their separation and not on their velocities and accelerations (e.g., the gravitational force). Show that under Galilean transformation, the force law (e.g. $F = GMm/r^2$) remains invariant.

2 **(a)** Two events occur at the same place in a certain inertial frame and are separated by a time interval of 4 seconds. What is the spatial separation between these two events in an inertial frame in which the events are separated by a time interval of 6 seconds?

(b) Two events occur at the same time in a certain inertial frame S and are separated by a distance of 1 km along the x axis. What is the time difference between these two events as measured in a frame S' moving with constant velocity along x and in which their separation is measured as 2 km?

Ans: 1.34×10^{9} m, 5.77×10^{-6} s

3 **(a)** An archer releases an arrow that travels at a speed of 50 m s^{-1} towards a target 25 m away. At what velocity must an observer move so as to see the releasing of the arrow and the arrow hitting the target happening at the same position?

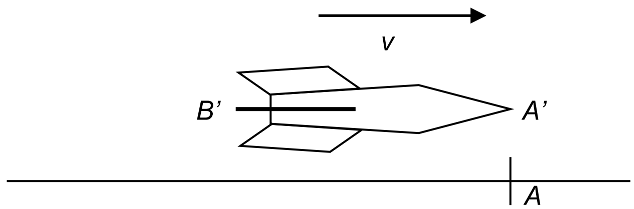
(b) Two stars that are 3×10^{10} m apart are observed to undergo nova explosions at times t_1 and t_2, with $t_2 = t_1 + 1$ sec. With what velocity must a space shuttle be moving so that the astronaut on board will observe the two nova explosions happening simultaneously?

Ans:

4 A physics professor on Earth gives an examination to students who are on a rocket ship traveling at speed v relative to Earth. The moment the ship passes the professor, he signals the start of the exam. If he wishes his students to have time T_o (rocket time) to complete the exam, How long does he have to wait before sending a light signal telling them to stop.

Ans: $T = T_o\sqrt{\dfrac{1-v/c}{1+v/c}}$

5 A rocketship of proper length l_o travels at constant velocity v relative to frame S (see diagram below). The nose of the ship (A') passes the point A in S at $t = t' = 0$, and at this instant a light signal is sent out from A' to B'.



(a) When, by rocketship time (t'), does the signal reach the tail B' of the ship?

(b) At what time t_1, as measured in S, does the signal reach the tail of the ship?

(c) At what time t_2, as measured in S, does the tail of the ship pass the point A?

Ans: $t' = l_o/c$, $t_1 = \dfrac{l_o}{c}\sqrt{\dfrac{1-v/c}{1+v/c}}$, $t_2 = l_o/\gamma v$

6 At noon a rocketship passes the Earth with a velocity 0.8c. Observers on the ship and on Earth agree that it is noon.

(a) At 12.30 pm as read by a rocketship clock, the ship passes an interplanetary navigational station that is fixed relative to the Earth and whose clock read Earth time. What time is it at the station?

(b) How far from Earth (in Earth coordinates) is the station?

(c) At 12.30 pm rocketship time the ship reports by radio back to Earth. When (by Earth time) does the Earth receive the signal?

(d) The station on Earth replies immediately. When (by rocketship time) is the reply received?

Ans: (a) 12.50pm, (b) 7.2×10^{11} m, (c) 1.30pm, (d) 4.30pm

7 The rockets of the Goths and the Huns are each 1000 m long in their respective rest frame. The rockets pass each other, virtually touching, at relative speed of 0.8c. The Huns have a laser cannon at the rear of their rocket that shoots a deadly laser beam at right angles to the motion. The captain of the Hun rocket wants to send a threatening message to the Goths by “firing a shot across their bow.” He tells his first mate, “The Goths rocket is length contracted to 600 m. Fire the laser cannon at the instant the nose of our rocket passes the tail of their rocket. The laser beam will cross 400 m in front of them.” But things are different from the Goths’ perspective. The Goth captain muses, “The Huns’ rocket is length contracted to 600 m, 400 m shorter than our rocket. If they fire the laser cannon as their nose passes the tail of our rocket, the lethal laser blast will go right through our side.” The first mate on the Hun rocket fires as ordered. Does the laser beam blast the Goths or not? Resolve this paradox. Show that, when properly analyzed, the Goths and the Huns agree on the outcome. Your analysis should contain both quantitative calculations and written explanation.

8 Two neutrons, A and B, are approaching each other along a common straight line. Each has a constant speed βc as measured in the laboratory. Show that the total energy of neutron B, as observed in the rest frame of neutron A, is

$$E = \frac{1+\beta^2}{1-\beta^2} M_o c^2$$

where M_o is the rest mass of the neutron.

9 **(a)** A photon of energy E collides with a stationary particle of rest mass m_o and is absorbed. What is the velocity of the resulting composite particle?

(b) A particle of rest mass m_o moving at a speed of $0.8c$ collides with a similar particle at rest and forms a composite particle. What is the rest mass of the composite particle and what is its speed?

Ans: (a) $\left(\frac{E}{E+m_o c^2}\right)c$, (b) $\frac{4m_o}{\sqrt{3}}; \frac{c}{2}$

11 (Morin Page 615 Exercise 12.22)

A mass m travels at speed $3c/5$, and another mass m sits at rest.

(a) Find the total energy and momentum of the two particles in the lab frame.

(b) Find the speed of the CM of the system, by using a velocity-addition argument.

(c) Find the total energy and momentum of the two particles in the CM frame, without using the Lorentz Transformations (for energy and momentum).

(d) Verify that the E's and the p's (of the two frames) are related by the relevant Lorentz Transformations.

(e) Verify that $E^2 - p^2c^2$ for each mass is the same in both frames. Likewise for $E_{\text{total}}{}^2 - p_{\text{total}}{}^2 c^2$

MORE QUESTIONS:

12 (Morin Page 550 Problem 11.6)

A train and a tunnel both have proper lengths L. The train moves toward the tunnel at speed v. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, the sensor tells the bomb to disarm itself.

Does the bomb explode?

13 (Modified from Morin Page 557 Problem 11.37)

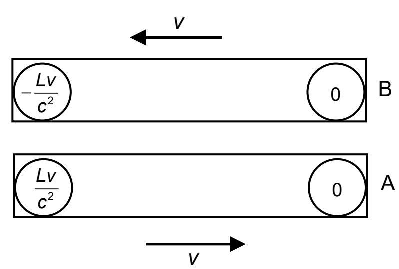
A and B leave from a common point (with their clocks both reading zero) and travel in opposite directions each with speed v relative to a stationary observer on Earth. When B's clock reads T, he sends out a light signal to A. When A receives the signal, what time does A's clock read? Answer this question by doing the calculations entirely in

(a) A's frame, and

(b) B's frame.

14 (Morin Page 559 Exercise 11.43)

An observer on the ground sees two trains, A and B, both of proper length L, move in opposite directions at speed v with respect to the ground. She notices that when the trains "overlap", clocks at the front of A and rear of B both read zero. From the "rear clock ahead" effect, she therefore also notices that clocks at the rear of A and front of B read Lv/c^2 and $-Lv/c^2$, respectively



Now imagine riding along on A. When the rear of B passes the front of your train (A), clocks at both these places read zero (as stated above). Explain, by working only in the frame of A, why clocks at the back of A and the front of B read Lv/c^2 and $-Lv/c^2$, respectively, when these points coincide.

Solutions

1 **(a)** $x' = x - vt$

$y' = y$

$z' = z$

$t' = t$

(b) Galilean Transformation: $x' = x - vt$

$$\Rightarrow \frac{dx'}{dt} = \frac{dx}{dt} - v$$

$$\Rightarrow u_x' = u_x - v$$

Hence, if an observer in the lab frame observes a photon travelling at speed c, an observer in the moving (primed) frame will observe the light travelling at $c - v$, which violates the second postulate.

(c) We will use the gravitational force as an example. In the unprimed frame,

$$F = \frac{GMm}{r^2} = \frac{GMm}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In the primed frame, the gravitational force becomes:

$$\begin{aligned} F' = \frac{GMm}{r'^2} &= \frac{GMm}{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2} \\ &= \frac{GMm}{((x_2 - vt) - (x_1 - vt))^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \frac{GMm}{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= F \end{aligned}$$

Hence, the gravitational force is invariant under Galilean transformation.

2 **(a)** $x_2 = x_1, \quad t_2 - t_1 = 4$

$$6 = t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right] = 4\gamma$$

Hence, $\gamma = 3/2$ and $\beta = \sqrt{5}/3$.

$$\begin{aligned} x_2' - x_1' &= \gamma\left[(x_2 - x_1) - v(t_2 - t_1)\right] \\ &= \frac{3}{2}\left[0 - 4\beta c\right] = 2\sqrt{5}c = 1.34 \times 10^9 \text{ m} \end{aligned}$$

(b) In this case, $\gamma = 2$ and $\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{3}/2$.

Therefore, $t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right] = 2\left[\frac{\sqrt{3}}{2} \cdot \frac{1000}{3\times 10^8}\right] = 5.77 \times 10^{-6}$ s

3 **(a)** Intuitively, the required speed will be 50 m s^{-1} because at this speed, the arrow will appear stationary to the observer and it will be the bow and the target that move pass the arrow. We can also use Lorentz Transformations to show this, though it will be an overkill:

Given $x_2 - x_1 = 25$ and $t_2 - t_1 = 25 \div 50 = 0.5$, we require $x_2' - x_1' = 0$. Therefore,

$$x_2' - x_1' = 0 = \gamma\left[(x_2 - x_1) - v(t_2 - t_1)\right]$$

$$\Rightarrow \qquad v = \frac{x_2 - x_1}{t_2 - t_1} = \frac{25}{0.5} = 50 \text{ m s}^{-1}$$

(b) Given $x_2 - x_1 = 3\times 10^{10}$ and $t_2 - t_1 = 1$, we require $t_2' - t_1' = 0$. Therefore,

$$t_2' - t_1' = 0 = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$$

$$\Rightarrow \qquad v = c^2\frac{t_2 - t_1}{x_2 - x_1} = 9\times 10^{16} \times \frac{1}{3\times 10^{10}} = \underline{\mathbf{3 \times 10^6 \text{ m s}^{-1}}}$$

4 Assume that time = 0 when the spaceship passes the professor and let *T* be the required waiting time of the professor. When he sends out the 'stop writing' signal at time = *T*, the spaceship will have moved forward by *vT*. The signal will chase and catch up with the spaceship at time = *t* measured in the Earth frame. The time elapsed in the spaceship's frame will be t/γ and we require $t/\gamma = T_o$.

Since the spaceship will have travelled at a distance of *vt* when the signal catches up with it, the time that elapses between the sending out of the signal and its arrival at the spaceship is *vt/c*. Hence,

$$t = T + vt/c$$

$$\Rightarrow \qquad t = T/(1-\beta)$$

$$\Rightarrow \qquad T_o = \frac{t}{\gamma} = T\frac{(1-\beta^2)^{1/2}}{1-\beta} = T\sqrt{\frac{1+\beta}{1-\beta}}$$

$$\Rightarrow \qquad T = T_o\sqrt{\frac{1-\beta}{1+\beta}}$$

5 **(a)** $t' = l_o / c$

(b) If we denote the events of the light signal being emitted at A' and its arrival as B' as A and B respectively, then $t_A' = 0, t_B' = l_o/c, x_A' = 0, x_B' = -l_o$.

Hence, $t_1 = \gamma\left[t_B' + \frac{v}{c^2}x_B'\right] = \frac{1}{\sqrt{1-\beta^2}}\left[\frac{l_o}{c} + \frac{\beta}{c}(-l_o)\right] = \frac{l_o}{c}\left[\frac{1-\beta}{\sqrt{1-\beta^2}}\right] = \frac{l_o}{c}\sqrt{\frac{1-\beta}{1+\beta}}$

(c) The spaceship passes by A at time = l_o/v according to passengers on the spaceship, i.e., $t' = l_o/v$.

Hence,
$$\begin{aligned} t &= \gamma\left[t' + \frac{v}{c^2}(-l_o)\right] \\ &= \gamma\left[\frac{l_o}{v} + \frac{v}{c^2}(-l_o)\right] \\ &= \frac{\gamma l_o}{v}\left[1 - \frac{v^2}{c^2}\right] \\ &= \frac{l_o}{\gamma v} \end{aligned}$$

6 **(a)** $\beta = 0.8 \Rightarrow \gamma = 5/3$.

$$\Delta t = \gamma \Delta t' = \frac{5}{3}(30) = 50 \text{ min}$$

Hence, the station time is 12.50pm.

(b) $x = 0.8c \times 50 \times 60 =$ **7.2×10^{11} m**

(c) The light signal will require $7.2 \times 10^{11} \div (3 \times 10^8)$ = 2400 sec = 40 min to travel from the station to Earth. Hence, the Earth receives the signal at 12.50pm + 40min = 1.30pm.

(d) Using the result of Q4, $T_o = T\sqrt{\frac{1+\beta}{1-\beta}} = 1.5 \text{ hours} \times \sqrt{\frac{1+0.8}{1-0.8}} = 4.5 \text{ hours}$

Therefore, the rocket will receive the signal at 4.30pm rocketship time.

7 In the rest frame of neutron A, $u_B' = \dfrac{u_B - v}{1 - u_B v/c^2} = \dfrac{-\beta c - \beta c}{1-(-\beta c)\beta c/c^2} = \dfrac{-2\beta c}{1+\beta^2}$. That is,

$$\beta' = \frac{2\beta}{1+\beta^2} \qquad \Rightarrow \gamma = \frac{1}{\sqrt{1-\beta'^2}}$$

$$= \frac{1}{\sqrt{1-4\beta^2/\left(1+\beta^2\right)^2}}$$

$$= \frac{1+\beta^2}{\sqrt{1-2\beta^2+\beta^4}}$$

$$= \frac{1+\beta^2}{1-\beta^2}$$

Hence, $E = \gamma M_0 c^2 = \dfrac{1+\beta^2}{1-\beta^2} M_0 c^2$

8 **(a)** We will show that this is an impossible scenario. Let the velocity of particle after absorbing the photon be *v*. If we switch to this frame, then the particle will appear stationary after absorbing the photon. However, the speed of light cannot be transformed away which means that before the absorption, the photon will have some energy. On the other hand, the particle should be moving at *v* in the opposite direction. The total energy of the system will include the rest-mass energy of the particle, its KE, and the energy of the photon. This is more than the total energy of the system after the absorption, which is equal to the rest-mass energy of the particle only. Hence, if the scenario was indeed possible, we will violate the law of conservation of energy if we make our observation in the rest frame of the particle after the absorption.

(b) $\gamma = \dfrac{1}{\sqrt{1-0.8^2}} = \dfrac{5}{3}$

Conservation of Energy: $\left(m_o + \dfrac{5}{3}m_o\right)c^2 = \dfrac{8}{3}m_o c^2 = \gamma_1 M_o c^2$ —— (1)

Conservation of Momentum: $\dfrac{5}{3} \times m_o \left(\dfrac{4}{5}c\right) = \gamma_1 M_o \beta c$ ——— (2)

(2) ÷ (1), $\dfrac{1}{2c} = \dfrac{\beta}{c} \;\Rightarrow\; \beta = \dfrac{1}{2}$

Hence, the velocity of the composite particle is $\dfrac{1}{2}c$.

The corresponding value of γ_1 is $\dfrac{1}{\sqrt{1-1/4}} = \dfrac{2}{\sqrt{3}}$. Hence $M_o = \dfrac{8}{3}m_o \times \dfrac{\sqrt{3}}{2} = \dfrac{4}{\sqrt{3}}m_o$

10 1 MeV = 1.6 × 10^{-19} × 10^{6} = 1.6 × 10^{-13} J. Hence,

$$\frac{1}{2}\left(9.11\times10^{-31}\right)v^2 = 1.6\times10^{-13}$$

$$v = \sqrt{3.51\times10^{17}} = \mathbf{\underline{5.93 \times 10^8\ m\ s^{-1}}}$$

This is larger than the speed of light, which means that the formula KE = ½ mv^2 is not valid due to the high speed of the electron. Using relativistic approach, we have:

$$(\gamma - 1)mc^2 = 1.6\times10^{-13}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 2.95$$

$$\beta = 0.941$$

Hence, v = 0.941 × (3 × 10^8) = **<u>2.82 × 10^8 m s^{-1})</u>**

11 **(a)** $$\beta = \frac{3}{5} \Rightarrow \quad \gamma = \frac{1}{\sqrt{1-(3/5)^2}} = \frac{5}{4}$$

Energy in lab frame $= mc^2 + \frac{5}{4}mc^2 = \frac{9}{4}mc^2$

Momentum in lab frame $= \gamma mv + 0 = \frac{5}{4}m \times \frac{3}{5}c = \frac{3}{4}mc$

(b) Let v be the velocity of the CM frame. In this frame, the two particles move with equal but opposite velocity. In particular, since the target particle is stationary in the lab frame, it should be moving with a velocity of $-v$ in the CM frame. This in turn implies that particle 1 (the incident particle) has velocity v in the CM frame. To summarise,

$$u_1' = v \quad \text{and} \quad u_2' = -v$$

Substitute $u_1' = v$ into the velocity addition formula:

$$v = \frac{u_1 - v}{1 - u_1 v/c^2} = \frac{0.6c - v}{1 - 0.6v/c}$$

Solving for v, we get

$$v = \frac{1}{3}c$$

(c) In the CM frame, the speed of both particles are $v = \frac{1}{3}c$. Hence, $\gamma = \frac{1}{\sqrt{1-(1/3)^2}} = \frac{3}{\sqrt{8}}$

Total energy $= 2 \times \gamma mc^2 = \frac{3}{\sqrt{2}} mc^2$

Total momentum = 0

(d) Lorentz Transformation for Energy and Momentum:

$E = \gamma(E' + vp_x')$ and $p_x = \gamma\left(p_x' + \frac{v}{c^2}E'\right)$

Hence, Energy in the lab frame $= \frac{3}{\sqrt{8}}\left(\frac{3}{\sqrt{2}}mc^2 - \frac{1}{3}c \times 0\right) = \frac{9}{4}mc^2$

Momentum in the lab frame $= \frac{3}{\sqrt{8}}\left(0 + -\frac{1}{3c} \times \frac{3}{\sqrt{2}}mc^2\right) = \frac{3}{4}mc$

(e) <u>Particle 1:</u>

Lab frame: $E^2 - p^2c^2 = \left(\frac{5}{4}mc^2\right)^2 - \left(\frac{3}{4}mc\right)^2 c^2 = m^2c^4$

CM frmae: $E'^2 - p'^2 c^2 = \left(\frac{3}{\sqrt{8}}mc^2\right)^2 - \left(\frac{1}{\sqrt{8}}mc\right)^2 c^2 = m^2c^4$

<u>Particle 2:</u>

Lab frame: $E^2 - p^2c^2 = (mc^2)^2 - (0)^2 c^2 = m^2c^4$

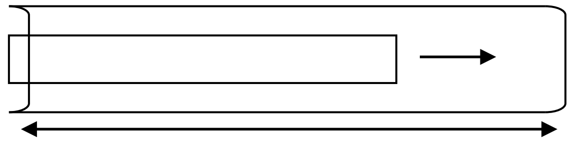
CM frmae: $E'^2 - p'^2 c^2 = \left(\frac{3}{\sqrt{8}}mc^2\right)^2 - \left(\frac{1}{\sqrt{8}}mc\right)^2 c^2 = m^2c^4$

<u>Total:</u>

Lab frame: $E^2 - p^2c^2 = \left(\frac{9}{4}mc^2\right)^2 - \left(\frac{3}{4}mc\right)^2 c^2 = \frac{9}{2}m^2c^4$

CM frame: $E'^2 - p'^2 c^2 = \left(\frac{3}{\sqrt{2}}mc^2\right)^2 - (0)^2 c^2 = \frac{9}{2}m^2c^4$

12



From the train's perspective, the tunnel is contracted to L/γ. Clearly, when the front of the train reaches the far end of the tunnel, the rear end is still a distance from the tunnel. Hence, the bomb will explode. We will show that the same result can be obtained by doing our calculations in the tunnel's frame.

The diagram above shows the train being contracted into L/γ as it fully enters the tunnel. At this moment, the deactivation signal is sent toward its front end. The time required to reach the front end is L/c, assuming he uses an EM wave as the signal. The front end of the train is now $L - L/\gamma$ from the far end of the tunnel and will take $\left[L - L/\gamma\right]/v$ to reach there. The difference in time of the two events are

$$
\begin{aligned}
\left[L - \frac{L}{\gamma}\right]\frac{1}{v} - \frac{L}{c} &= \frac{L}{c}\frac{1}{\beta}\left[1 - \sqrt{1-\beta^2}\right] - \frac{L}{c} \\
&= \frac{L}{\beta c}\left[1 - \sqrt{1-\beta^2} - \beta\right] \\
&= \frac{L}{\beta c}\sqrt{1-\beta}\left[\sqrt{1-\beta} - \sqrt{1+\beta}\right]
\end{aligned}
$$

Since $\sqrt{1-\beta} - \sqrt{1+\beta}$ is clearly less than 0, $\left[L - \frac{L}{\gamma}\right]\frac{1}{v}$ is less than $\frac{L}{c}$, which means that the signal cannot reach the front end of the train to deactivate the bomb in time.

13 (a) To A, velocity of B $= \dfrac{-v-v}{1+v^2/c^2} = \dfrac{-2v}{1+v^2/c^2}$. Therefore, $\beta' = \dfrac{2\beta}{1+\beta^2}$ and

$$\gamma' = \left[1-\beta'^2\right]^{-1/2} = \left[1-\frac{4\beta^2}{\left(1+\beta^2\right)^2}\right]^{-1/2} = \frac{1+\beta^2}{1-\beta^2}$$

We define:

Event 1: B sends out signal. Hence, $t_1' = T$ and $x_1' = 0$.

Therefore,

$$t_1 = \gamma'\left[t_1 + \left(\frac{-2v}{1+\beta^2}\right)\frac{x_1'}{c^2}\right] = \gamma' T = \left[\frac{1+\beta^2}{1-\beta^2}\right]T$$

By then, the distance between A and B is $\dfrac{2\beta c}{1+\beta^2} \times \left[\dfrac{1+\beta^2}{1-\beta^2}\right]T = \left[\dfrac{2\beta c}{1-\beta^2}\right]T$, which means that the signal will take $\left[\dfrac{2\beta}{1-\beta^2}\right]T$ to reach A. Hence, time at which A receives the signal is $t_1 + \left[\dfrac{2\beta}{1-\beta^2}\right]T = \left[\dfrac{(1+\beta)^2}{1-\beta^2}\right]T = \left[\dfrac{1+\beta}{1-\beta}\right]T$.

If we use the Doppler effect formula,

The time is $\left[\dfrac{1+\beta'}{1-\beta'}\right]^{1/2} T = \left[\dfrac{(1+\beta)^2}{(1-\beta)^2}\right]^{1/2} T = \left[\dfrac{1+\beta}{1-\beta}\right]T$

13 **(b)** In B's frame, A is moving away at $2\beta c \big/ \left(1+\beta^2\right)$. The time at which the signal catches up with A is t_2'. Hence,

$$c\left(t_2' - T\right) = \left(\frac{2\beta c}{1+\beta^2}\right) t_2'$$

Rearranging, $t_2' = \dfrac{1+\beta^2}{\left(1-\beta\right)^2} T$, and $x_2' = \beta' c \times t_2' = \dfrac{2\beta c}{\left(1-\beta\right)^2} T$

Therefore, $t_2 = \gamma' \left[t_2' - \beta' \dfrac{x_2'}{c} \right]$

$$= \frac{1+\beta^2}{1-\beta^2} \left[\frac{1+\beta^2}{\left(1-\beta\right)^2} T - \frac{2\beta}{1+\beta^2} \times \frac{2\beta}{\left(1-\beta\right)^2} T \right]$$

$$= \frac{1+\beta^2}{1-\beta^2} \frac{T}{\left(1-\beta\right)^2} \left[\frac{\left(1+\beta^2\right)^2 - 4\beta^2}{1+\beta^2} \right]$$

$$= \frac{T}{\left(1-\beta^2\right)\left(1-\beta\right)^2} \left[1-\beta^2\right]^2$$

$$= T \left[\frac{1+\beta}{1-\beta} \right]$$

14 Using the result of Q13, $\beta' = \dfrac{2\beta}{1+\beta^2}$ and $\gamma' = \dfrac{1+\beta^2}{1-\beta^2}$.

To A, length of B is contracted to $\dfrac{L}{\gamma'} = \left(\dfrac{1-\beta^2}{1+\beta^2}\right)L$. Hence, B will need to travel $L - \dfrac{L}{\gamma'}$ $= L\left[\dfrac{2\beta^2}{1+\beta^2}\right]$ for the other ends to coincide. The time needed is

$$\begin{aligned} t &= L\left[\frac{2\beta^2}{1+\beta^2}\right] \div \left[\frac{2\beta c}{1+\beta^2}\right] \\ &= \frac{L\beta}{c} = \frac{Lv}{c^2} \end{aligned}$$

On the other hand,

$$\begin{aligned} t_2' &= \gamma'\left(t_2 - \left(-\beta'/c\right)x_2\right) \\ &= \frac{1+\beta^2}{1-\beta^2}\left[\frac{L\beta}{c} + \frac{2\beta/c}{1+\beta^2}\left(-L\right)\right] \\ &= \frac{1+\beta^2}{1-\beta^2}\left(\frac{L\beta}{c}\right)\left[1 - \frac{2}{1+\beta^2}\right] \\ &= \frac{1+\beta^2}{1-\beta^2}\left(\frac{L\beta}{c}\right)\left[\frac{\beta^2-1}{1+\beta^2}\right] \\ &= -\frac{Lv}{c^2} \end{aligned}$$