RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2019 Year 6

Chapter 13D : Eigenvalues and Eigenvectors

SYLLABUS INCLUDES

- understand the terms 'eigenvalue' and 'eigenvector', as applied to square matrices; find eigenvalues and eigenvectors of 2×2 and 3×3 matrices (restricted to cases where the eigenvalues are real and distinct);
- express a matrix $\mathbf{A}$ in the form $\mathbf{QDQ}^{-1}$, where $\mathbf{D}$ is a diagonal matrix of eigenvalues and $\mathbf{Q}$ is a matrix whose columns are eigenvectors, and use this expression, e.g. in calculating powers of matrices.
- prove certain properties involving square matrices and their eigenvalues with corresponding eigenvectors.

PRE-REQUISITES

- Vectors, Matrices, Recurrence Relations

CONTENT

1 Eigenvalues and Eigenvectors
2 Diagonalisation of a Matrix
3 Computing the Powers of a Matrix
4 Other properties of Eigenvalues and Eigenvectors

1 Eigenvalues and Eigenvectors

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation given by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

If there exists a **nonzero** vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some $\lambda \in \mathbb{R}$, then we say that $\mathbf{x}$ is called an **eigenvector** of **A** and the corresponding λ is called an **eigenvalue** of **A** corresponding to the eigenvector.

For example, the vector $\mathbf{x} = \begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}=\begin{pmatrix}3 & 0\\8 & -1\end{pmatrix}$ corresponding to the eigenvalue $\lambda=3$, since $\mathbf{A}\mathbf{x} = \begin{pmatrix}3 & 0\\8 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix} = 3\mathbf{x}$.

In fact, any scalar multiple of $\begin{pmatrix}1\\2\end{pmatrix}$ is also an eigenvector of **A** corresponding to the eigenvalue 3.

Let $\mathbf{v} = t\begin{pmatrix}1\\2\end{pmatrix}$

$$\mathbf{A}\mathbf{v} = \mathbf{A}\left(t\begin{pmatrix}1\\2\end{pmatrix}\right) = t\,\mathbf{A}\begin{pmatrix}1\\2\end{pmatrix} = t\begin{pmatrix}3\\6\end{pmatrix} = 3t\begin{pmatrix}1\\2\end{pmatrix} = 3\mathbf{v}.$$

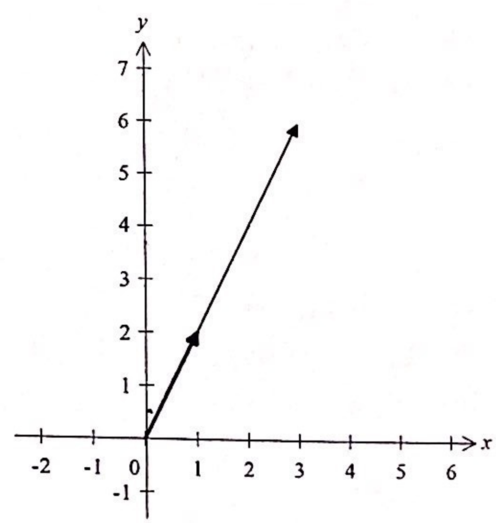
NOTE: The eigenvectors of an eigenvalue is **NOT** unique (scalar multiple).

(i) If there exists a point $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}\mathbf{x} = \mathbf{x}$ (that is, with eigenvalue 1), then this point is called an **invariant point** of T.

(ii) If for some $\lambda \in \mathbb{R}$, $\mathbf{x}$ is a nonzero vector such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, then this point is mapped onto another point on the line $\mathbf{r} = t\mathbf{x}$. This line is called an **invariant line** of T.

For the above example that $\mathbf{x} = \begin{pmatrix}1\\2\end{pmatrix}$ is an eigenvector of $\mathbf{A}=\begin{pmatrix}3 & 0\\8 & -1\end{pmatrix}$ corresponding to the eigenvalue $\lambda=3$.

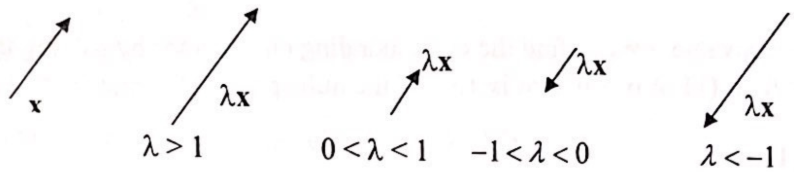
The point $\begin{pmatrix}1\\2\end{pmatrix}$ is mapped to another point (in this case, $\begin{pmatrix}3\\6\end{pmatrix}$), but still lies on the line $\mathbf{r} = t\begin{pmatrix}1\\2\end{pmatrix}$.



Remark

The points on invariant lines of T are <u>**NOT**</u> necessary invariant points of T.

Eigenvalues and eigenvectors have a useful geometric interpretation in $\mathbb{R}^2$ and $\mathbb{R}^3$. If λ is an eigenvalue of A corresponding to x, then $\mathbf{Ax} = \lambda\mathbf{x}$, so it means that multiplication by A dilates x, contracts x, or reverse the direction of x, depending on the value of λ.



Remarks :

1) By definition, an eigenvector is a non-zero vector.

2) Eigenvalues are also called **proper values or characteristic values or latent roots.**

3) In the current syllabus, we are only interested in computing the eigenvalues and eigenvectors of 2×2 and 3×3 matrices.

How to find eigenvalues and eigenvectors of a matrix?

Consider the linear transformation in the above example, $\mathrm{T}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3&0\\8&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

To find the eigenvectors and eigenvalues of $\begin{pmatrix}3&0\\8&-1\end{pmatrix}$, we do the following

$$\mathbf{Ax}=\lambda\mathbf{x}$$

$$\begin{pmatrix}3&0\\8&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\lambda\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}3&0\\8&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\left[\begin{pmatrix}3&0\\8&-1\end{pmatrix}-\lambda\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}3-\lambda&0\\8&-1-\lambda\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$A\underset{\sim}{x}=\lambda\underset{\sim}{x}$$
$$A\underset{\sim}{x}-\lambda\underset{\sim}{x}=\underset{\sim}{0}$$
$$(A-\lambda I)\underset{\sim}{x}=\underset{\sim}{0}$$
$$\det(A-\lambda I)=0$$

This is a homogeneous linear system.
Since $\mathbf{x}\neq\mathbf{0}$, the equation has non-trivial solution. Therefore,

$$\det\begin{pmatrix}3-\lambda&0\\8&1-\lambda\end{pmatrix}=0$$

In general, to find the eigenvalues and eigenvectors of a $n{\times}n$ matrix **A** we rewrite $\mathbf{Ax}=\lambda\mathbf{x}$ as

$$\mathbf{Ax}=\lambda\mathbf{Ix} \quad\Rightarrow\quad (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0} \quad\text{or}\quad (\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}.$$

Then, solve the following equation for λ

$$\det(\mathbf{A}-\lambda\mathbf{I})=0 \quad\text{or}\quad \det(\lambda\mathbf{I}-\mathbf{A})=0$$

This equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$ is known as the **characteristic equation**. Since this equation is a polynomial of degree n, **A** has at most n real eigenvalues. **However, for the current syllabus, the characteristic equation will be n distinct real roots.**

For each eigenvalue, we can find the corresponding eigenvector by solving the equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$ or $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$, that is, to find the null space of the matrix $\mathbf{A}-\lambda\mathbf{I}$.

Example 1

Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}=\begin{pmatrix}0 & -1 & -2\\ 2 & 3 & 2\\ 1 & 1 & 3\end{pmatrix}$.

Solution:

$$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}0 & -1 & -2\\ 2 & 3 & 2\\ 1 & 1 & 3\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}-\lambda & -1 & -2\\ 2 & 3-\lambda & 2\\ 1 & 1 & 3-\lambda\end{pmatrix}$$

1. solve for $\det(A-\lambda I)=0$
2. get eigenvalues ($\lambda_1, \lambda_2, \lambda_3$)
3. sub the λ's into $(A-\lambda I)\underset{\sim}{x}=0$
4. solve for $\underset{\sim}{x}$ (rref, like null space)

The characteristic equation of **A** is det(**A**–λ**I**)=0

$-\lambda((3-\lambda)^2-2)+(2(3-\lambda)-2)-2(2-(3-\lambda)$

The eigenvalues of **A** are $1, 2, 3$

NOTE: $(A-\lambda I)\underset{\sim}{x}=\underset{\sim}{0}$.

for $\lambda_1=1$:

$\begin{pmatrix}-1 & -1 & -2\\ 2 & 2 & 2\\ 1 & 1 & 2\end{pmatrix}\underset{\sim}{x}=\underset{\sim}{0}$. $\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$

$\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

$x_1+x_2=0$

$x_3=0$

$\begin{pmatrix}x_1\\ -x_1\\ 0\end{pmatrix}=x_1\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\begin{pmatrix}-\lambda & -1 & -2\\ 2 & 3-\lambda & 2\\ 1 & 1 & 3-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Thus,

Method 1 (Use GC – Plysmlt2 App)

When $\lambda=1$, $-x_1 - x_2 - 2x_3 = 0$

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$x_1 + x_2 + 2x_3 = 0$$

Using GC, we get $x_1 = -t,\ x_2 = t,\ x_3 = 0,\ ,\ t \in \mathbb{R}$

Thus, an eigenvector corresponding to the eigenvalue $\lambda=1$ is

Method 2 (Guess)

When $\lambda=2$, $-2x_1 - x_2 - 2x_3 = 0$ ------(1)

$2x_1 + x_2 + 2x_3 = 0$ ------(2)

$x_1 + x_2 + x_3 = 0$ ------(3)

Note: Observe that Eqn and are the same. We just need to use Eqn and to solve the equations, and we expect to the equations. However, we just need

Subst $x_1 =$ into and , we obtain

Thus, an eigenvector corresponding to the eigenvalue $\lambda=2$ is

Question: *Would using* $x_1 = 0$ *works?* $x_2 = 0\,?\ x_2 = 1\,?$

Method 3 (Row Operations)

When $\lambda=3$,

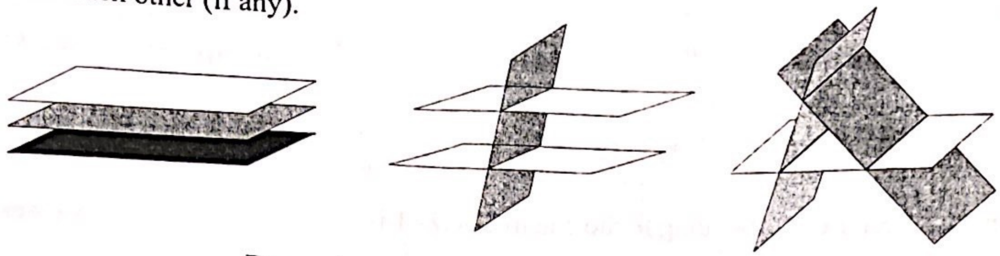
$$\begin{aligned} -3x_1 - x_2 - 2x_3 &= 0 \\ 2x_1 + 0x_2 + 2x_3 &= 0 \\ x_1 + x_2 + 0x_3 &= 0 \end{aligned} \Rightarrow \begin{pmatrix} -3 & -1 & -2 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

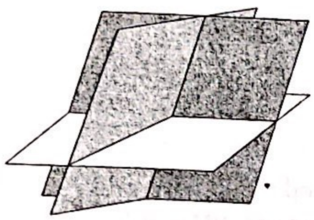
Thus, $x_1 = -x_3$ and $x_2 = x_3$

So, an eigenvector corresponding to the eigenvalue $\lambda=3$ is $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

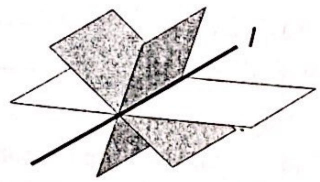
Recall from Chapter 5c (Vectors III) that there are three different cases on how 3 planes intersect each other (if any).



Do not intersect at any common point



Intersect at one common point



Intersect on a line

From Example 1, we can see that we will obtain a system of 3 linear equations when finding the eigenvectors. The eigenvectors would be the solution of these 3 equations for different values of λ:

$$\begin{aligned} -\lambda x_1 - x_2 - 2x_3 &= 0 \\ 2x_1 + (3-\lambda)x_2 + 2x_3 &= 0 \\ x_1 + x_2 + (3-\lambda)x_3 &= 0 \end{aligned}$$

Question: when finding the eigenvectors, the system of 3 linear equations would correspond to which geometrical interpretation for the intersection of the 3 planes?

Answer:

Observe that for the case when $\lambda=1$,

$$-x_1 - x_2 - 2x_3 = 0 \quad \Rightarrow$$

$$2x_1 + 2x_2 + 2x_3 = 0 \quad \Rightarrow$$

$$x_1 + x_2 + 2x_3 = 0 \quad \Rightarrow$$

Since the planes

An eigenvector would then be

$AX = \lambda X$. $AX - \lambda X = 0$

Example 2

Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}=\begin{pmatrix}1 & 2 & -1\\ 1 & 0 & 1\\ 4 & -4 & 5\end{pmatrix}$.

+ - +

Solution: $\begin{pmatrix}1 & 2 & -1\\ 1 & 0 & 1\\ 4 & -4 & 5\end{pmatrix} - \begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix} = \begin{pmatrix}1-\lambda & 2 & -1\\ 1 & -\lambda & 1\\ 4 & -4 & 5-\lambda\end{pmatrix}$

The characteristic equation of **A** is $(1-\lambda)(-\lambda(5-\lambda)+4) - 2(5-\lambda - 4) - (-4-(-4\lambda))$

$= (1-\lambda)(4-5\lambda+\lambda^2) - 2(1-\lambda) + 4 - 4\lambda$

$= (1-\lambda)(4-5\lambda+\lambda^2 - 2 + 4)$

The eigenvalues of **A** are 1, -3, -2 $= (1-\lambda)(\lambda^2 - 5\lambda + 6)$

Consider $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$

One eigenvector is

When $\lambda = 1$, its eigenvector is $x_3\begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 1\end{pmatrix}$. $\begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}$.

When $\lambda = -3$, its eigenvector is $x_2\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$

When $\lambda =$, its eigenvector is

Example 3 (9649/HCI/2017/Promos/Q4a)

A linear transformation $T: \mathbf{R}^3 \to \mathbf{R}^3$ is represented by the matrix $\mathbf{A}$. Let $\mathbf{x}$ be an eigenvector of the matrix $\mathbf{A}$ with corresponding eigenvalue λ.

(i) Describe the geometrical significance of the eigenvectors of $\mathbf{A}$ in relation to the linear transformation T when $\lambda = 0, 1, -1$. [3]

(ii) Describe the geometrical significance of the eigenvectors of $\mathbf{A}$ in relation to the linear transformation T when $\lambda \neq 0, 1, -1$. [1]

Solution:

$\mathbf{Ax} = \lambda\mathbf{x}$

(i) When $\lambda = 0, \mathbf{Ax} = \mathbf{0}$

When $\lambda = 1, \mathbf{Ax} = \mathbf{x}$

When $\lambda = -1, \mathbf{Ax} = -\mathbf{x}$

(ii) $\mathbf{Ax} = \lambda\mathbf{x},\ \lambda \neq 0, 1, -1$

2 Diagonalisation of a Matrix

A **diagonal matrix D** is one where all off-diagonal elements are zero. The following are 2×2 and 3×3 diagonal matrices:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Diagonal matrices have beautiful properties that make computation involving these matrices simple and convenient. Some of these properties are:

(i) $\det(\mathbf{D})$ is the product of its diagonal elements.

(ii) If $\det(\mathbf{D}) \neq 0$, then $\mathbf{D}^{-1}$ is a diagonal matrix with the corresponding reciprocals in the diagonal, that is $\mathbf{D}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}$ and $\mathbf{D}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$ for the 2×2 and 3×3 diagonal matrices.

(iii) Observe that if $\mathbf{D}$ is a 3×3 diagonal matrices, then $\mathbf{D}^2 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$.

In general, $\mathbf{D}^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$ where n is a positive integer. This can be generalized to any dimension.

> A square matrix **A** is said to be **diagonalisable** if there is an invertible matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. Then the matrix **P** is said to diagonalise **A**, that is
>
> $$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} \Leftrightarrow \mathbf{A} = \mathbf{PDP}^{-1}.$$

Let **A** be a matrix with eigenvectors $\mathbf{x}_1=\begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}$, $\mathbf{x}_2=\begin{pmatrix}x_{12}\\x_{22}\\x_{32}\end{pmatrix}$ and $\mathbf{x}_3=\begin{pmatrix}x_{13}\\x_{23}\\x_{33}\end{pmatrix}$ and with corresponding eigenvalues λ_1, λ_2 and λ_3 respectively.

Then $\mathbf{A}\mathbf{x}_1=\lambda_1\mathbf{x}_1$, $\mathbf{A}\mathbf{x}_2=\lambda_2\mathbf{x}_2$, $\mathbf{A}\mathbf{x}_3=\lambda_3\mathbf{x}_3$.

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}=\lambda_1\begin{pmatrix}x_{11}\\x_{21}\\x_{31}\end{pmatrix}$$

$$\begin{pmatrix}a_{11}x_{11}+a_{12}x_{21}+a_{13}x_{31}\\a_{21}x_{11}+a_{22}x_{21}+a_{23}x_{31}\\a_{31}x_{11}+a_{32}x_{21}+a_{33}x_{31}\end{pmatrix}=\begin{pmatrix}\lambda_1x_{11}\\\lambda_1x_{21}\\\lambda_1x_{31}\end{pmatrix}$$

Thus,

$$\mathbf{A}(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)=\mathbf{A}\begin{pmatrix}x_{11}&x_{12}&x_{13}\\x_{21}&x_{22}&x_{23}\\x_{31}&x_{32}&x_{33}\end{pmatrix}$$

$$=\begin{pmatrix}\lambda_1x_{11}&\lambda_2x_{12}&\lambda_3x_{13}\\\lambda_1x_{21}&\lambda_2x_{22}&\lambda_3x_{23}\\\lambda_1x_{31}&\lambda_2x_{32}&\lambda_3x_{33}\end{pmatrix}$$

$$=\begin{pmatrix}x_{11}&x_{12}&x_{13}\\x_{21}&x_{22}&x_{23}\\x_{31}&x_{32}&x_{33}\end{pmatrix}\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}=(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$$

So, $\mathbf{A}=(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)^{-1}=\mathbf{PDP}^{-1}$

or $(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)^{-1}\mathbf{A}(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$

i.e. $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$, where $\mathbf{P}=(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \mathbf{x}_3)$ and $\mathbf{D}=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}$.

Through this, we managed to diagonalise **A**.

Question: *Are all square matrices diagonalisable? Are the matrices* **P** *and* **D** *unique?*

$A = PDP^{-1} = (\underset{\sim}{e_1}\ \ \underset{\sim}{e_2}\ \ \underset{\sim}{e_3})\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}(\underset{\sim}{e_1}\ \ \underset{\sim}{e_2}\ \ \underset{\sim}{e_3})^{-1}$

Theorem 2.1

If a square matrix **A** of order n has n distinct eigenvalues, then A is diagonalisable.

$A = PDP^{-1}$

$P^{-1}A = \cancel{P^{-1}P}DP^{-1}$

$P^{-1}AP = D$

Example 4

Diagonalise the matrix $\begin{pmatrix} 0 & -1 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix}$.

Solution:

From Example 1, we know that $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ are the eigenvectors corresponding to the eigenvalues 1, 2, 3 respectively.

$$\Rightarrow \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \mathbf{A} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

3 Computing the Powers of a Matrix

By using the diagonalization of a matrix **A**, it is relatively easy to compute any power of **A** without going through the traditional process of computing which can be very tedious.

Computing the power of a diagonalisable matrix
If **A** is a $n \times n$ diagonalisable matrix, then for any positive integer k, $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$.

Since **A** is diagonalizable, then there exists invertible matrix **P** and diagonal matrix **D** such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} \iff \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

$$\begin{aligned}
\mathbf{A}^k &= \mathbf{A}\mathbf{A}\ldots\mathbf{A} \\
&= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})\ldots(\mathbf{PDP}^{-1}) && (k \text{ copies of } \mathbf{A}) \\
& && (k \text{ copies of } \mathbf{PDP}^{-1}) \\
&= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\ldots\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} && (k-1 \text{ copies of } \mathbf{D}(\mathbf{P}^{-1}\mathbf{P})) \\
&\quad \text{(Matrix Multiplication is associative)} \\
&= \mathbf{P}(\mathbf{DIDIDI}\ldots\mathbf{DI\ D})\mathbf{P}^{-1} \\
&= \mathbf{PDDD}\ldots\mathbf{D\ DP}^{-1} && (k-1 \text{ copies of } \mathbf{DI}. \text{ Note } (\mathbf{P}^{-1}\mathbf{P})=\mathbf{I}) \\
&= \mathbf{PD}^k\mathbf{P}^{-1}. && (k \text{ copies of } \mathbf{D})
\end{aligned}$$

Example 5

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^5 = \begin{pmatrix} 1^5 & & \\ & 2^5 & \\ & & 3^5 \end{pmatrix}$$

Given that $\mathbf{A} = \begin{pmatrix} 0 & -1 & -2 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix}$, find a matrix **P** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

Find also a diagonal matrix **E** such that $\mathbf{A}^{10} = \mathbf{PEP}^{-1}$.

Solution:

From Example 3, we have diagonalised **A** to be $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1}$,

with $\mathbf{P} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Then $\mathbf{A}^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 3^{10} \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1}$

Therefore $\mathbf{E} = \begin{pmatrix} 1 & & \\ & 2^{10} & \\ & & 3^{10} \end{pmatrix}$

4 Other Properties of Eigenvalues and Eigenvectors

Let **A** and **B** be square matrices of the same size and **x** be an eigenvector of both **A** and **B** with corresponding eigenvalue λ and μ respectively.

Example 6
Show that $\mathbf{A}^{-1}\mathbf{x}=\lambda^{-1}\mathbf{x}$, where $\lambda \neq 0$.

Solution: $A\underset{\sim}{x}=\lambda \underset{\sim}{x}$

$$A^{-1}A\underset{\sim}{x} = A^{-1}(\lambda \underset{\sim}{x})$$

$$\underset{\sim}{x}=\lambda A^{-1}\underset{\sim}{x}$$

$$A^{-1}\underset{\sim}{x}=\frac{1}{\lambda}\underset{\sim}{x}=\lambda^{-1}\underset{\sim}{x}$$

so **x** is an eigenvector of $\mathbf{A}^{-1}$ with corresponding eigenvalue λ^{-1}.

Example 7
Show that **x** is an eigenvector of
- (i) **A+B** with eigenvalue $\lambda+\mu$.
- (ii) $k\mathbf{A}$ where $k\in R$, with corresponding eigenvalue $k\lambda$.
- (iii) $\mathbf{A}^n$ with eigenvalue λ^n, n is a positive integer.

Solution:
(i)

(ii)

(iii)

Example 8

The matrix **A** has an eigenvalue λ with corresponding eigenvector **x**. Given that a non-singular matrix **E** is of the same order as **A**, then show that **Ex** is an eigenvector of the matrix **B** where $\mathbf{B} = \mathbf{E}\mathbf{A}\mathbf{E}^{-1}$ and λ is the corresponding eigenvalue.

Solution:

4 Applications of Eigenvalues and Eigenvectors

Example 9

A sequence of numbers $a_0, a_1, a_2, \ldots$ is defined by the linear recurrence relation

$$a_n = a_{n-1} + 6a_{n-2}, \; n \geq 2,$$

where $a_0 = a_1 = 1$. Find an explicit formula for a_n.

Solution:

Consider the vector $\mathbf{u}_n = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$.

Then, $\mathbf{u}_n = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} a_n + 6a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \mathbf{A}\mathbf{u}_{n-1}$, where $\mathbf{A} = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix}$

Thus, $\mathbf{u}_n = \mathbf{A}^n \mathbf{u}_0$ for $n \geq 0$.

For the eigenvalues of $\mathbf{A}$, consider $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det \begin{pmatrix} 1-\lambda & 6 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda + 2)(\lambda - 3) = 0$$

When $\lambda =$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

Its eigenvector is .

When $\lambda =$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

Its eigenvector is .

Thus,

$\mathbf{A} =$

$\mathbf{A}^n =$

Hence, $\mathbf{u}_n = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \mathbf{A}^n \mathbf{u}_0 = \mathbf{A}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$

$=$

So, $a_n =$

SUMMARY