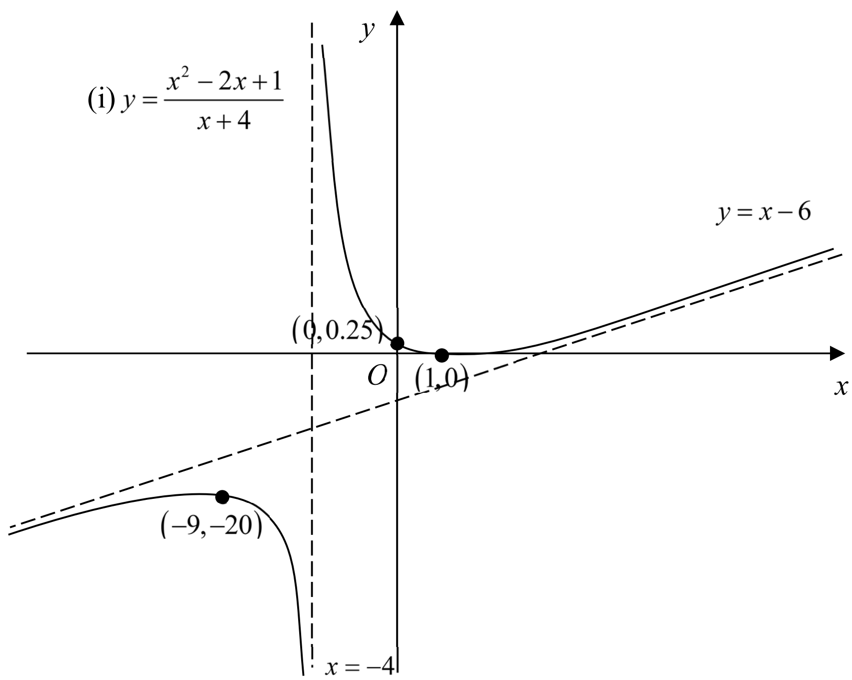
Mastery Questions

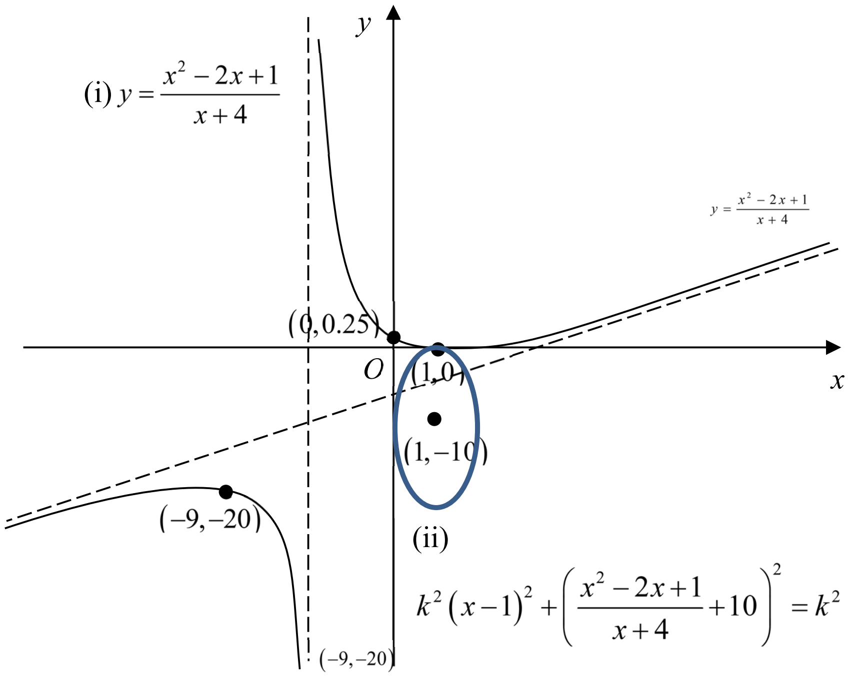
1. [CJC/17/Promos/Q3]

(i) Sketch the curve with equation $y=\dfrac{x^2-2x+1}{x+4}$, stating the equations of any asymptotes, the coordinates of any turning points and any points of intersection with the axes. **[3]**

(ii) By drawing a suitable graph on the same diagram in part (i), find the range of values of k, where $k>0$, such that $k^2(x-1)^2+\left(\dfrac{x^2-2x+1}{x+4}+10\right)^2=k^2$ has at least one real root. **[3]**

[Ans: (ii) $k \geq 10$]

1(i)	

1(ii)	 Draw $k^2(x-1)^2+(y+10)^2=k^2$ or $(x-1)^2+\dfrac{(y+10)^2}{k^2}=1$ on the same diagram, ellipse centre $(1,-10)$ and horizontal axis fixed ±1 and vertical axis $\pm k$ variable. For at least one root, $(0,0.25)$

2. (Tutors can highlight this question for students to try as self-practice)

A curve is defined parametrically by the equations,

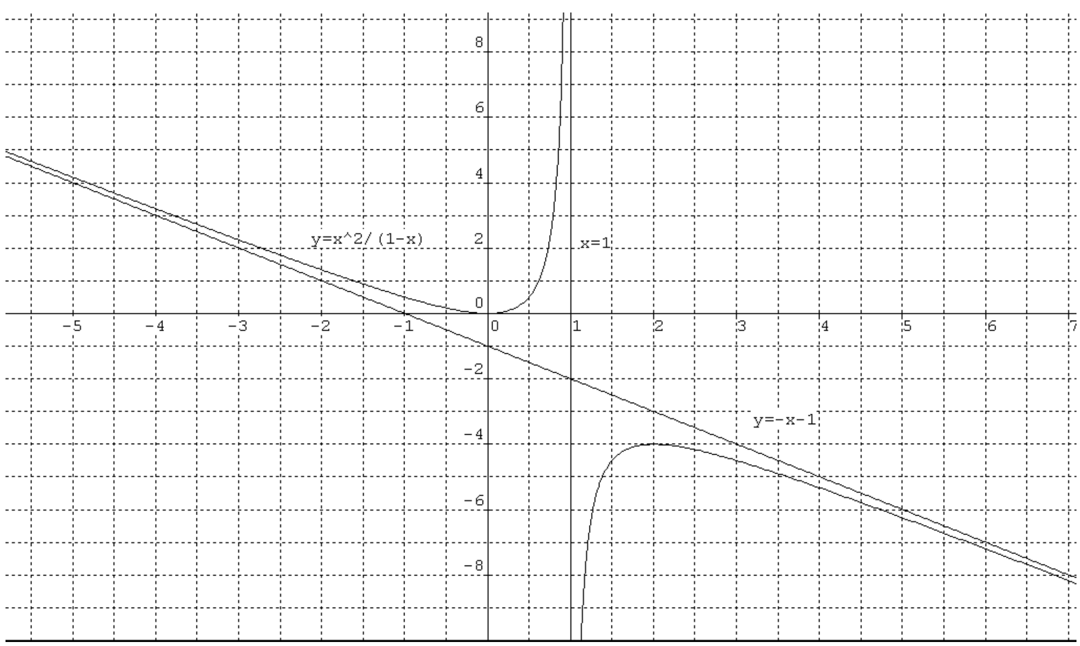
$$x=\frac{t}{1+t} \quad ; \quad y=\frac{t^2}{1+t} \qquad t\in\mathbb{R},\ t\neq -1$$

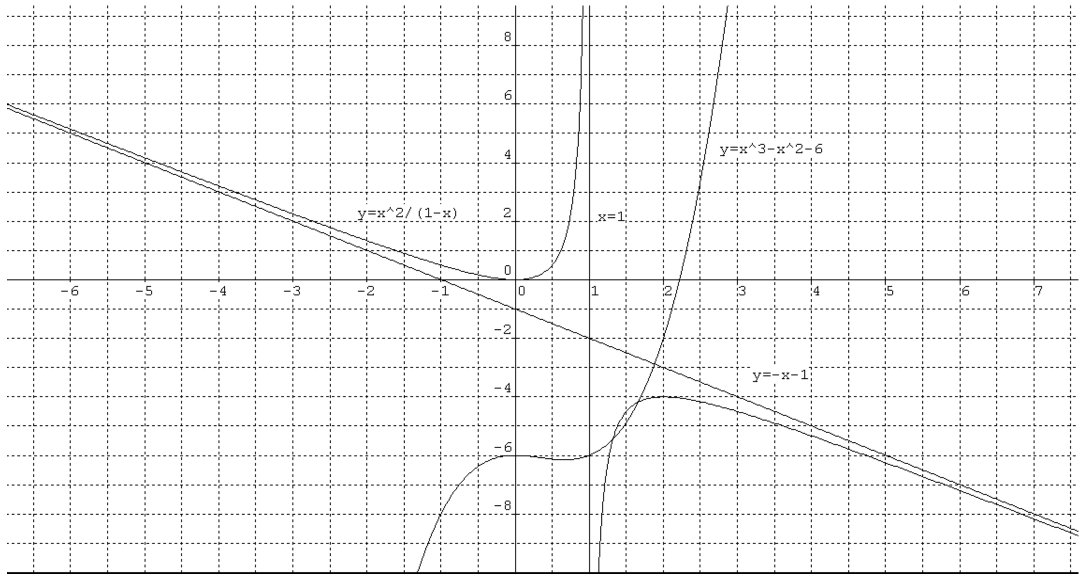
(i) Find the Cartesian equation of the curve, expressing your answer in the form $y=\mathrm{f}(x)$.

(ii) Sketch the curve. Label your graph clearly, indicating any asymptote(s) and stationary point(s).

(iii) By sketching another suitable graph on the same diagram as in (ii), determine the number of real roots of the equation $\mathrm{f}(x)+6=x^3-x^2$.

[Ans : (i) Cartesian Equation : $y=\dfrac{x^2}{1-x}$ (iii) Number of real roots = 2]

(i)	$x=\dfrac{t}{1+t}\quad,\qquad y=\dfrac{t^2}{1+t}$ $\Rightarrow x+xt=t$ $\Rightarrow t=\dfrac{x}{1-x}$ $y=\dfrac{\dfrac{x^2}{(1-x)^2}}{1+\dfrac{x}{1-x}}=\dfrac{x^2}{(1-x)^2}\left[\dfrac{1-x}{1}\right]=\dfrac{x^2}{1-x}$ $\therefore\quad y=\dfrac{x^2}{1-x} \longleftarrow$ Cartesian Equation of the curve
(ii)	To find the equation of the asymptotes of the curve $y=\dfrac{x^2}{1-x}$ $\qquad\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1+\dfrac{1}{(1-x)^2}$ $y=-x-1+\dfrac{1}{1-x}$ $\qquad\dfrac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\dfrac{1}{(1-x)^2}=1\Rightarrow x=0$ or $x=2$ $x\to\infty, y\to -x-1$ $\qquad$ When $x=0, y=0\,(\text{min})$. When $x=2, y=-4\,(\text{max})$. Equation of the asymptotes: $y=-x-1$ and $x=1$ 
(iii)	$\mathrm{f}(x)+6=x^3-x^2\Rightarrow \mathrm{f}(x)=x^3-x^2-6$. We need to find the number of intersections between the curve representing $y=x^3-x^2-6$ and $y=\mathrm{f}(x)$



From the sketch, there are 2 intersection points between the graphs, hence, 2 real roots to the equation.

3. [CJC/06/Promo/Q8]

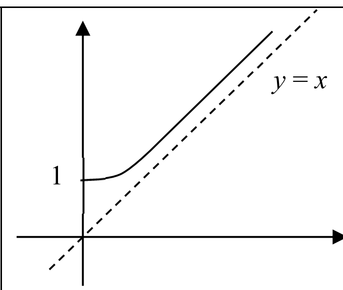
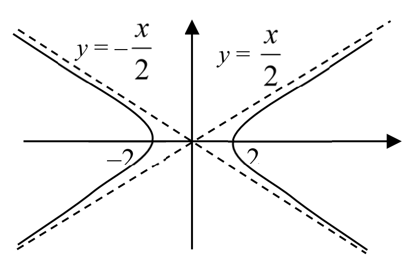
Find the cartesian equations and coordinates of the intersections of the following curves with the x and y–axes (if any):

(i) $x = t^2, y = \sqrt{t^4 + 1}$ **[2]**

(ii) $x = -2\sec\theta, y = \tan\theta$ **[2]**

On separate diagrams, sketch the curves in (i) and (ii), indicating clearly the equation(s) of any asymptotes. **[4]**

[Ans : i) $y = \sqrt{x^2 + 1}$; (0, 1), ii) $y^2 + 1 = \dfrac{x^2}{4}, (-2, 0), (2, 0)$]

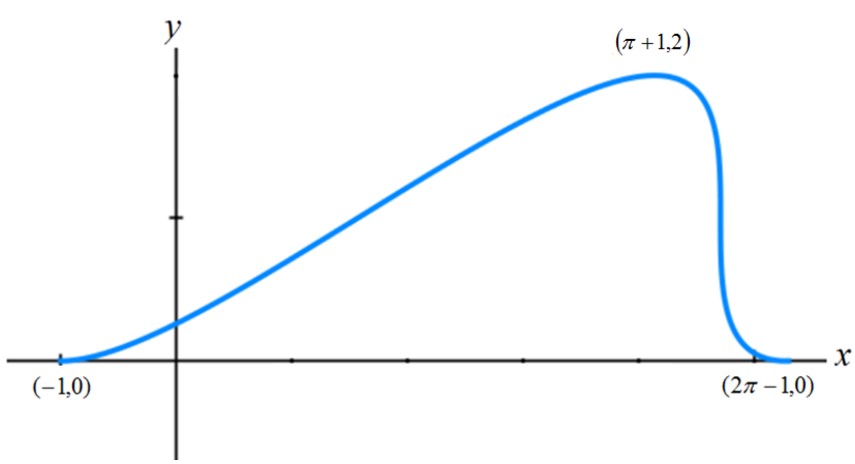
i)	$y = \sqrt{x^2 + 1}$; (0, 1) As $x \to \pm\infty$, $y \to \sqrt{x^2} \to \lvert x \rvert$ Note: $\sqrt{x^2} = \lvert x \rvert$! Since $x = t^2$, it means $x > 0$, hence $\lvert x \rvert = x$. Therefore the oblique asymptote is $y = x$	
ii)	$\dfrac{x^2}{4} = \sec^2\theta, y^2 = \tan^2\theta$ $\tan^2\theta + 1 = \sec^2\theta$ $y^2 + 1 = \dfrac{x^2}{4}$ Intercepts : (–2, 0) , (2, 0)	

4. [N2016/H2 Maths/2/3 (part)]

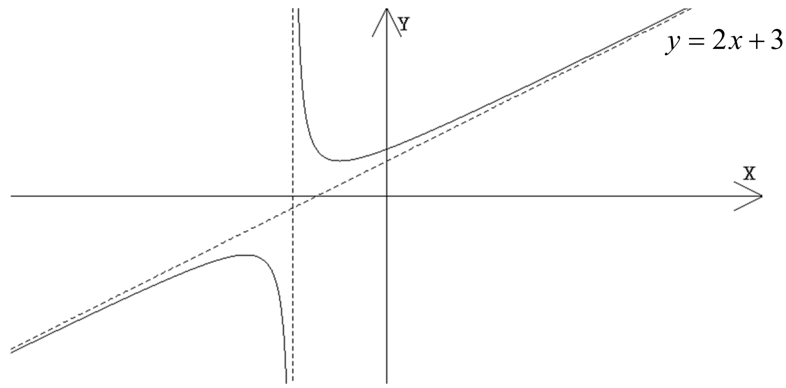
A curve D has parametric equations

$$x = t - \cos t, \quad y = 1 - \cos t, \quad \text{for } 0 \le t \le 2\pi \ .$$

Sketch the graph of D. Give in exact form the coordinates of the points where D meets the x-axis, and also give in exact form the coordinates of the maximum point on the curve.

4	$x = t - \cos t$, $y = 1 - \cos t$ **(i)** On the x-axis, $1 - \cos t = 0 \Rightarrow \cos t = 1 \Rightarrow t = 0$ or $t = 2\pi$ When $t = 0$, $x = 0 - 1 = -1$ When $t = 2\pi$, $x = 2\pi - 1$ Hence the coordinates of points on the x-axis are $(-1,0)$ and $(2\pi - 1,0)$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 + \sin t} = 0$ for maximum or minimum $\Rightarrow \sin t = 0$ $\Rightarrow t = 0, \pi, or 2\pi$ Since $t = 0$ and $t = 2\pi$ correspond to the two points on the x-axis, the maximum point occurs when $t = \pi$. When $t = \pi$, $x = \pi - \cos\pi = \pi - (-1) = \pi + 1$, $y = 1 - \cos\pi = 1 - (-1) = 2$ Hence the coordinates of the maximum point are $(\pi + 1, 2)$. The following diagram shows the graph of D: 

5.



A sketch of the curve $y=\dfrac{ax^2+bx+c}{x+d}$, where a,b,c and d are constants, is shown, not to scale in the diagram. The equations of the asymptotes, also shown in the diagram, are $x=-2$ and $y=2x+3$.

(i) Write down the value of d. [1]

(ii) Find the value of a and show that $b=7$. [3]

(iii) Given that the curve has a stationary point where $x=-1$, find the value of c and the x-coordinates of the other stationary point. [4]

(iv) Copy the above sketch and, by drawing a sketch of another suitable curve in the same diagram, find the number of real roots for the equation

$$2x^4+7x^3+8x^2=x+2.$$ [3]

(i)	$x=-2$ is the vertical asymptote implies the denominator is $(x+2)$. Therefore, $d=2$.	
(ii)	$y=\dfrac{ax^2+bx+c}{x+2}$ By long division $y=ax+(b-2a)+\dfrac{c-2(b-2a)}{x+d}$ As $x\to\pm\infty$, $y\to ax+(b-2a)$. $\therefore ax+(b-2a)=2x+3$	Comparing coefficients: $a=2$ $b-2a=3$ $b-2(2)=3$ $b=7$
(iii)	$y=2x+3+\dfrac{c-2(7-2\times2)}{x+2}$ $=2x+3+\dfrac{c-6}{x+2}$ $\dfrac{dy}{dx}=2-\dfrac{(c-6)}{(x+2)^2}$ When $x=-1$, $\dfrac{dy}{dx}=0$	Hence $\dfrac{dy}{dx}=2-\dfrac{(8-6)}{(x+2)^2}=2-\dfrac{2}{(x+2)^2}$ When $\dfrac{dy}{dx}=0$ $0=2-\dfrac{2}{(x+2)^2}$ $(x+2)^2=1$ $x+2=1$ or -1 $x=-1$ or -3 The other stationary point is at $x=-3$.

<table>
<tr>
<td></td>
<td>$$0 = 2 - \frac{(c-6)}{(-1+2)^2}$$
$$\frac{(c-6)}{(1)^2} = 2$$
$$c = 8$$</td>
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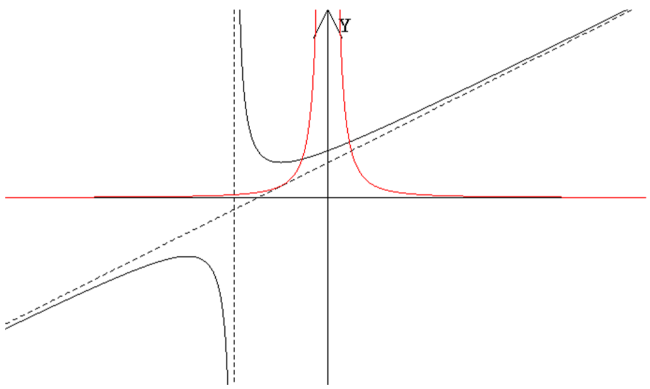
<tr>
<td>(iv)</td>
<td colspan="2">$$2x^4 + 7x^3 + 8x^2 = x + 2$$
$$x^2\left(2x^2 + 7x + 8\right) = x + 2$$
$$\frac{x^2\left(2x^2 + 7x + 8\right)}{(x+2)} = 1$$
$$\frac{\left(2x^2 + 7x + 8\right)}{(x+2)} = \frac{1}{x^2}$$
$y = 2x + 3$
x
$x = -2$

By sketching the curve $y = \frac{1}{x^2}$ on the same diagram as $y = \frac{ax^2 + bx + c}{x + d}$, we can find the number of real roots by counting the number of intersection points.



From the graph, we see two intersection points. Hence there are 2 solutions.</td>
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