Work, Energy and Power

1 **A Level S Paper N97 Q3**

Fig.1.1 shows a uniform cylindrical cork of height h and cross-sectional area A floating with its axis vertical in water of density ρ in a measuring cylinder. When floating, the cork displaces its own weight of water. The area of cross-section of the cylinder is only slightly-greater than A. The depth of water below the bottom of the cork is h, and the free water is h/4 above the bottom of the cork. The difference between the areas of cross-section is so small that the volume of water above the bottom of the cork is negligible.

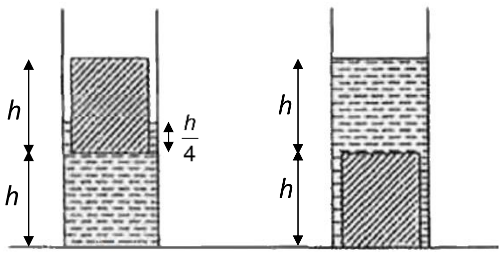


Fig. 1.1 **Fig.1.2**

The cork is pushed downwards slowly until it is on the base of the measuring cylinder (Fig. 1.2). Neglecting viscosity and surface tension effects, find an expression for the total work W expended in pushing the cork downwards.

[6]

[$\frac{3}{4}A\rho gh^2$]

2 **A Level S Paper N05 Q2**

This question is about a simplified model of the operation of an electric fan.
The blades of the fan are of length r. When the fan is rotating, it is assumed that it accelerates air from rest to velocity v.
This forms a long cylinder of air of radius r moving at velocity v, as illustrated in Fig. 2.1.
The density of air is ρ.

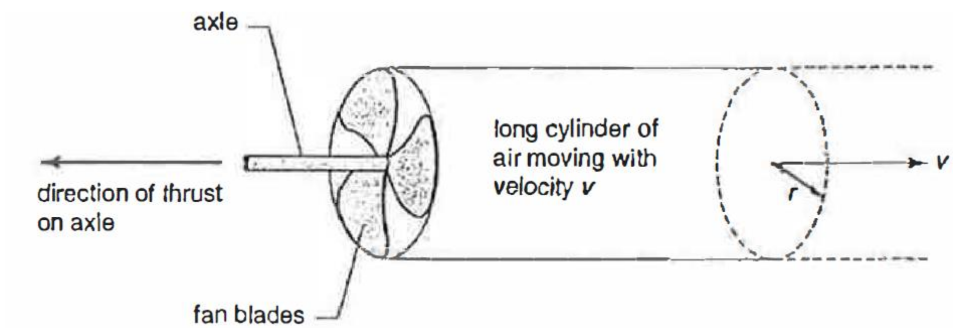


Fig. 2.

(a) Obtain expressions for

(i) The rate at which momentum is transferred to the air passing into the fan, [2]

[$\pi r^2 \rho v^2$]

(ii) The magnitude of the thrust on the axle of the fan, [1]

(iii) The rate at which the fan motor is working, neglecting any losses in the drive of the blades, [1]

[$\pi r^2 \rho v^3$]

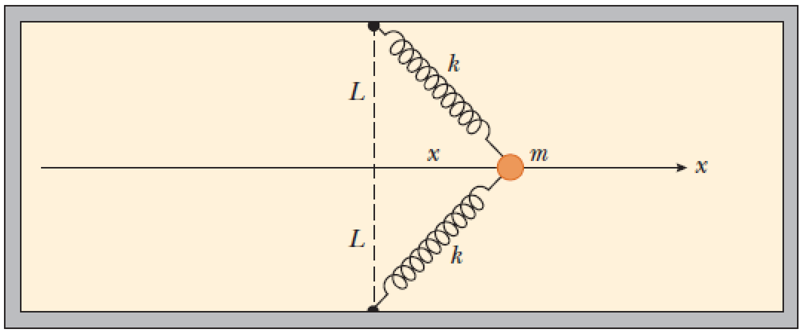
(iv) The rate at which kinetic energy is supplied to the air. [1]

[$\frac{1}{2}\pi r^2 \rho v^3$]

(b) The answers to **(a)(iii)** and **(iv)** should be different. Suggest one reason for this other than the losses in the drive to the blades. [1]

3 Serway 7th Ed P7.49

A particle of mass 1.18 kg is attached between two identical springs on a horizontal, frictionless table top. Both springs have spring constant k and are initially unstressed.



Top View

(a) The particle is pulled a distance x along a direction perpendicular to the initial configuration of the springs as shown in the figure.
Show that the force exerted by the springs on the particle is

$$\vec{F} = -2kx\left(1 - \frac{L}{\sqrt{x^2 + L^2}}\right)\hat{i}$$

(b) Show that the potential energy of the system is

$$U(x) = kx^2 + 2kL\left(L - \sqrt{x^2 + L^2}\right)$$

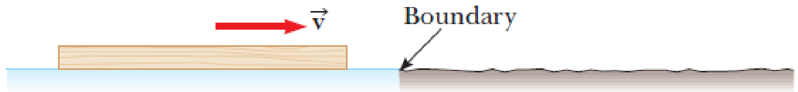
(c) Make a plot of $U(x)$ versus x and identify all equilibrium points. Assume L = 1.20 m and k = 40.0 Nm^{-1}.

(d) If the particle is pulled 0.500 m to the right and then released, what is its speed when it reaches the equilibrium point at x = 0?

[0.823 ms^{-1}]

4 Serway 7th Ed P8.27

A uniform board of length L is sliding along a smooth (frictionless) horizontal plane as shown in Fig 6(a). The board then slides across the boundary with a rough horizontal surface. The coefficient of kinetic friction between the board and the second surface is μ_k.



(a)

(b)

(a) Find the acceleration of the board at the moment its front end has travelled a distance x beyond the boundary.

$$[a = \frac{\mu_k g x}{L}]$$

(b) The board stops at the moment its back end reaches the boundary as shown in Fig 6(b). Find the initial speed v of the board.

$$v = \sqrt{\mu_k g L}$$

5 A light bucket contains a mass M of sand. It is connected to a wall by a light spring with a constant tension T, as shown in Fig. 7.1. Initially, the bucket is held at a distance of L from the wall. The ground is frictionless.

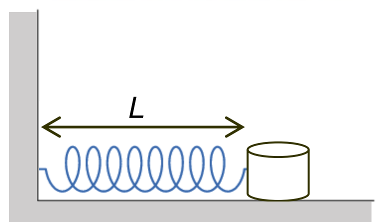


Fig 7.1

The bucket is released. On its way towards the wall, the bucket leaks sand at a rate proportional to its acceleration.

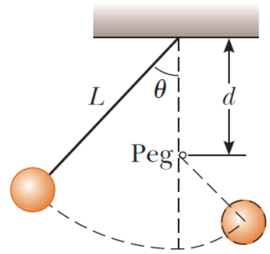
(a) Find the mass of the sand in the bucket as a function of time, $m(t)$.

(b) Find the velocity $v(t)$ and distance from the wall $x(t)$ for time t when there is still sand in the bucket. What is the speed right before all the sand leaves the bucket?

(c) What is the maximum value of the bucket's kinetic energy?

6 Serway 7th Ed P8.62

A pendulum, comprising a light string of length L and a small sphere, swings in the vertical plane. The string hits a peg located a distance d below the point of suspension.



(a) Show that if the sphere is released from a height below that of the peg, it will return to this height after the string strikes the peg.

(b) Show that if the pendulum is released from the horizontal position ($\theta = 90°$) and is to swing in a complete circle centered on the peg, the minimum value of d must be $\frac{3L}{5}$.

WEP

1 A Level S Paper N97 Q3

Archimede's Principle states that a body immersed in fluid experiences an upthrust equal in magnitude to the weight of the fluid it displaces. Furthermore, the Principal of Floatation states that a floating body displaces a quantity of fluid of a weight equal to its own weight.

When a body is floating (Fig. 1.1), the

$$\text{weight}_{\text{body}} = \text{upthrust} = \text{weight of fluid displaced} = V\rho g = (\tfrac{1}{4}Ah)\rho g$$

where A is the cross sectional area of the cork, ρ the density of the fluid and $(\tfrac{1}{4}h)$ the depth of the cork immersed.

Since the cross-sectional area of the cork is almost equal to that of the cylinder, the water will fully immerse the cork the instant that the cork is pushed slighly downwards. Once that happens, then

$$\text{upthrust} = \text{weight of fluid displaced} = V\rho g = (Ah)\rho g$$

The resultant force acting on the body is now

$$(Ah)\rho g - (\tfrac{1}{4}Ah)\rho g = (\tfrac{3}{4}Ah)\rho g \qquad \text{(upwards)}$$

In order to push the cork downwards slowly without accelerating it, a constant downward force of $(\tfrac{3}{4}Ah)\rho g$ must be applied by the external agent doing the work.

Work done by a constant force is the product of the force $(\tfrac{3}{4}Ah)\rho g$ and the displacement h.

$$W = Fd = (\tfrac{3}{4}Ah)\rho g \times h = \tfrac{3}{4}A\rho g h^2$$

2 (a) (i) The volume of air accelerated from rest per unit time is $(\pi r^2)v$. The mass of air accelerated from rest per unit time is $(\pi r^2)\rho v$. The change in velocity of this volume of air is $(v-0)=v$.
Hence the rate of change of momentum of the air column is $(\pi r^2)\rho v v = (\pi r^2)\rho v^2$

(ii) The rate of change of momentum of the air column is equal to the resultant force acting on it (Newton's Second Law of motion). By Newton's third law of motion, the air column will exert an equal but opposite backward thrust $(\pi r^2)\rho v^2$ on the axle of the fan.

(iii) Rate at which work is done = Force exerted $\times$ Velocity = $(\pi r^2)\rho v^2 \times v = (\pi r^2)\rho v^3$.

(iv) Rate at which kinetic energy is supplied to the air = $\tfrac{1}{2}(\dfrac{m}{t})v^2 = \tfrac{1}{2}(\pi r^2)\rho v v^2 = \tfrac{1}{2}(\pi r^2)\rho v^3$.

(b) The volume of air is accelerated linearly from 0 to v in time t. (iii) calculate the power given to the air when it is moving at v and thus is the maximum power. Hence, (iii) will always be larger than (iv). To be correct, it should be the power over the time t and thus the power needed to accelrate the air from 0 to v should be Force x $\dfrac{1}{2}$v .

3 Serway 7th Ed P7.49

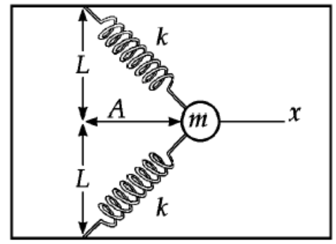


FIG. P7.49

(a) The new length of each spring is $\sqrt{x^2+L^2}$, so its extension is $\sqrt{x^2+L^2}-L$ and the force it exerts is $k\left(\sqrt{x^2+L^2}-L\right)$ toward its fixed end. The y components of the two spring forces add to zero. Their x components add to

$$\vec{\mathbf{F}} = -2\hat{\mathbf{i}}k\left(\sqrt{x^2+L^2}-L\right)\frac{x}{\sqrt{x^2+L^2}} = \boxed{-2kx\hat{\mathbf{i}}\left(1-\frac{L}{\sqrt{x^2+L^2}}\right)}$$

(b) Choose $U=0$ at $x=0$. Then at any point the potential energy of the system is

$$U(x) = -\int_0^x F_x dx = -\int_0^x\left(-2kx+\frac{2kLx}{\sqrt{x^2+L^2}}\right)dx = 2k\int_0^x x dx - 2kL\int_0^x \frac{x}{\sqrt{x^2+L^2}}dx$$

$$U(x) = \boxed{kx^2+2kL\left(L-\sqrt{x^2+L^2}\right)}$$

(c) $U(x) = 40.0x^2+96.0\left(1.20-\sqrt{x^2+1.44}\right)$

For negative x, $U(x)$ has the same value as for positive x. The only equilibrium point (i.e., where $F_x=0$) is $\boxed{x=0}$.

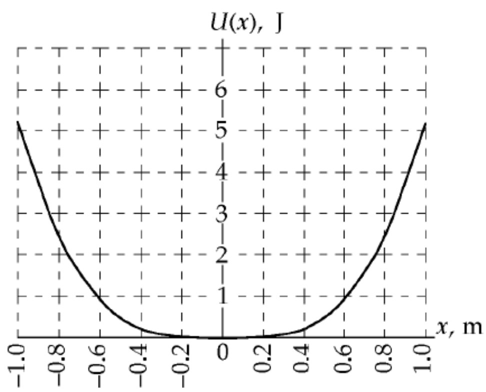


FIG. P7.49(c)

(d) $K_i + U_i + \Delta E_{\text{mech}} = K_f + U_f$

$$0 + 0.400\text{ J} + 0 = \frac{1}{2}(1.18\text{ kg})v_f^2 + 0$$

$$v_f = \boxed{0.823\text{ m/s}}$$

4 Serway 7th Ed P8.27

(a) Let m be the mass of the whole board. The portion on the rough surface has mass $\frac{mx}{L}$. The normal force supporting it is $\frac{mxg}{L}$ and the frictional force is $\frac{\mu_k mgx}{L} = ma$. Then

$$\boxed{a = \frac{\mu_k gx}{L}\text{ opposite to the motion}}.$$

(b) In an incremental bit of forward motion dx, the kinetic energy converted into internal energy is $f_k dx = \frac{\mu_k mgx}{L}dx$. The whole energy converted is

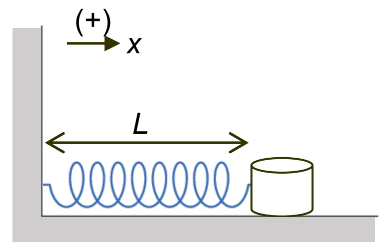
$$\frac{1}{2}mv^2 = \int_0^L \frac{\mu_k mgx}{L}dx = \frac{\mu_k mg}{L}\frac{x^2}{2}\bigg|_0^L = \frac{\mu_k mgL}{2}$$

$$\boxed{v = \sqrt{\mu_k gL}}$$

5 **(a)** Note the unusual (and unphysical) assumption that the tension in the spring is constant T, independent of time or the length of the spring. In other words, this is not a question on SHM. Note, also, that as the mass m of the sand decreases, its acceleration a increases, but the product $ma = T$ remains constant, by virtue of the previous assumption.

By Newton's 2nd Law,

$$F_{\text{net}} = ma$$
$$-T = ma \text{ ------- (1)}$$



where the negative sign is according to the sign convention (to the right is positive).

Given $\frac{dm}{dt} \propto a$, $\frac{dm}{dt} = ka$ ------- (2), k being the proportionality constant.

Combining (1) and (2)

$$-kTdt = mdm$$

Notice that the above equation is consistent with the fact that dm is a negative quantity (because m is decreasing in time). Integrating both sides,

$$\int mdm = \int -kTdt$$
$$\frac{m^2}{2} = -kTt + \text{const.}$$

At $t = 0$, $m = M$, $\text{const.} = \frac{M^2}{2}$.

Therefore,

$$m(t) = \sqrt{M^2 - 2kTt}$$

(b) Using $\dfrac{dm}{dt} = ka = k\dfrac{dv}{dt}$ and integrating both sides,

$$\int \frac{dv}{dt} = \int \frac{1}{k}\frac{dm}{dt}$$
$$v = \frac{m}{k} + D$$

When $\text{v} = 0,\ m = M,\ D = -\dfrac{M}{k}$. Therefore,

$$v(m) = \frac{m - M}{k} \ \Rightarrow\ v(t) = \frac{\sqrt{M^2 - 2kTt} - M}{k}$$

Integrating on both sides,

$$\int dx = \int \left(\frac{\sqrt{M^2 - 2kTt}}{k} - \frac{M}{k}\right)dt$$
$$x = -\frac{(M^2 - 2kTt)^{\frac{3}{2}}}{3k^2T} - \frac{M}{k}t + E$$

When $t = 0,\ x = L,\ E = L + \dfrac{M^3}{3k^2T}$. Hence,

$$x(t) = \frac{-(M^2 - 2kTt)^{\frac{3}{2}}}{3k^2T} - \frac{M}{k}t + L + \frac{M^3}{3k^2T}$$

Just before all the sand leaves the bucket, $m = 0$,

$$v(0) = -\frac{M}{k}.$$

(c)

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{m - M}{k}\right)^2$$

At maximum point, $\dfrac{dE_k}{dm} = 0$. Hence,

$$\frac{dE_k}{dm} = \frac{1}{2k^2}(3m^2 - 4Mm + M^2) = \frac{1}{2k^2}(3m - 1)(m - M) = 0$$
$$m = \frac{M}{3} \text{ or } m = M \text{ (rejected)}$$

When $m = \dfrac{M}{3}$,

$$E_k = \frac{M}{2k^2 3}\frac{4M^2}{9} = \frac{2M^3}{27k^2}.$$

6 Serway 7th Ed P8.62

(a) Energy is conserved in the swing of the pendulum, and the stationary peg does no work. So the ball's speed does not change when the string hits or leaves the peg, and the ball swings equally high on both sides.

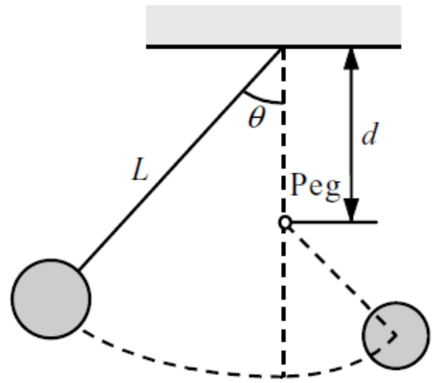


FIG. P8.62

(b) Relative to the point of suspension,

$$U_i = 0,\ U_f = -mg\left[d - (L - d)\right]$$

From this we find that

$$-mg(2d - L) + \frac{1}{2}mv^2 = 0$$

Also for centripetal motion,

$$mg = \frac{mv^2}{R} \text{ where } R = L - d$$

Upon solving, we get $\boxed{d = \frac{3L}{5}}$.